Output Regulation of Uncertain Nonlinear Systems with Nonlinear Exosystems

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***Abstract*— An adaptive control algorithm is proposed for output regulation of uncertain nonlinear systems in output feedback form under disturbances generated from nonlinear exosystems. A new nonlinear internal model is proposed to generate the desired input term for suppression of the disturbances. The proposed internal model design is based on boundedness of the disturbance, high gain design and saturation and the designed internal model is capable to tackle disturbances in any specified initial conditions. The uncertainty in the system allows the all the parameters to be unknown, except the high frequency gain, and it is specified in a vector of constant unknown parameters, which is tackled using nonlinear dominant functions and an adaptive control coefficient. The proposed control algorithm ensures the global convergence of the state variables to the invariant manifold, which implies that the measurement or the tracking error approaches to zero asymptotically.**

***Index Terms*— Output regulation, Internal model principle, Uncertainty, Nonlinear systems, Adaptive control, Backstepping**

I. INTRODUCTION

Output regulation concerns with stabilization of dynamic systems as well rejecting the disturbances or tracking the desired trajectories. The output regulation problem is well posed and solved for linear systems in [1], [2]. For nonlinear systems, an important contribution to output regulation is reported in [3], [4] that the necessary and sufficient conditions for the existence of a local full information solution are specified as that the linearized system is stabilizable and there exists a certain invariant manifold. A semiglobal extension to these results for a class of feedback linearizable systems is reported in [5] using a saturated high-gain observer [6]. Output regulation of error feedback is solved [7], [8] with the application of system immersion technique. The uncertainty parameterized by unknown constant parameters are treated as special cases of exogenous signals and the solution, extended from the error feedback regulation, is referred to as structurally stable regulation. A semiglobal adaptive output feedback control is presented in [9] for nonlinear systems represented by input-output models, using a high-gain observer. Global solutions for output regulation using state or partial state feedback are shown for strict feedback systems in [10] and for extended strict feedback systems in [11]. Global output regulation for output feedback system is reported in [12]. Global adaptive output regulation for the output feedback systems is shown in [13].

A common assumption in the global output regulation results via measurement feedback [12], [13] is that the exosystem is linear. The linear dynamics of the exosystem are convenient for the transformation and reformulation of the exosystem model so that a suitable internal model can then be designed for output regulation [14], [15], [16]. In this paper, we consider global output regulation of uncertain nonlinear systems with a class of nonlinear exosystems. The difficulty encountered in designing internal models for output regulation is that the disturbances are not measured directly, and they are always mixed with the state variables of the system. The nonlinear dynamics in the exosystem make the design of the internal model more difficult. We propose a novel design of a high-gain internal model which exploits the boundedness of the disturbances and the high gain dominance with delicately defined saturation levels. The proposed internal model design is then used together with nonlinear adaptive control techniques to provide a solution to the output regulation with nonlinear exosystems. The proposed control algorithm guarantees the convergence of the state variable to the invariant manifold, which also implies that the tacking error converges to zero.

II. PROBLEM FORMULATION

We consider a single-input-single-output nonlinear system which can be transformed into the output feedback form

$$\begin{aligned} \dot{x} &= A_c x + \phi(y, w, a) + bu \\ y &= Cx \\ e &= y - q(w) \end{aligned} \tag{1}$$

with

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_\rho \\ \vdots \\ b_n \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix},$$

where $x \in R^n$ is the state vector, $u \in R$ is the control, $y \in R$ is the output, e is the measurement output, $a \in R^q$ and $b \in R^n$ are vectors of unknown parameters, $\phi(y, w, a)$ is a smooth vector field with each element being polynomials of its variables and satisfying $\phi(0, w, a) = 0$, q is an unknown polynomial of w, $w \in R^m$ are disturbances, and they are generated from an exosystem

$$\dot{w} = s(w) \tag{2}$$

Assumption 1: The system is minimum phase, i.e., the polynomial $\mathcal{B}(s) = \sum_{i=\rho}^{n} b_i s^{n-i}$ is Hurwitz, and the high frequency gain b_ρ is known.

Remark 1: We assume that b_ρ is known to simplify the presentation. In the case of unknown b_ρ, a Nussbaum gain may be designed in a similar way as in [17].

Assumption 2: The flows of vector field $s(w)$ are bounded and converge to periodic solutions.

The adaptive output regulation problem that we are going to solve is to find a finite dimensional system

$$\begin{aligned} \dot{\mu} &= \nu(\mu, e(t)), \quad \mu \in R^s, \\ u &= u(\mu, e(t)) \end{aligned} \tag{3}$$

such that for every $x(0) \in R^n$, $w(0) \in \Omega \subset R^m$, $\mu(0) \in R^s$, $x(t)$, $\mu(t)$ and $u(t)$ are bounded $\forall t \geq 0$, and $\lim_{t\to\infty} e(t) = 0$.

III. State Transformation

For the system (1) with relative degree $\rho > 1$, the following filter is introduced [18]

$$\begin{aligned} \dot{\xi}_1 &= -\lambda_1 \xi_1 + \xi_2 \\ &\vdots \\ \dot{\xi}_{\rho-1} &= -\lambda_{\rho-1}\xi_{\rho-1} + u \end{aligned} \tag{4}$$

where $\lambda_i > 0$ for $i = 1, \ldots, \rho-1$ are the design parameters. Define the filtered transform

$$\bar{z} = x - [\bar{d}_1 \ldots \bar{d}_{\rho-1}]\xi \tag{5}$$

where $\xi = [\xi_1 \ldots \xi_{\rho-1}]^T$, $\bar{d}_i \in R^n$ for $i = 1, \ldots, \rho-1$ and they are recursively generated by $\bar{d}_{\rho-1} = b$ and $\bar{d}_i = [A_c + \lambda_{i+1} I]\bar{d}_{i+1}$ for $i = \rho-2, \ldots, 1$. The system (1) is then transformed to

$$\begin{aligned} \dot{\bar{z}} &= A_c \bar{z} + \phi(y, w, a) + d\xi_1 \\ y &= C\bar{z} \end{aligned} \tag{6}$$

where $d = [A_c + \lambda_1 I]\bar{d}_1$. It can be shown that $d_1 = b_\rho$ and

$$\mathcal{D}(s) := \sum_{i=1}^{n} d_i s^{n-i} = \mathcal{B}(s) \prod_{i=1}^{\rho-1}(s + \lambda_i) \tag{7}$$

With ξ_1 as the input, the system (6) is with relative degree one and minimum phase. We introduce another state transform to extract the internal dynamics of (6) with $z \in R^{n-1}$ given by

$$z = \bar{z}_{2:n} - \frac{d_{2:n}}{d_1} y \tag{8}$$

where $(\cdot)_{2:n}$ refers to the vector or matrix formed by the 2nd row to the nth row. With the coordinates $\{z, y\}$, (6) is rewritten as

$$\begin{aligned} \dot{z} &= Dz + \psi(y, w, \theta) \\ \dot{y} &= z_1 + \psi_y(y, w, \theta) + b_\rho \xi_1 \end{aligned} \tag{9}$$

where the unknown parameter vector $\theta = [a^T, b^T]^T$, and D is the companion matrix of d given by

$$D = \begin{bmatrix} -d_2/d_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -d_n/d_1 & 0 & \ldots & 0 \end{bmatrix},$$

and

$$\begin{aligned} \psi(y, w, \theta) &= D\frac{d_{2:n}}{d_1} y + \phi_{2:n}(y, w, a) \\ &\quad - \frac{d_{2:n}}{d_1}\phi_1(y, w, a) \\ \psi_y(y, w, \theta) &= \frac{d_2}{d_1} y + \frac{d_{2:n}}{d_1}\phi_1(y, w, a) \end{aligned}$$

Notice that D is Hurwitz, from Assumption 1 and (7), and that the dependence of d on b is reflected in the parameter θ in $\psi(y, w, \theta)$ and $\psi_y(y, w, \theta)$ and it is easy to check that $\psi(0, w, \theta) = 0$ and $\psi_y(0, w, \theta) = 0$.

IV. Internal Model

The output regulation problem considered in this paper is well posed if the following assumption is satisfied.

Assumption 3: There exists an invariant manifold $\pi(w) \in R^{n-1}$ satisfying

$$\frac{\partial \pi(w)}{\partial w} s(w) = D\pi(w) + \psi(q(w), w, \theta) \tag{10}$$

Based on Assumption 3, we have

$$\frac{\partial q(w)}{\partial w} s(w) = \pi_1(w) + \psi_y(q(w), w, \theta) + b_\rho \alpha \tag{11}$$

With ξ_1 bing viewed as the input, α is the feedforward term used for output regulation to tackle the disturbances, and it given by

$$\alpha(w, \theta) = b_\rho^{-1}\left[\frac{\partial q(w)}{\partial w} s(w) - \pi_1(w) - \psi_y(q(w), w, \theta)\right] \tag{12}$$

We now introduce the last transform based on the invariant manifold with

$$\tilde{z} = z - \pi \tag{13}$$

Finally we have the model for the control design

$$\begin{aligned} \dot{\tilde{z}} &= D\tilde{z} + \tilde{\psi} \\ \dot{e} &= \tilde{z}_1 + \tilde{\psi}_y + b_\rho(\xi_1 - \alpha(w)) \\ \dot{\xi}_1 &= -\lambda_1 \xi_1 + \xi_2 \\ &\vdots \\ \dot{\xi}_{\rho-1} &= -\lambda_{\rho-1}\xi_{\rho-1} + u \end{aligned} \tag{14}$$

where

$$\begin{aligned}\tilde{\psi} &= \psi(y,w,\theta)-\psi(q(w),w,\theta)\\ \tilde{\psi}_y &= \psi_y(y,w,\theta)-\psi_y(q(w),w,\theta)\end{aligned}$$

To solve the problem, we need an assumption on the structure of the exosystem.

Assumption 4: There exists an immersion of the exosystem

$$\begin{aligned}\dot{\eta} &= F\eta+\varphi(\alpha)\\ \alpha &= H\eta \qquad (15)\end{aligned}$$

where $\eta \in R^r$ and

$$F=\begin{bmatrix}0&1&0&\dots&0\\0&0&1&\dots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&1\\0&0&0&\dots&0\end{bmatrix},\quad H=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}^T,$$

Remark 2: An example of exosystem in (15) is a Van de Pol equation

$$\begin{aligned}\dot{\eta}_1 &= \eta_2-\eta_1^3+\eta_1\\ \dot{\eta}_2 &= -\eta_1\end{aligned}$$

Since the state in the internal model η is unknown, we design the internal model

$$\begin{aligned}\dot{\hat{\eta}} &= (F-gGkH)(\hat{\eta}-b_\rho^{-1}gGke)\\ &\quad+\bar{\varphi}(\hat{\eta}_1-b_\rho^{-1}gHGke)+gGk\xi_1 \qquad (16)\end{aligned}$$

where g is a design parameter, $G=\text{diag}\{1,g,\dots,g^{r-1}\}$, $k \in R^r$ is chosen such that $F_o=F-kH$ is Hurwitz, $\bar{\varphi}(\cdot): R^r \to R^r$ is a decentralized $C^{\rho-1}$ soft saturation function with

$$\bar{\varphi}_i(\mu)=\begin{cases} l_{u,i}(\mu) & \text{if } \varphi_i(\mu)\ge M_{l,i}\\ \varphi_i(\mu) & \text{if } |\varphi_i(\mu)|<M_{l,i}\\ l_{l,i}(\mu) & \text{if } \varphi_i(\mu)\le -M_{l,i}\end{cases} \qquad (17)$$

The saturation levels are chosen such that

$$\begin{aligned}&\max_{w\in\Omega}|\varphi_i(\alpha(w))|<M_{l,i}\\ &\quad<M_{u,i}<\max_{w\in\Omega,\mu\in R}|\varphi_i(\alpha(w)-\mu)| \qquad (18)\end{aligned}$$

with $M_{u,i}$ denoting another positive real design parameter. The functions $l_{u,i}$ and $l_{l,i}$, for $i=1,\dots,r$, are $(\rho-1)$ order Hermite-Birkoff interpolation functions [19] between the points of $(\mu_{p,i},M_{l,i})$ and $(\mu_{p,i}+\Delta_i,M_{u,i})$, and between $(\mu_{n,i},-M_{l,i})$ and $(\mu_{n,i}-\Delta_i,-M_{u,i})$ respectively with Δ_i being a positive real design parameter, and $\mu_{p,i}$ and $\mu_{n,i}$ being the values such that $\varphi_i(\mu_{p,i})=M_{l,i}$ and $\varphi_i(\mu_{n,i})=-M_{l,i}$. The Hermite-Birkoff interpolation functions ensure that the derivatives match up to the specified order. The boundary conditions for $(\mu_{p,i},M_{l,i})$ are the derivatives of φ_i at $\mu_{p,i}$, and at the upper point $(\mu_{p,i}+\Delta_i,M_{u,i})$, the derivatives are set to zero. Similar settings apply to the points $(\mu_{n,i},-M_{l,i})$ and $(\mu_{n,i}-\Delta_i,-M_{u,i})$. Therefore the soft saturation function $\bar{\varphi}_i$ has its values bounded in $[-M_{u,i},M_{u,i}]$ and it has continuous derivatives up to order $\rho-1$.

Remark 3: If φ_i is upper bounded or lower bounded, then the respective soft saturation function is not needed. Note that the right hand side of (18) can be infinity. In such a case, we only need to ensure $\max_{w\in\Omega}|\varphi_i(\alpha(w))|<M_{l,i}<M_{u,i}$.

If we define the auxiliary error

$$\tilde{\eta}=\eta-\hat{\eta}+b_\rho^{-1}gGke \qquad (19)$$

it can be shown that

$$\begin{aligned}\dot{\tilde{\eta}} &= (F-gGkH)\tilde{\eta}+\varphi(\eta_1)-\bar{\varphi}(\eta_1-\tilde{\eta}_1)\\ &\quad+b_\rho^{-1}gGk(\tilde{z}_1+\tilde{\psi}_y) \qquad (20)\end{aligned}$$

To analyze the property of the internal model, we define a scaled auxiliary error

$$\zeta=G^{-1}\tilde{\eta} \qquad (21)$$

It can be shown that

$$\begin{aligned}\dot{\zeta} &= g(F-kC)\zeta+G^{-1}[\varphi(\eta_1)-\bar{\varphi}(\tilde{\eta}_1-\eta_1)]\\ &\quad+b_\rho^{-1}gk(\tilde{z}_1+\tilde{\psi}_y) \qquad (22)\end{aligned}$$

To complete the internal model design, we need a result given in the following lemma.

Lemma 1: For a smooth φ_i and the saturation function defined in (18), there exists an upper bound of the expression $\left|\frac{\varphi_i(\eta_1)-\bar{\varphi}_i(\eta_1-\mu)}{\mu}\right|$, for all $w\in\Omega,\mu\in R$, and $\eta_1=\alpha(w)$ being a smooth function of w.

Proof: Let us consider in two cases.

Case 1, $|\varphi_i(\alpha(w)-\mu)|\ge M_{l,i}$. From the definition of $M_{l,i}$ we have $\max_{w\in\Omega}|\varphi_i(\alpha(w))|<M_{l,i}$. Since φ_i is continuous, which is implied by the smoothness, there exists a positive real value ϵ, such that $|\mu|>\epsilon$. In this case we have

$$\left|\frac{\varphi_i(\eta_1)-\bar{\varphi}_i(\eta_1-\mu)}{\mu}\right|<2\frac{M_{u,i}}{\epsilon} \qquad (23)$$

Case 2, the saturation function is not activated, ie, $|\varphi_i(\alpha(w)-\mu)|<M_{l,i}$. In this case, we have

$$\begin{aligned}&\varphi_i(\eta_1)-\bar{\varphi}_i(\eta_1-\mu)\\ &= \varphi_i(\eta_1)-\varphi_i(\eta_1-\mu))\\ &= -\int_0^1\frac{\partial\varphi_i(\eta_1-v\mu)}{\partial v}dv\\ &= \mu\int_0^1\left[\frac{\partial\varphi_i(\eta_1+s)}{\partial s}\right]_{s=-v\mu}dv \qquad (24)\end{aligned}$$

Hence, we have

$$\begin{aligned}&\frac{\varphi_i(\eta_1)-\sigma(\varphi_i(\eta_1-\mu))}{\mu}\\ &= \int_0^1\left[\frac{\partial\varphi_i(\eta_1+s)}{\partial s}\right]_{s=-v\mu}dv \qquad (25)\end{aligned}$$

From the smoothness of φ_i, the integration in (25) exists for any finite value of μ, and therefore the integral exists for $|\mu| \le \epsilon$. For $|\mu| > \epsilon$, we still have

$$\left| \frac{\varphi_i(\eta_1) - \bar{\varphi}_i(\eta_1 - \mu)}{\mu} \right| < 2\frac{M_{u,i}}{\epsilon} \quad (26)$$

Combining both the cases, we conclude the proof of the lemma.

With the result shown in Lemma 1, we complete the design of the internal model by setting the gain g as

$$g > 2\lambda_{\max}(P_F) r\bar{g} + 2 \quad (27)$$

where

$$\bar{g} = \max_i \max_{w\in\Omega, \mu\in R} \left| \frac{\varphi_i(\eta_1) - \bar{\varphi}_i(\eta_1 - \mu))}{\mu} \right| \quad (28)$$

and P_F is a positive definite matrix, satisfying

$$P_F(F - kH) + (F - kH)^T P_F = -I$$

V. Control Design

If the system (1) is of relative degree one, then ξ_1 in (14) is the control input. For the systems with higher relative degrees, adaptive backstepping will be used to find the final control output u from the desirable value of ξ_1. Supposing that $\hat{\xi}_1$ is desirable value for ξ_1, we define

$$\hat{\xi}_1 \quad = \quad b_\rho^{-1}\bar{\xi}_1 \quad (29)$$

From (14) we have

$$\dot{e} = \tilde{z}_1 + \bar{\xi}_1 + b_\rho\tilde{\xi}_1 - b_\rho\eta_1 + \tilde{\psi}_y \quad (30)$$

where $\tilde{\xi}_1 = \xi_1 - \hat{\xi}_1$. Sine the nonlinear functions involved in $\tilde{\psi}$ and $\tilde{\psi}_y$ are polynomials with $\tilde{\psi}(0, w, \theta) = 0$ and $\tilde{\psi}_y(0, w, \theta) = 0$, w is bounded, and the unknown parameters are constants, it can be shown that

$$|\tilde{\psi}| \quad < \quad \bar{r}_z(|e| + |e|^p) \quad (31)$$
$$|\tilde{\psi}_y| \quad < \quad \bar{r}_y(|e| + |e|^p) \quad (32)$$

where p is a known positive integer, depending on the polynomials in $\tilde{\psi}$ and $\tilde{\psi}_y$, and $\bar{r}_z$ and $\bar{r}_y$ are unknown positive real constants. We now design the virtual control $\hat{\xi}_1$ as, with $c_0 > 0$,

$$\bar{\xi}_1 = -c_0 e - \hat{l}_0(e + e^{2p-1}) + b_\rho\hat{\eta}_1 - gk_1 e \quad (33)$$

where $\hat{l}_0$ is an adaptive coefficient with $\hat{l}_0(0) = 0$. Using (19), we have the resultant error dynamics

$$\begin{aligned} \dot{e} \quad &= \quad -c_0 e - \hat{l}_0(e + e^{2p-1}) + \tilde{z}_1 \\ &\quad +b_\rho\tilde{\xi}_1 - b_\rho\tilde{\eta}_1 + \tilde{\psi}_y \end{aligned} \quad (34)$$

The adaptive law is given by

$$\dot{\hat{l}}_0 \quad = \quad \gamma_l(e^2 + e^{2p}) \quad (35)$$

where γ_l is a positive real design parameter. If the relative degree $\rho = 1$, we set $u = \hat{\xi}_1$. For $\rho > 1$, adaptive backstepping can be used to obtain the following results:

$$\begin{aligned} \hat{\xi}_2 \quad &= \quad -b_\rho e - c_1\tilde{\xi}_1 - l_1\left(\frac{\partial\hat{\xi}_1}{\partial e}\right)^2 \tilde{\xi}_1 \\ &\quad +\frac{\partial\hat{\xi}_1}{\partial e}[b_\rho(\xi_1 - \hat{\eta}_1) + gk_1 e] \\ &\quad +\frac{\partial\hat{\xi}_1}{\partial\hat{\eta}}\dot{\hat{\eta}} + \frac{\partial\hat{\xi}_1}{\partial\hat{l}_0}\dot{\hat{l}}_0 \qquad (36) \\ \hat{\xi}_i \quad &= \quad -\tilde{\xi}_{i-2} - c_{i-1}\tilde{\xi}_{i-1} - l_{i-1}\left(\frac{\partial\hat{\xi}_{i-1}}{\partial e}\right)^2 \tilde{\xi}_{i-1} \\ &\quad +\frac{\partial\hat{\xi}_{i-1}}{\partial e}[b_\rho(\xi_1 - \hat{\eta}_1) + gk_1 e] \\ &\quad +\frac{\partial\hat{\xi}_{i-1}}{\partial\hat{\eta}}\dot{\hat{\eta}} + \frac{\partial\hat{\xi}_{i-1}}{\partial\hat{l}_0}\dot{\hat{l}}_0 \qquad \text{for } i = 3, \ldots, \rho \end{aligned}$$

where $\tilde{\xi}_i = \xi_i - \hat{\xi}_i$ for $i = 1, \ldots, \rho - 2$, c_i and l_i, $i = 2, \ldots, \rho - 1$, are positive real design parameters, and τ_i, for $i = 1, \ldots, \rho - 2$, are tuning functions. Finally we design the control input as

$$u = \hat{\xi}_\rho \quad (37)$$

VI. Stability Analysis

In this section, we shall establish the boundedness of all the variables and the convergence to zero of the measurement output. We start with the analysis of the internal model by defining

$$V_0 = \zeta^T P_F \zeta \quad (38)$$

Using (22), we have

$$\begin{aligned} \dot{V}_0 \quad &= \quad -g\zeta^T\zeta + 2\zeta^T P_f\{G^{-1}[\varphi(\eta_1) - \bar{\varphi}(\eta_1 - \tilde{\eta}_1)] \\ &\quad +b_\rho^{-1}gk(\tilde{z}_1 + \tilde{\psi}_y)\} \\ &\le \quad -g\zeta^T\zeta + 2\lambda_{\max}(P_F)r\bar{g}\zeta^T\zeta \\ &\quad +2|b_\rho^{-1}|g\|P_F k\|\|\zeta\|(|\tilde{z}_1| + |\tilde{\psi}_y|) \\ &\le \quad -\zeta^T\zeta + (b_\rho^{-1}g\|P_F k\|)^2(|\tilde{z}_1| + |\tilde{\psi}_y|)^2 \\ &\le \quad -\zeta^T\zeta + 2(b_\rho^{-1}g\|P_F k\|)^2|\tilde{z}_1|^2 \\ &\quad +2(b_\rho^{-1}g\|P_F k\|\bar{\gamma}_y)^2(|e| + |e|^p)^2 \\ &\le \quad -\zeta^T\zeta + 2(b_\rho^{-1}g\|P_F k\|)^2|\tilde{z}_1|^2 \\ &\quad +4(b_\rho^{-1}g\|P_F k\|\bar{\gamma}_y)^2(e^2 + e^{2p}) \end{aligned} \quad (39)$$

Define a Lyapunov function candidate

$$\begin{aligned} V \quad &= \quad \beta_1 V_0 + \beta_2\tilde{z}^T P_z\tilde{z} \\ &\quad +\frac{1}{2}[e^2 + \sum_{i=1}^{\rho-1}\tilde{\xi}_i^2 + \gamma_l^{-1}(l_0 - \hat{l}_0)^2] \end{aligned} \quad (40)$$

where β_1 and β_2 are two positive reals, P_z is a positive definite matrix satisfying

$$P_z D + D^T P_z \quad = \quad -I$$

With the design of $\hat{\xi}_i$, for $i = 1, \ldots, \rho$, the dynamics of $\tilde{\xi}_i$ can be easily evaluated. From the dynamics of $\tilde{z}$ in (14) and the dynamics of $\tilde{\eta}$ in (20), virtual controls and adaptive laws designed in the previous section, we have the derivative of V as

$$\begin{aligned}\dot{V} &= \beta_1[-\zeta^T\zeta + 2(b_\rho g\|P_F k\|)^2|\tilde{z}_1|^2 \\ &\quad +4(b_\rho g\|P_F k\|\bar{\gamma}_y)^2(e^2+e^{2p})] \\ &\quad +\beta_2[-\tilde{z}^T\tilde{z} + 2\tilde{z}^T P_z\tilde{\psi}] \\ &\quad +(\hat{l}_0 - l_0)(e^2+e^{2p}) \\ &\quad -c_0e^2 - \hat{l}_0(e^2+e^{2p}) \\ &\quad +e\tilde{z}_1 + e\tilde{\psi}_y - eb_\rho\zeta_1 \\ &\quad +\sum_{i=1}^{\rho-1}[-c_i\tilde{\xi}_i^2 - k_i(\frac{\partial\hat{\xi}_i}{\partial e})^2\tilde{\xi}_i^2 \\ &\quad -\tilde{\xi}_i\frac{\partial\hat{\xi}_i}{\partial e}\tilde{z}_1 - \tilde{\xi}_i\frac{\partial\hat{\xi}_i}{\partial e}\tilde{\psi}_y + \tilde{\xi}_i\frac{\partial\hat{\xi}_i}{\partial e}b_\rho\zeta_1] \end{aligned} \quad (41)$$

The stability analysis can be proceeded by using the inequalities $2xy < rx^2 + y^2/r$ or $xy < rx^2 + y^2/(4r)$ for $x > 0$, $y > 0$ and r being any positive real, to tackle the cross terms between the variables $\tilde{z}$, ζ, e, $\tilde{\xi}_i$, for $i = 1, \ldots, \rho - 1$. It can be shown that there exists a sufficiently big positive real β_1 and then a sufficiently big positive real β_2 finally the sufficient big l_0 such that the following result holds

$$\dot{V} \le -\frac{1}{2}\beta_1\zeta^T\zeta - \frac{1}{4}\beta_2\tilde{z}^T\tilde{z} - c_0e^2 - \sum_{i=1}^{\rho-1} c_i\tilde{\xi}_i^2 \quad (42)$$

The boundedness of V further implies ζ, $\tilde{z}$, e, $\tilde{\xi}_i \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ for $i = 1, \ldots, \rho - 1$, and the boundedness of $\hat{l}_0$. Since the disturbance w is bounded, e, $\tilde{z}$, $\zeta \in \mathcal{L}_\infty$ implies the boundedness of y, z and $\hat{\eta}$, which further implies the boundedness of $\hat{\xi}_1$ and then the boundedness of ξ_1. The boundedness of $\hat{\xi}_1$ and ξ_1, together with the boundedness of e, $\hat{\eta}$, $\hat{l}_0$ and $\hat{b}_\rho$ implies the boundedness of $\hat{\xi}_2$ and then the boundedness of ξ_2 follows the boundedness of $\tilde{\xi}_2$. Applying the above reasoning recursively, we can establish the boundedness of $\hat{\xi}_i$ for $i > 2$ to $i = \rho - 1$. We then conclude that all the variables are bounded.

The boundedness of all the variables implies the boundedness of $\dot{\zeta}$, $\dot{\tilde{z}}$, $\dot{e}$, and $\dot{\tilde{\xi}}_i$, which further implies, together with ζ, $\tilde{z}$, e, $\tilde{\xi}_i \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and Barbalat's lemma, $\lim_{t\to\infty}\zeta = 0$, $\lim_{t\to\infty}\tilde{z} = 0$, $\lim_{t\to\infty}e(t) = 0$ and $\lim_{t\to\infty}\tilde{\xi}_i = 0$ for $i = 1, \ldots, \rho - 1$.

The above analysis proves the following result.

Theorem 1: For a system (1) satisfying the Assumptions 1 to 4, the output regulation problem is globally solvable. In particular, the feedback control system consists of the ξ-filters (4), the adaptive internal model (16), the parameter adaptive law (35) and the feedback control (37).

VII. An Example

We use a simple example to illustrate the proposed control design, concentrating on the design of nonlinear internal model. Consider a first order system

$$\begin{aligned}\dot{y} &= \theta y^2 - w_1 + u \\ e &= y\end{aligned} \quad (43)$$

where θ is an unknown parameter, the disturbance w_1 is generated by

$$\begin{aligned}\dot{w}_1 &= w_2 + w_1 - w_1^3 \\ \dot{w}_2 &= -w_1\end{aligned} \quad (44)$$

with $|w_1| \le \Delta_\eta$, a known constant. It is easy to see that $q(w) = 0$, $\pi(w) = 0$ and

$$\alpha(w) = w_1 \quad (45)$$

From the exosystem and the desired feedforward input α, it can be seen that Assumption 4 is satisfied with $w = \eta$, and that the exosystem (44) is in the format of (15), with

$$\begin{aligned}\phi_1(\eta_1) &= \eta_1 - \eta_1^3 \\ \phi_2(\eta_1) &= -\eta_1\end{aligned} \quad (46)$$

Following (16), we design the internal model

$$\begin{aligned}\begin{bmatrix}\dot{\hat{\eta}}_1 \\ \dot{\hat{\eta}}_2\end{bmatrix} &= \begin{bmatrix}-\bar{k}_1 & 1 \\ -\bar{k}_2 & 0\end{bmatrix}\begin{bmatrix}\hat{\eta}_1 - \bar{k}_1 y \\ \hat{\eta}_2 - \bar{k}_2 y\end{bmatrix} \\ &\quad + \begin{bmatrix}\bar{\phi}_1(\hat{\eta}_1 - \bar{k}_1 y) \\ \bar{\phi}_2(\hat{\eta}_1 - \bar{k}_1 y)\end{bmatrix} + \begin{bmatrix}\bar{k}_1 \\ \bar{k}_2\end{bmatrix} u\end{aligned} \quad (47)$$

where $\bar{k}_1 = gk_1$, $\bar{k}_2 = g^2k_2$. Since the system is of relative degree one, we design the functions $\bar{\phi}_i(\mu)$ as $\sigma_i(\phi_i(\mu))$ for $i = 1, 2$ with σ_i being the saturation functions with saturation levels at $|\Delta_\eta - \Delta_\eta^3|$ and Δ_η respectively. The control input is given by

$$u = -cy - \hat{l}_0(y + y^3) + \hat{\eta}_1 - gk_1 y \quad (48)$$

where

$$\dot{\hat{l}}_0 = y^2 + y^4, \text{ with } \hat{l}_0(0) = 0 \quad (49)$$

Finally we decide the value of g. From the functions ϕ_i, $i = 1, 2$, we have $\bar{g} = 1 + 3\Delta_\eta^2$. The matrix P_F can be decided once the gain k is decided. For this simple example, we have

$$P_F = \frac{1}{2}\begin{bmatrix}\frac{1}{k_1} + \frac{k_2}{k_1} & -1 \\ -1 & \frac{1}{k_1} + \frac{k_1}{k_2} + \frac{1}{k_1k_2}\end{bmatrix} \quad (50)$$

It is then straightforward to calculate g using (27) with $r = 2$.

For simulation study, we set $c = 1$, $k_1 = 3$, $k_2 = 2$, and we take $\Delta_\eta = \sqrt{2}$ based on the characteristics of the exosystem. With the chosen k and Δ_η, we obtain $g = 20.3$. In the simulation study, we found that this g value is very conservative, and the results shown in this paper were obtained with $g = 2$. The system output and input are shown in Figure 1, while the disturbance generated from the exosystem and its estimate generated from the internal model are shown in Figure 2. As shown in the figures, the internal model successfully reproduces the feedforward control needed after a transient period, and the system output is regulated to zero, as required.

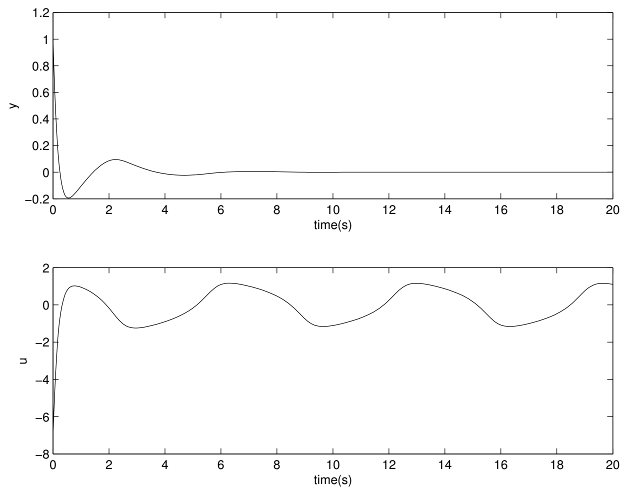

Fig. 1. System output y and input u.

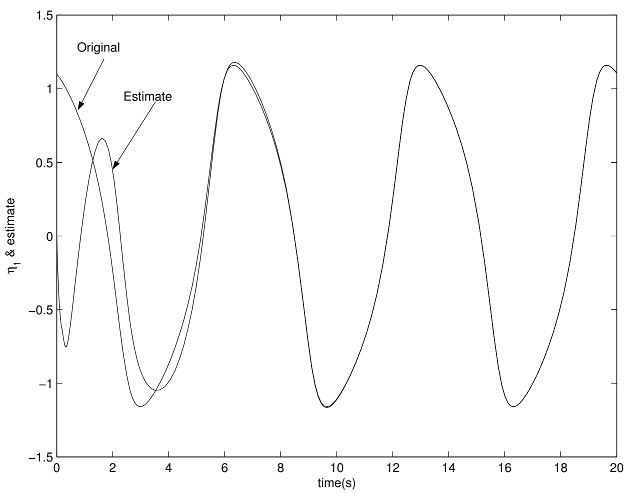

Fig. 2. Disturbance η_1 and its estimate $\hat{\eta_1}$.

VIII. Conclusions

We have proposed a control design for output regulation of uncertain systems with nonlinear exosystems. The success of the proposed design relies on the design of a new nonlinear internal model, which incorporates in the high gain design with saturation. The proposed control design only uses the structural information of the system, and the high frequency gain, and no knowledge is needed of any other system parameters. Even though the adaptive control techniques are used for the control design, the unknown system parameters are not estimated, unlike a completely adaptive treatment in [17]. The system uncertainties are tackled by using high order polynomials which dominate the uncertainties. This robust way of tackling uncertainties makes it possible to deal with the unknown disturbances in the measurement, which is not considered in [17]. With one coefficient being made adaptive, other control design coefficients involved in the proposed algorithm can be any positive reals. The proposed control algorithm ensures the convergence of state variables to the invariant manifold globally, and the measurement output approaches zero asymptotically.

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