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Optimal State Estimation with Continuous, Multirate and Randomly Sampled Measurements

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***Abstract*— This paper presents an optimal filter for a continuous dynamic system with continuous, multirate and randomly sampled measurements. Using the optimal filtering theory for the Ito-Volterra systems with discontinuous measure, the optimal filter for linear state space model with continuous and discrete measurements is rigorously derived, and several known results are recovered, including the Kalman-Bucy and Jazwinski filters. A previously unknown optimal filter for the continuous systems with continuous and sampled measurements, including the case of multirate and random sampling, is obtained. Using the Monte Carlo simulations, the derived filter is compared with the previously reported alternatives. The comparison shows that the developed filter gives the least-mean-squares estimates of the states and the correct estimation error covariance. The alternative filters produce less than optimal estimates, and, at the same time, tend to overestimate the quality of the obtained estimations. Numerical simulations demonstrate that the proposed approach is more convenient in practice: It allows one to simultaneously handle analog and sampled measurements without approximations, and is particularly convenient in the case of the multirate and randomly sampled measurements, often present with a human-in-the-loop and networked data acquisition.**

I. INTRODUCTION

Most processes of practical interest are continuous in nature, while the available measurements used to probe the current state of the process are either sampled (discrete), or a combination of sampled and continuous measurements. There are three fundamental options in approaching the problem of state estimation of a continuous process with the combination of continuous and discrete measurements, summarized in Figure 1: (1) Discrete state estimator approach requires the sampling of the continuous model of the process and the approximation of the available continuous measurements. Subsequently, one of the known state estimators for discrete systems (i.e. discrete Kalman filter, KF) can be applied. (2) A second alternative is to lift the discrete measurements into the space of continuous functions (e.g. by using a polynomial fit of discrete measurements in a sliding window) and then apply one of the known results for state estimation for the continuous system (i.e. Kalman-Bucy filter). (3) The final option is to directly consider the state estimation problem with a continuous model and the combination of the discrete and continuous measurements. The simplicity and the applicability of the classical methods of the state estimation resulted in a wide acceptance of the

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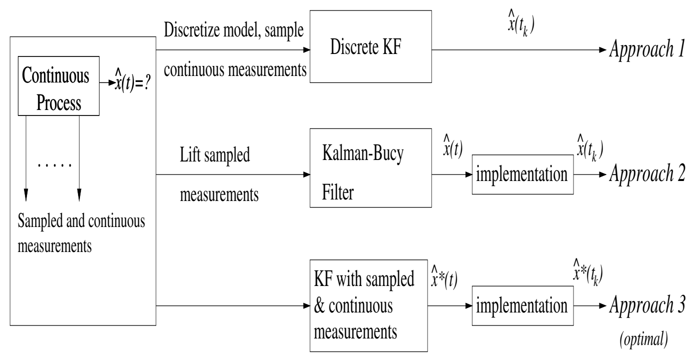

Fig. 1. State estimation problem for continuous process with continuous and sampled measurements.

first two approaches. However, the last approach must be followed to obtain the optimal state estimation, and is more theoretically challenging since it leads to continuous filter equations with discontinuous inputs.

Most of the existing algorithms on multirate state estimation are based on the application of the discrete KF (Approach 1). For example, a multirate extended KF (EKF) was used in [1] to estimate the unmeasurable process states using frequently available measurements of temperature and density and the infrequent and delayed measurements of average molecular weights. Shah et al. [2] implemented a multirate formulation of the iterated EKF on a bioreactor. Mutha et al. [3] proposed fixed-lag smoothing-based EKF algorithm. The Kalman filter has also been a basis of multirate digital filters (decimators and interpolators) and filter banks [4]. A lifting technique was used in [6] to transform a multirate single-input single-output system to a single-rate MIMO system, which allowed them to use slow-rate measurements to generate high rate control inputs.

In this paper, we first present the description of the stochastic linear systems in the integral form of the Ito-Volterra (IV) equations. To allow for the case of discontinuities in controls, measurements and states, we modify the standard Ito-Volterra model by introducing an integral model with discontinuous measure. We then show that the optimal filter for the IV systems with discontinuities in measurements can be specialized for the case of state space systems. Several well known results were recovered including Kalman-Bucy and Jazwinski (continuous process with discrete measurements) filters. A previously unknown optimal filter for the continuous systems with continuous and sampled measurements, including the case of multirate and random sampling, is obtained. The paper is concluded with a numerical illustration.

II. Ito-Volterra Description of Dynamic Systems

Let (Ω, F, P) be a complete probability space with an increasing right-continuous family of σ-algebras $F_t, t \geq 0$, and let $(W(t), F_t, t \geq 0)$ and $(V(t), F_t, t \geq 0)$ be independent Wiener processes. Here Ω is the sample space, F is a set of subsets on which the probability measure (or, simply, probability) is defined, and P is the probability defined on F. All subsets of F form a σ-algebra, and F_t denotes a family of subsets (σ-algebra) for each t such that for $t_1 < t_2$, $F_{t_1} \subset F_{t_2}$. The partly observed F_t-measurable random process $(x(t), z(t))$ can be described using the Ito-Volterra equations:

$$x(t)=\int_0^t [A(t,s)x(s)+B(t,s)u(s)]ds+\int_0^t G(t,s)dW(s) \quad (1)$$

$$z(t)=\int_0^t C(t,s)x(s)ds + \int_0^t H(t,s)dV(s) \quad (2)$$

where $x(t) \in R^n$ is the state vector, and $z(t) \in R^m$ is a vector of measurements integrated over the time interval $[0,t]$. The vector-valued function $B(t,s)u(s)$ describes the effect of known system inputs. Matrix functions $A(t,s)$ and $G(t,s)$ of appropriate dimension and vector-function $B(t,s)u(s)$ are smooth functions of t uniformly in s. Functions $C(t,s)$, and $H(t,s)$ of appropriate dimensions are continuous in t and s and $H(t,s)H^T(t,s) > 0$. Both t and s are independent (e.g. time) variables with $t \geq s \geq 0$ and can be used to assign a variable number of time-varying delays in both states and measurements. All coefficients in (1) and (2) are deterministic functions. Without loss of generality, we assume zero initial conditions.

The estimation problem is to find the estimate of the system state $x(t)$ described by the Ito–Volterra model (1) based on the observation process $Z(t) = \{z(s), 0 \leq s \leq t\}$, which minimizes the Euclidean 2-norm

$$J = E[(x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t))] \quad (3)$$

at each time moment t. In alternative formulation, our objective is to find conditional expectation $m(t) = \hat{x}(t) = E(x(t) \mid F_t^Z)$. As usual, the matrix function $P(t) = E[(x(t) - m(t))(x(t) - m(t))^T \mid F_t^Z]$ is the estimation error covariance.

The above formulation is, in fact, the Kalman filtering problem for the integral Ito-Volterra system. The standard state space formulation is recovered by making all functional parameters in (1) and (2) dependent on s only.

It can be shown that the variance $P(t)$ alone is not sufficient to completely characterize the state estimation process and obtain closed-form filtering equations for dynamic systems in the integral form. However, the explicit solution can be obtained in terms of the integral cross-correlation function $f(t,s)$, which characterizes the deviation of the optimal estimate $m(t)$ from an unknown true state $x(t)$, and defined as:

$$f(t,s) = E[(x_s^t - m_s^t)(x(s) - m(s))^T \mid F_{t,s}^Z] \quad (4)$$

where x_s^t can be viewed as a state with independent (time) variable s and parameter t, and is equal to

$$x_s^t=\int_0^s [A(t,r)x(r)+B(t,r)u(r)]dr+\int_0^s G(t,r)dW(r) \quad (5)$$

The governing equation for x_s^t can be differentiated with respect to s to yield the state space form of equation (5). $F_{t,s}^Z$ is the σ-algebra generated by the stochastic process z_s^t:

$$z_s^t = \int_0^s C(t,s)x(s)ds + \int_0^s H(t,s)dV(s) \quad (6)$$

and $m_s^t = E[x_s^t \mid F_{t,s}^Z]$, where we treat t as a parameter. Note that function f is a generalization of the variance P since $f_t^t = P(t)$. Furthermore for $s = t$, $x_s^t = x(t)$ and $z_s^t = z(t)$.

III. Optimal Filtering for Systems with Bounded Discontinuities

Consider a nondecreasing vector-valued function of bounded variation: $\mu(t) = (\mu_1(t), \ldots, \mu_m(t)) \in R^m$. In essence, $\mu(t)$ is an arbitrary function, and it is only required that it remains bounded on each finite subinterval of its definition, and $\mu(t_1) \leq \mu(t_2)$ if $t_1 \leq t_2$. Continuity of $\mu(t)$ is not required. We can express it as

$$\mu(t) = \{\mu_k^c(t) + \sum_{i=1}^N \Delta\mu_{ki}\chi(t - t_{ki}),\ k = \overline{1,m}\} \quad (7)$$

where $\mu_k^c(t)$ is a continuous nondecreasing function, and the second term describes bounded jumps in k-th components of $\mu(t)$ at times t_{ki}, where χ is the Heaviside unit step function and $\Delta\mu_{ki}$ is the size of the jump. The sampled measurements are modelled assuming $\Delta\mu_{ki} = 1$.

The discontinuous measure μ can be used to describe discontinuities in states and measurements. Only discontinuity in the measurements are considered in this paper, resulting in the following model for the k-th measurement channel:

$$z_k(t)=\int_0^t C_k(t,s)x(s)d\mu_k(s)+\int_0^t H_k(t,s)dV_k(\mu_k(s)) \quad (8)$$

where μ_k is the k-th component of μ, $k = \overline{1,m}$.

Theorem 1: [7] The optimal in Kalman sense estimate $m(t)$ of the states of system (1) based on discontinuous integral measurements (8) satisfies the filter equation

$$\begin{aligned} m(t) &= \int_0^t (A(t,s)m(s) + B(t,s)u(s))ds \\ &+ \int_0^t K(t,s)[dz_s^t - C(t,s)m(s)d\mu(s)] \end{aligned} \quad (9)$$

where $K(t,s) = f(t,s)C^T(t,s)(H(t,s)H^T(t,s))^{-1}$, and function $f(t,s)$ satisfies the Riccati-like equation

$$\begin{aligned} f(t,s) &= \int_0^s [A(s,r)f^T(t,r) + f(s,r)A^T(t,r) \\ &+\frac{1}{2}(G(t,r)G^T(s,r) + G(s,r)G^T(t,r))]dr \\ &- \int_0^s [K_{tsss}C(s,r)f^T(s,r) + K_{sttt}C(t,r)f^T(t,r) \\ &-\frac{1}{2}K_{ttts}C(s,r)f^T(s,r) - \frac{1}{2}K_{ssst}C(t,r)f^T(t,r)]d\mu(r) \end{aligned}$$

where $K_{tsss}(t,s,r) = f(t,r)A^T(s,r)[B(s,r)B^T(s,r)]^{-1}$, with similar expressions used to define the gains $K_{\times\times\times\times}$

with different subscripts, and the multiplication by an m-dimensional measure $\mu(t)$ is in the componentwise sense, as in equation (8).

In the next section, the result for the IV systems with discontinuous measure in the measurement model is specialized for the case of continuous state space systems with an arbitrary combination of discrete and continuous measurements.

IV. Optimal Filter for State Space Systems with Continuous and Discrete Measurements

Consider a particular case of the Ito-Volterra system with A, B, G independent of t. Differentiation of (1) with respect to t yields the standard state space system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)w(t) \qquad (10)$$

where $\omega dt = dW$, $\omega(t) \sim N(0, \tilde{Q}(t))$ is the $l \times 1$ white Gaussian process. Without loss of the generality, assume $\tilde{Q}(t) = I$. Further assume that the observation process is also memoryless (C and H are independent of t). Then the measurement model is given by the following integral equation with discontinuous measure:

$$z_k(t) = \int_0^t C_k(s)x(s)d\mu_k(s) + \int_0^t H_k(s)dV_k(\mu_k(s)) \qquad (11)$$

Obviously, if $\mu(t) = t$ we have a case of the continuous system with continuous measurements in the integral form. If $\mu^c \equiv 0$, the observation model given by equation (11) describes the case of a continuous process with only sampled measurements. The general case of equation (7) will describe the dynamic system with an arbitrary combination of discrete and continuous measurements. Since Theorem 1 gives the optimal filter for the most general case, it is now only a matter of specializing the main result to different cases of practical interest. We begin by re-stating the result of the Theorem 1 for the state space systems (10) (11). When $A(t,s)$, $B(t,s)$, $C(t,s)$, $G(t,s)$, $H(t,s)$ are independent of t, $x_s^t = x(s)$, $m_s^t = \mathrm{E}[x(s)|F_s^Y] = m(s)$, and $f(t,s) = P(s)$. Then the optimal filter takes the following form:

$$\begin{aligned} m(t) = &\int_0^t (A(s)m(s) + B(s)u(s))ds + \int_0^t P(s^-) \\ &\times \left[I + C^T(s)\big(H(s)H^T(s)\big)^{-1}C(s)P(s^-)\Delta\mu(s)\right]^{-1} \\ &\times C^T(s)\big(H(s)H^T(s)\big)^{-1}[dz(s) - C(s)m(s^-)d\mu(s)] \quad (12) \end{aligned}$$

$$\begin{aligned} P(t) = &\int_0^t \left[A(s)P(s) + P(s)A^T(s) + G(s)G^T(s)\right] ds \\ &- \int_0^t P(s^-)\left[I + C^T(s)\big(H(s)H^T(s)\big)^{-1}C(s)P(s^-)\Delta\mu(s)\right]^{-1} \\ &\times C^T(s)\big(H(s)H^T(s)\big)^{-1}C(s)P(s^-)d\mu(s) \quad (13) \end{aligned}$$

where $\Delta\mu(s)$(=1) is a jump of the function $\mu(s)$ at s. At the point of discontinuity $m(t^-) = \lim m(s)$ as $s \to t$ from the left. A similar definition is used for $P(t^-)$. Multiplication by an m-dimensional measure $d\mu(s)$ is understood in a component-wise sense. For purely continuous measurements in k-th channel $z_k^d \equiv 0$, and if only discrete measurements are available, then $z_k^c \equiv 0$. The discontinuity in z leads to discontinuity in estimate $m(t)$ and the covariance function $P(t)$. At the point of discontinuity t_{ki} where a new discrete measurement becomes available in k-th measurement channel, the optimal value of $m(t)$ and $P(t)$ can be explicitly calculated from (12)–(13). Therefore, the optimal filter has a form of a differential equation with discontinuities at the time of arrival of discrete measurements. The solution is sought as a vibrosolution [7] with explicit expressions for the jumps in m and P. The fusion of the sampled and continuous measurements in calculations of the optimal state estimates is direct and explicit. There is no need for multirate filters, because no matter what the sampling rate is, discrete measurements are used in the stimulation process immediately as they become available. The resulting filter is not only optimal, but is computationally more attractive than filters obtained following approaches 1 and 2.

The following two simple cases further demonstrate the application of the general result to the state space systems.

A. Continuous system with discrete measurements

The model of discrete measurements is obtained by setting $\mu^c \equiv 0$ in equation (7):

$$\mu(s) = \left(\sum_{i=1}^{N_i} \Delta\mu_{ki}\chi(s - t_{ki}(t)),\ k = \overline{1,m}\right) \qquad (14)$$

In this case, the differential measure $d\mu(s)$ is equal

$$d\mu(s) = \left(\sum_{i=1}^{N_i} \Delta\mu_{ki}\delta(s - t_{ki}(t)),\ k = \overline{1,m}\right) ds \qquad (15)$$

where δ is the Dirac-delta function.

For a continuous dynamic system with discrete measurements at different and time-varying sampling rates in different measurement channels, the optimal filter is obtained from equations (12), and (13). Between the last discrete measurement at $t = t_{i-1}$ and the next measurement $t = t_i$ in any of the measurement channels, the estimate of the state and the error covariance matrix are given by:

$$m(t) = m(t_{i-1}^+) + \int_{t_{i-1}^+}^{t_i} (A(s)m(s) + B(s)u(s))ds \qquad (16)$$

$$P(t) = P(t_{i-1}^+) + \int_{t_{i-1}^+}^{t_i} [A(s)P(s) + P(s)A^T(s) + G(s)G^T(s)]ds$$

At t_i when a discrete measurement becomes available, the state estimate and the covariance are equal to

$$m(t_i^+) = m(t_i^-) + \delta m(t_i) \qquad (17)$$

$$P(t_i^+) = P(t_i^-) + \delta P(t_i) \qquad (18)$$

where $\delta m(t_i)$ and $\delta P(t_i)$ are the jumps caused by the arrival of a discrete measurement. To *explicitly* calculate the expression for δm, the equation (12) is integrated with respect to $dz(s)$ and $d\mu(s)$. Integration with respect to $dz(s)$ yields:

$$\begin{aligned} &\int_{t_{i-1}}^{t_i+} P(s^-)[I + C^T(s)(H(s)H^T(s))^{-1}C(s)P(s-)\Delta\mu(s)]^{-1} \\ &\quad \times C^T(s)\big(H(s)H^T(s)\big)^{-1}dz(s) \\ &\quad = K(t_i)(z(t_i^+) - z(t_i^-)) = K(t_i)\delta z(t_i) \qquad (19) \end{aligned}$$

where $\delta z(t_i)$ is the discrete measurement at t_i and

$$K(t_i) = P(t_i^-) \times \left\{ I + C^T(t_i) \left[H(t_i) H^T(t_i) \right]^{-1} C(t_i) P(t_i^-) \Delta\mu(t_i) \right\}^{-1} \times C^T(t_i) \left[H(t_i) H^T(t_i) \right]^{-1} \quad (20)$$

The integration of (12) with respect to $d\mu(s)$ gives:

$$\int_{t_{i-1}}^{t_i^+} P(s-) \left[I + C^T(s) \left(H(s) H^T(s) \right)^{-1} C(s) P(s-) \Delta\mu(s) \right]^{-1} \times C^T(s) \left(H(s) H^T(s) \right)^{-1} \left[C(s) m(s-) d\mu(s) \right] = K(t_i) C(t_i) m(t_i^-) \quad (21)$$

We have thus obtained that

$$\delta m(t_i) = K(t_i) \left[\delta z(t_i) - C(t_i) m(t_i^-) \right] \quad (22)$$

Similarly, by integrating (13) w.r.t $d\mu(s)$, obtain

$$\delta P(t_i) = -K(t_i) C(t_i) P(t_i^-) \quad (23)$$

Note that the optimal filter derived in this section, equations (16)–(20), is applicable to all practically important cases of the continuous processes with discrete measurements, including the case of multirate measurements (sections VI-A and VI-B), and randomly or non-uniformly sampled measurements (section VI-C). It can be shown that the derived filter is identical to the Jazwinski filter [8], which is the Kalman-Bucy filter for continuous process with sampled measurements.

B. Continuous system with continuous measurements

This is the case when $\mu(t) = t$, yielding the following optimal filter equations:

$$\begin{aligned} m(t) &= \int_0^t (A(s)m(s) + B(s)u(s))ds \\ &+ \int_0^t P(s) C^T(s) \left(H(s) H^T(s) \right)^{-1} \\ &\times \left[dz(s) - C(s) m(s) ds \right] \end{aligned} \quad (24)$$

$$P(t) = \int_0^t \left[A(s)P(s) + P(s)A^T(s) + G(s)G^T(s) \right] ds - \int_0^t P(s) C^T(s) \left(H(s) H^T(s) \right)^{-1} C(s) P(s) ds \quad (25)$$

which are equivalent to the Kalman-Bucy filter.

C. Continuous system with continuous and sampled measurements

The general case of the differential measure (7) allows us to formally describe dynamic systems with any combination of continuous and discrete measurements. In particular, it allows us to apply Theorem 1 to the case of state space systems with both discrete and continuous measurements (Case 4 in the numerical example), the case for which the optimal filter was previously unknown. Specifically, consider the continuous process described by (10)–(11) where both discrete and continuous measurements are present simultaneously. Then the optimal filter is given by equations (12)–(13), and the jumps in the state estimation and estimation error covariance are explicitly given by equations (22)–(23).

Note that the class of continuous systems, for which the result of Theorem 1 is relevant, is significantly broader than state space systems, and includes systems with discontinuities in states, systems with memory and the distributed parameter systems.

V. Alternative Methods of State Estimation

To obtain the state estimates using the discrete Kalman filter (Approach 1), the continuous model of the process and continuous measurements must be discretized. The discrete process model is equal

$$\begin{aligned} x(t_{i+1}) &= \Phi(t_{i+1}, t_i) x(t_i) + \Lambda(t_{i+1}, t_i) u(t_i) \\ &+ \Gamma(t_{i+1}, t_i) \omega(t_i) \end{aligned} \quad (26)$$

where the white Gaussian sequence $\omega(t_i) \sim N(0, Q(t_i))$, and

$$\begin{aligned} \Phi(t_{i+1}, t_i) &= e^{A(t_i)(t_{i+1} - t_i)} \\ \Lambda(t_{i+1}, t_i) &= \left(\Phi(t_{i+1}, t_i) - I \right) A(t_i)^{-1} B(t_i) \\ \Gamma(t_{i+1}, t_i) &= \left(\Phi(t_{i+1}, t_i) - I \right) A(t_i)^{-1} G(t_i) \end{aligned}$$

The covariance matrix $Q(t_i)$ of the discrete system is related to the covariance of the continuous system $\tilde{Q}$ as

$$Q(t_i) = \int_{t_i}^{t_{i+1}} \left[\Phi(t_{i+1}, \tau) G(\tau) \tilde{Q}(\tau) G^T(\tau) \Phi^T(t_{i+1}, \tau) \right] d\tau$$

indicating the direct dependence of $Q(t_i)$ (and therefore the Kalman gain of the discrete KF) on the discretization step. In particular, for small $\Delta t = t_{i+1} - t_i$, $Q(t_{i+1}) \approx G(t_i) \tilde{Q}(t_i) G^T(t_i) \Delta t$. With sufficiently small Δt the filter gain will be very small, making the discrete KF largely insensitive to the incoming measurements.

To apply the continuous Kalman filter (Approach 2), the discrete measurements[1] $y(t_i)$ must be fitted to a continuous function. For example, in the subsequent simulations, the following piecewise continuous approximation

$$y(t) = C(t) x(t) + H \nu(t) \quad (27)$$

is used, obtained by a linear extrapolation between the latest two available sampled measurements. Similar ideas can be found in [9] to get the intersample estimation for slow measurements. The obtained continuous measurements $\{y(t), t \geq t_0\}$ are assumed to be corrupted by the continuous white Gaussian noise process $\nu(t) \sim N(0, \tilde{R}(t))$. The relationship between covariances of the sampled measurements and their continuous approximation is given by

$$\tilde{R}(t) = R(t_i) \Delta t \quad (28)$$

where Δt is the time between two consecutive sampled measurements. Note that for time-varying sampling $\tilde{R}$ is the function of time even if $R =$ const. With large Δt, the described approach leads to a relatively small effect of measurements on the state estimation, which is clearly not the right way to utilize infrequent measurements.

[1] In the standard differential notation, the i-th continuous measurement $y_i(t) = \dot{z}_i(t)$. The relationship between j-th discrete measurement in differential and integral forms is given by $z_j(t_k) = \sum_{l=0}^{k} y_j(t_l)$, and $y_j(t_k) = \delta z_j(t_k)$.

VI. Examples

Consider a stable continuous linear time-invariant system modelled as:

$$\bar{A} = \begin{bmatrix} -1 & -.02 & -.03 \\ -.03 & -2 & .05 \\ -.05 & -6 & -3 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 10 \\ 2.5 \\ 1 \end{bmatrix} \bar{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The process itself is described by

$$A = \begin{bmatrix} -0.9 & -.02 & -.03 \\ -.03 & -1.8 & .05 \\ -.05 & -6 & -2.85 \end{bmatrix} \quad B = \begin{bmatrix} 11 \\ 2.75 \\ 1.1 \end{bmatrix} \quad C = [\bar{C}] \quad (29)$$

with $\mathrm{E}[x(0)] = [5 \ \ -6 \ \ 6]^T$. The plant-model mismatch is introduced to illustrate the effect of the approximation on the performance of the filters derived following approaches 1 and 2. We assume $\omega(t) \sim N(0, \tilde{Q})$ with $\tilde{Q} = \mathrm{diag}(0.25 x_i^{ss})^2, \ i = \overline{1,3}$, where x^{ss} is the steady state value. In the following cases 1–3, the sampled measurements of y_1 and y_2 are assumed to be available; the covariance of the measurement noise sequence $\nu(t_i)$ is known: $R = \mathrm{diag}(0.1 x_j^{ss})^2, \ j = 1, 2$.

In the following simulations, mean value of state $x(t)$ is obtained as $\dot{m}(t) = Am(t) + Bu$, with $m(t_0) = \mathrm{E}[x(0)]$. True states are calculated from

$$dx(t) = (Ax + Bu)dt + GdW(t) \quad (30)$$

where $G = I$ and the Brownian process $W(t)$ is approximated as a random walk [10].

The numerical experiment is performed following the Monte Carlo approach. Multiple realizations (N=1000) of state trajectories are calculated from the stochastic differential equation (30). Measurement y, used as an input to all filters, is obtained as $y(t) = Cx + Du + H\omega(t)$, where $\omega(t)$ is Gaussian white noise with zero mean and covariance R.

A. Case 1: Continuous process with single-rate sampled measurements

Case 1 is the base case: y_1 and y_2 are sampled with the same interval Δt. The state estimates are obtained following all three approaches. The root mean square errors (RMSE) for each filter are calculated as:

$$RMSE \ x_i = \sqrt{\tfrac{1}{N} \textstyle\sum_{i=1}^{N} (x_i(t) - \hat{x}_i(t))^2}$$

and plotted in Figure 2. The RMSE of the optimal filter is the smallest, as expected. Note that

$$\begin{aligned} P(t) &= \mathrm{E}[(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T] \\ &\approx \tfrac{1}{N} \textstyle\sum_{i=1}^{N} [(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T] \quad (31) \end{aligned}$$

where $x(t)$ is the true state from equation (30), and $\hat{x}(t)$ is the estimate obtained with different filters. Therefore, $\mathrm{diag}(P(t)) \approx RMSE^2$.

Each filter generates $P(t)$ during its operation, so it is instructive to compare the diagonal elements of the filter-generated error covariance matrix (not shown) with the actual value, equal to $RMSE^2$, obtained using Monte Carlo simulations. In the case of the optimal filter, the filter-generated values give a good approximation (less than 20% error at steady state for x_1) to the result obtained with ensemble averaging. The two alternative filters overestimate the quality of the generated state estimations. The discrete Kalman filter is giving the largest overestimation: For x_1 the steady state RMSE with the discrete KF is ≈ 2.7, while P generated by the filter gives the RMSE of less than 0.25.

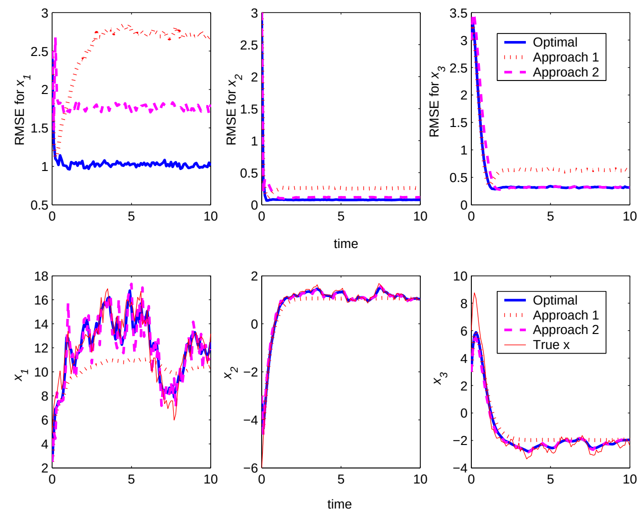

Fig. 2. Case 1: State estimations with single-rate sampled measurements.

Suboptimal filters predict the values of P smaller than the least theoretically possible (i.e. optimal) because of the effect of Δt-dependent approximation. For example, the discrete KF (Approach 1) predicts the smallest error covariance, while in reality the RMSE with the discrete KF is the largest. This behavior is due to the approximation of $Q(t_i)$ (which makes $Q(t_i)$ too small), leading to unjustifiably low Kalman gain and an excessive reliance of the erroneous process model in generating state estimations.

Figure 2 (bottom row) shows *typical* results of the state estimation with different filters. It is clear that a poor (and biased) estimate is obtained with the discrete KF. Though by "tuning" (in this case, substantially increasing) the covariance matrix Q the discrete KF could yield satisfactory state estimates, the *ad hoc* tuning is not desirable when the statistics of the process disturbances are known. Furthermore, to correctly "tune" the discrete and Kalman-Bucy filters, the result of the optimal filtering must be known, which defeats the purpose of considering the alternatives.

B. Case 2: Continuous process with multirate measurements

In this case, y_1 and y_2 are sampled with Δt and $10 \times \Delta t$, respectively. The optimal filter accounts for each available discrete measurement according to equations (17)–(18) with properly adjusted C immediately after the measurement becomes available. A multirate discrete filter is implemented following [2], which required the adjustment of the Kalman gain matrix at the moment when both fast and

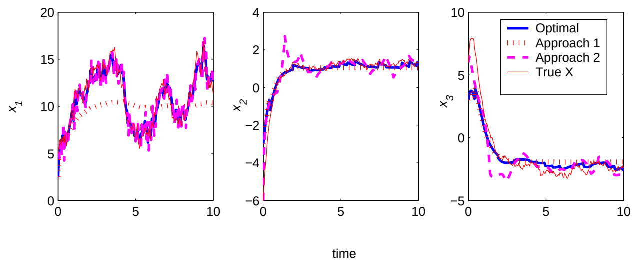

Fig. 3. Case 2: State estimation with multirate measurements.

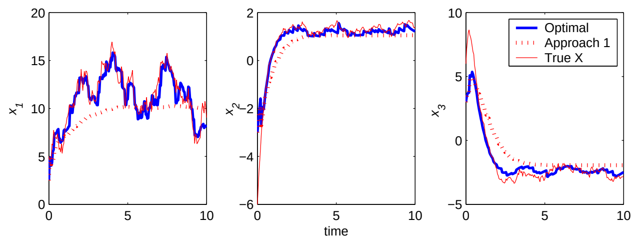

Fig. 4. Case 3: State estimation with randomly sampled measurements.

slow measurements are simultaneously available. To apply the continuous KF, the linear extrapolation is again used to approximate both y_1 and y_2 as piecewise linear functions. The continuous approximation of infrequent measurements y_2 will generally introduce significant errors, clearly visible in Figure 3, especially from the $\hat{x}_2$ plot. As in the previous case, Approach 1 leads to a biased estimation because of the approximation used to obtain $Q(t_i)$, and unjustifiably low values of the estimation error covariance. The optimal filter shows the best performance with the Kalman-Bucy filter giving reasonably accurate results.

C. Case 3: Continuous process with randomly sampled measurements

In this case, y_1 is available every $2 \times \Delta t$, while y_2 is sampled randomly. Figure 4 illustrates the performance of the optimal and the discrete KF. Note the effect of the arrival of a randomly sampled y_2 on the estimates $\hat{x}_2$ and $\hat{x}_3$, and the biased estimate produced by the discrete KF.

D. Case 4: Continuous process with both continuous and sampled measurements

In all previous cases, the optimal filter is equivalent to the Jazwinski filter. However, when both sampled and continuous measurements are present simultaneously, the developed optimal filter, to our knowledge, was previously unknown. The simulation results of Figure 5 show the state estimation with the optimal filter for the case of continuous measurement of x_1 and infrequently sampled x_2. The effect of infrequent measurements is especially pronounced from the examination of $\hat{x}_2$ and $\hat{x}_3$.

VII. CONCLUSIONS

The problem of the state estimation for continuous processes with continuous and sampled measurements, including multirate and randomly sampled cases, attracted considerable attention because of its practical importance. Most existing methods implement the ideas of either Approach 1 or 2, and require that the state estimation problem is approximated as either the state estimation for discrete or continuous systems. In this paper, we develop an optimal (in Kalman sense) filter without reverting to an approximation as a first step in the state estimation procedure. The resulting optimal filter is the continuous system with discontinuous inputs appearing every time a new sampled measurement becomes available. The developed filter is both optimal and convenient in practical applications, since each sampled measurement is processed immediately and explicitly when it becomes available without need for multirate filters. Numerical examples indicate that the developed filter provides the smallest state estimation errors and an accurate indication of the goodness of the obtained results by correctly estimating the error covariance. The alternative methods tend to suggest higher-quality estimates than actually achieved.

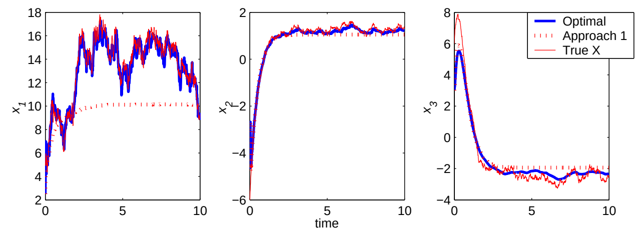

Fig. 5. Case 4: State estimation with continuous and sampled measurements.

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