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Washout Filters in Feedback Control: Benefits, Limitations and Extensions

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***Abstract*— Advantages and limitations of washout filters in feedback control of both continuous-time and discrete-time systems are discussed, and generalizations that alleviate the limitations are presented. The generalized washout filters presented involve feedback through auxiliary state variables that are intercoupled. This is in contrast to the more traditional washout filter-aided control, in which these variables couple to the system state but not to each other. The generalized washout filter results obtained in the paper include a systematic feedback control design procedure, which is not available for traditional washout filter-aided control. Some previously unpublished results in the Ph.D. dissertation of one of the authors (Lee, 1991) are presented in the context of their relation to the generalized results and to recent publications on delayed feedback control. We observe that delayed feedback control for discrete-time systems, used in a number of control of chaos studies, is a special case of washout filter-aided feedback. Moreover, the limitations of delayed feedback control can be overcome by the use of washout filter-aided feedback, which gives rise to the possibility of stabilizing a much larger class of systems.**

I. Introduction

It is a common practice in the analysis of nonlinear systems and in feedback control design to assume that the equilibrium point (or the operating point) of the system is accurately known or does not change over the operating regime. However, models of physical dynamical systems are in general uncertain. Therefore, static feedback control is ineffective in addressing problems where the operating point is not accurately known or there is parameter drift.

Consider the nonlinear system described by

$$\dot{x} = f(x,u) \qquad \text{(continuous-time)} \qquad (1)$$

or

$$x(k+1) = f(x(k),u(k)) \qquad \text{(discrete-time)} \qquad (2)$$

where $f(\cdot,\cdot)$ is uncertain, u is the scalar input and $x \in \Re^n$ is the state vector. Due to the uncertainty in f, the equilibrium points (if any) of the system (1) and the fixed points (if any) of (2) are also in general uncertain. Typically, one expands $f(\cdot,\cdot)$ about the operating point of interest, say x_o, and then applies linear feedback design techniques to the linearized model. Static state feedback, however, does not apply to problems in which the dynamics and the targeted operating point are uncertain. Moreover, static state feedback changes the operating conditions of the open-loop system. This results in wasted control effort and may also result in degrading system performance.

To overcome these problems, washout filters have been used in many applications (e.g., [1], [2], [3], [4], [5], [6], [7]). A washout filter (also sometimes called a washout circuit) is a high pass filter that washes out (rejects) steady state inputs, while passing transient inputs [1]. The main benefit of using washout filters is that all the equilibrium points of the open-loop system are preserved (i.e., their location isn't changed). In addition, washout filters facilitate automatic following of a targeted operating point, which results in vanishing control energy once stabilization is achieved and steady state is reached.

Although washout filters have been successfully used in many control applications, there is no systematic way to choose the constants of the washout filters and the control parameters. Recently, Bazanella, Kokotovic and Silva [8] proposed a technique to control continuous-time systems with unknown operating point. The operating point (or equilibrium point) was treated as an uncertain parameter and a certainty equivalence adaptive controller was proposed. In this work, we discuss benefits and limitations of washout filter-aided feedback for both continuous-time and discrete-time systems . We also discuss extensions of washout filter-aided feedback to overcome the limitations of washout filters and at the same time maintain their benefits. Our extensions are similar to that of [8], although we do not invoke a singular perturbation framework.

The paper proceeds as follows. In Sec. II, we discuss washout filters for both continuous-time and discrete-time systems. In Sec. III, we discuss linear washout filter-aided feedback control and present limitations of feedback through stable washout filters. In Sec. IV, we discuss delayed feedback control for discrete-time systems and its relation to washout filter-aided feedback. In Secs. V and VI, generalizations of washout filters are presented.

II. WASHOUT FILTERS

A washout filter is a high pass filter that washes out (rejects) steady state inputs, while passing transient inputs [1]. In continuous-time setting, the transfer function of a typical washout filter is

$$G(s) \quad = \quad \frac{y(s)}{x(s)} = \frac{s}{s+d}. \tag{3}$$

Here, d is the reciprocal of the filter time constant which is positive for a stable filter and negative for an unstable filter. With the notation

$$z(s) := \frac{1}{s+d}x(s) \tag{4}$$

the dynamics of the filter can be written as

$$\dot{z} = x - dz, \tag{5}$$

along with the output equation

$$y = x - dz. \tag{6}$$

In discrete-time, the dynamics of a washout filter can be written as

$$z(k+1) \quad = \quad x(k) + (1-d)z(k), \tag{7}$$

along with the output equation

$$y(k) = x(k) - dz(k). \tag{8}$$

For a stable washout filter, the filter constant satisfies $0 < d < 2$.

Note that the output of the washout filter (for both continuous-time and discrete-time cases) vanishes in steady state. Therefore, using washout filters in feedback control does not move the equilibrium points of the open-loop system. As will be discussed below, there are limitations in using stable washout filters in feedback control, and some of these limitations can be overcome using unstable washout filters.

III. LINEAR FEEDBACK THROUGH WASHOUT FILTERS

Below, we consider linear feedback through washout filters for both continuous-time and discrete-time systems, and we mention limitations of using stable washout filters. Some of these limitations, such as Lemma 3, are being reported in the current literature, although the results date back to the thesis of H.-C. Lee [3]. The results for the discrete-time case are new.

A. Continuous-time case

Suppose x_o is an unstable operating condition for system (1). In a small neighborhood of x_o, system (1) can be rewritten as

$$\dot{x} \quad = \quad Ax + bu + h(x,u) \tag{9}$$

where x now denotes $x - x_o$ (is the state vector referred to x_o), u is a scalar input, A is the Jacobian matrix of f evaluated at x_o, b is the derivative of f with respect to u evaluated at x_o, and $h(\cdot,\cdot)$ represents higher order terms, i.e., $h(0,0) = 0$ and $\frac{\partial h(0,0)}{\partial x} = 0$.

Next, washout filters are used in the feedback loop. The dynamic equations of the washout filters can be written as

$$\dot{z}_i \quad = \quad -d_i z_i + \sum_{j=1}^{n} c_{ij} x_j \tag{10}$$

where z_i is the state of the ith washout filter, $i = 1, \cdots, m$, and $m \leq n$ is a positive integer. Note that (10), where more than one state is used as an input to the washout filter, is more general than (5). The relationship between the operating point of interest of the open-loop system and the operating point of the washout filters is as follows:

$$z_{oi} \quad = \quad \frac{1}{d_i}\sum_{j=1}^{n} c_{ij} x_{oj} \tag{11}$$

In vector form, the closed-loop system can therefore be written as

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix} u + \begin{pmatrix} h(x,u) \\ 0 \end{pmatrix} \tag{12}$$

where $C = [c_{ij}]$ is an $m \times n$ matrix, which consists of nonzero row vectors, $D = diag(d_i)$, $i = 1, \cdots, m$.

The control input u is taken as a linear function of the washout filter' outputs obtained from the right side of (10)

$$y_i \quad = \quad -d_i z_i + \sum_{j=1}^{n} c_{ij} x_j. \tag{13}$$

The following two lemmas give general guidelines for choosing the matrices C and D based on controllability considerations.

Lemma 1: ([3]) If any two diagonal entries of the matrix D are the same, the linearization of the closed-loop system (12) is not controllable regardless of the controllability of the pair (A, b).

Proof: See [3], [9].

Note that controllability of the closed-loop system (12) does not imply that the eigenvalues of system (9) can be arbitrarily assigned by feedback through washout filters.

Lemma 2: ([3]) Suppose that λ_1 is an eigenvalue of both A and D, and that

$$\rho\begin{pmatrix} \lambda_1 I - A & b \\ C & 0 \end{pmatrix} \leq n. \tag{14}$$

Then, the linearization of the closed-loop system (12) is not controllable.

Proof: See [3], [9].

Since washout filter-aided feedback can be viewed as a form of output feedback (see [9]), where the outputs of the washout filters instead of the open-loop system states are used in the feedback, some of the capabilities of direct state feedback are lost. This is due to the restriction that $d_i \neq 0$.

The following lemma summarizes some of the capability limitations of feedback through stable washout filters.

Lemma 3: ([3]) If A has an odd number of eigenvalues with positive real part, then (9) cannot be stabilized using stable washout filters. This holds even if the eigenvalues of A with positive real part are linearly controllable.

Proof: See [3], [9].

Corollary 1: If the open-loop system possesses a zero eigenvalue, it cannot be moved using washout filter-aided feedback.

Proof: Follows from Lemma 3. ■

B. Discrete-time case

Suppose x_o is an unstable operating condition for system (2). In a small neighborhood of x_o, system (2) can be rewritten as

$$x(k+1) \quad = \quad Ax(k) + bu(k) + h(x(k), u(k)) \quad (15)$$

where x now denotes $x - x_o$ (is the state vector referred to x_o), u is a scalar input, A is the Jacobian matrix of f evaluated at x_o, b is the derivative of f with respect to u evaluated at x_o, and $h(\cdot, \cdot)$ represents higher order terms, i.e., $h(0,0) = 0$ and $\frac{\partial h(0,0)}{\partial x} = 0$.

Next, washout filters are used in the feedback loop. The dynamic equations of the washout filters can be written as

$$z_i(k+1) \quad = \quad (1 - d_i)z_i(k) + \sum_{j=1}^{n} c_{ij}x_j(k) \quad (16)$$

where z_i is the state of the ith washout filter, $i = 1, \cdots, m$, and $m \leq n$ is a positive integer. The relationship between the operating point of the open-loop system and the operating point of the washout filters is as follows:

$$z_{oi} \quad = \quad \frac{1}{d_i} \sum_{j=1}^{n} c_{ij}x_{oj} \quad (17)$$

In vector form, the closed-loop system can therefore be written as

$$\begin{pmatrix} x(k+1) \\ z(k+1) \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & I - D \end{pmatrix} \begin{pmatrix} x(k) \\ z(k) \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} h(x(k), u(k)) \\ 0 \end{pmatrix} \quad (18)$$

where $C = [c_{ij}]$ is an $m \times n$ matrix, which consists of nonzero row vectors, $D = diag(d_i)$, $i = 1, \cdots, m$.

The control input u is taken as a linear function of the washout filter' outputs obtained from the right side of (16)

$$y_i(k) \quad = \quad -d_i z_i(k) + \sum_{j=1}^{n} c_{ij}x_j(k). \quad (19)$$

The following two lemmas give general guidelines for choosing the matrices C and D based on controllability considerations. The results are analogous to the continuous-time results presented in the previous section.

Lemma 4: If any two diagonal entries of the matrix D are the same, the linearization of the closed-loop system (18) is not controllable regardless of the controllability of the pair (A, b).

Proof: See [9].

Note that controllability of the closed-loop system (18) does not imply that the eigenvalues of system (15) can be arbitrarily assigned by feedback through washout filters.

Lemma 5: Suppose that λ_1 is an eigenvalue of both A and $I - D$, and that

$$\rho \begin{pmatrix} \lambda_1 I - A & b \\ C & 0 \end{pmatrix} \leq n. \quad (20)$$

Then, the linearization of the closed-loop system (18) is not controllable.

Proof: See [9].

The following lemma summarizes some of the capability limitations of feedback through stable washout filters.

Lemma 6: If A possesses an odd number of real eigenvalues (counting multiplicities) in $(1, \infty)$ (i.e., if $\det(I - A) < 0$) then it cannot be stabilized using stable washout filters.

Proof: See [9].

Corollary 2: If the linearization of the open-loop state dynamics matrix A possesses an eigenvalue of 1 (i.e., $I - A$ is singular), then this eigenvalue cannot be moved using washout filter-aided feedback.

Proof: Follows from Lemma 6. ■

IV. Delayed Feedback Control as a Special Case of Washout Filter-Aided Feedback

Delayed feedback control (DFC) was proposed by Pyragas [10] as a technique for control of chaos. Since then, DFC has been used by many authors in control of chaos studies [11], [12], [13], [11], [14], [15], [16], [17], [18], [19]. We have shown that DFC in discrete-time systems is a special case of washout filter-aided feedback control, and that some of the results in the literature on limitations of DFC are actually special cases of results in the Ph.D. dissertation of one of the authors [3]. In addition, we have shown how these limitations of DFC may be overcome using washout filters and generalized washout filters (as introduced below). The details are not included here due to space limitations, but they are available on the web in the technical report [9].

We proceed in the next two sections to give a generalization of washout filter-aided feedback for both continuous-time and discrete-time systems.

V. Generalization of Continuous-Time Washout Filter-Aided Feedback

Next, we consider a generalization of washout filters in which the individual washout filters are coupled through a constant coupling matrix. Consider system (1) with x_o

as the operating condition. System (1) can be rewritten as follows in a small neighborhood of x_o:

$$\dot{x} = Ax + Bu + h(x,u) \tag{21}$$

We are interested in designing a control law that stabilizes this system while maintaining all its equilibrium points.

The generalized washout filter-aided feedback proposed here results in the closed-loop system

$$\dot{x} = Ax + Bu + h(x,u) \tag{22}$$

$$\dot{z} = P(x-z) \tag{23}$$

$$u = K(x-z) \tag{24}$$

Here P is a nonsingular matrix and K is a feedback gain matrix. Since in steady state, the control input vanishes (i.e., $u \equiv 0$), the equilibrium points of the open-loop system are not shifted by this type of feedback control. Suppose that the pair (A,B) is stabilizable. Are there matrices K and P such that the closed-loop system is stable?

To answer this question, we consider the effect of matrices K and P on the linearization of the closed-loop system

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A+BK & -BK \\ P & -P \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} =: A_c \begin{pmatrix} x \\ z \end{pmatrix} \tag{25}$$

Proposition 1: ([8]) The determinant of the closed-loop state dynamics matrix A_c satisfies

$$\det(A_c) = \det(A)\det(-P) \tag{26}$$

Proof: See [8], [9].

Corollary 3: If the matrix A has a zero eigenvalue, then the closed-loop system state dynamics matrix A_c will also have a zero eigenvalue.

Proof: Follows from Proposition 1. ∎

Since this type of feedback doesn't shift equilibria, it shouldn't be surprising that it can't modify a zero eigenvalue (this would also modify any stationary bifurcation in the system).

The following result gives some conditions on the controller matrix P for the controller to be stabilizing. This result is akin to Lemma 3 pertaining to washout filter-aided feedback. In words, the result means that if the open-loop system possesses an odd number of unstable eigenvalues, then a necessary condition for the closed-loop system to be stable is that the controller must also have an odd number of unstable eigenvalues.

Lemma 7: [8] Let the number of unstable eigenvalues of A be odd. Then, for the closed-loop matrix A_c to be Hurwitz, $-P$ must also have an odd number of unstable eigenvalues.

Proof: See [8], [9].

We will show that if A is nonsingular and (A,B) is stabilizable, then there is a pair P, K such that A_c is Hurwitz. Recall that eigenvalues are preserved under similarity transformations.

Let $T_1 = \begin{pmatrix} I & 0 \\ 0 & P^{-1} \end{pmatrix}$. Then we have

$$A_{c1} := T_1 A_c T_1^{-1} = \begin{pmatrix} A+BK & -BKP \\ I & -P \end{pmatrix} \tag{27}$$

Next, let $T_2 = \begin{pmatrix} I & M \\ 0 & I \end{pmatrix}$. It is easy to see that $T_2^{-1} = \begin{pmatrix} I & -M \\ 0 & I \end{pmatrix}$. Applying the transformation T_2 to A_{c1} gives

$$A_{c2} := T_2 A_{c1} T_2^{-1} = \begin{pmatrix} A+BK+M & -AM-BKM-M^2-BKP-MP \\ I & -M-P \end{pmatrix}$$

Consider the $(1,2)$ block term of A_{c2}. Suppose $P = \epsilon P_1$ and $M = M_0 + \epsilon M_1 + O(\epsilon^2)$ with $\epsilon > 0$ and sufficiently small. It is straightforward to show, after setting the $(1,2)$ block of A_{c2} to zero, i.e.,

$$AM + BKM + M^2 + BKP + MP = 0 \tag{28}$$

and collecting terms with same power in ϵ, that $O(1)$ terms:

$$(A+BK+M_0)M_0 = 0. \tag{29}$$

This holds if $M_0 = 0$ or $M_0 = -A - BK$. Taking $M_0 = -A - BK$ and finding the ϵ^1 terms gives

$$M_1(A+BK) + AP_1 = 0 \tag{30}$$

Since $A+BK$ can be guaranteed invertible (by restricting K so that $0 \notin \sigma(A+BK)$), we find that $M_1 = -AP_1(A+BK)^{-1}$. Since M_1 can be determined uniquely through matrix inversion, it is clear that the Implicit Function Theorem implies that (28) has a locally unique solution $M(\epsilon) = M_0 + \epsilon M_1 + O(\epsilon^2)$ near M_0. Therefore, $M = M_0 + \epsilon M_1 + O(\epsilon^2) = -A - BK - \epsilon AP_1(A+BK)^{-1} + O(\epsilon^2)$. Substituting M and P in A_{c2} yields

$$A_{c2} = \begin{pmatrix} -\epsilon AP_1(A+BK)^{-1} + O(\epsilon^2) & 0 \\ I & A_{c2}(2,2) \end{pmatrix}$$

where $A_{c2}(2,2) = A + BK + \epsilon(AP_1(A+BK)^{-1} - P_1) + O(\epsilon^2)$.

Assume that A has no zero eigenvalues. To make A_{c2} Hurwitz, we need to choose P_1 such that $-AP_1(A+BK)^{-1}$ is Hurwitz. Clearly such a P_1 exists (e.g., $P_1 = A^{-1}(A+BK)$). Also we need to choose K such that $A+BK$ is Hurwitz with eigenvalues away from zero. Such a K is guaranteed to exist since the pair (A,B) is assumed to be stabilizable.

Proposition 2: Consider the closed-loop system (25). Suppose that the matrix A has no eigenvalues at 0. Suppose also that the pair (A,B) is stabilizable. Then there exists a nonsingular $P \in R^{n\times n}$ and $K \in R^{m\times n}$ such that $(x_o^T, x_o^T)^T$ is asymptotically stable equilibrium point of (25).

VI. Generalization of Discrete-Time Washout Filter-Aided Feedback

The results of this section are counterparts of the continuous-time results of the previous section for the discrete-time case. Consider system (2) with x_o as the operating condition. System (2) can be rewritten as follows in a small neighborhood of x_o:

$$x(k+1) = Ax(k) + Bu(k) + h(x(k), u(k)) \quad (31)$$

We are interested in designing a control law that stabilizes this system while maintaining all its equilibrium points. The generalized washout filter-aided feedback proposed here results in the closed-loop system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + h(x(k), u(k)) && (32)\\ z(k+1) &= Px(k) + (I-P)z(k) && (33)\\ u(k) &= G(x(k) - z(k)) && (34) \end{aligned}$$

Here P is a nonsingular matrix and G is a feedback gain matrix. Since in steady state, the control input vanishes (i.e., $u \equiv 0$), the equilibrium points of the open-loop system are not shifted by this type of feedback control. Suppose that the pair (A, B) is stabilizable. Are there matrices G and P such that the closed-loop system is stable?

To answer this question, we consider the effect of matrices P and G on the linearization of the closed-loop system

$$\begin{aligned} \begin{pmatrix} x(k+1) \\ z(k+1) \end{pmatrix} &= \begin{pmatrix} A+BG & -BG \\ P & I-P \end{pmatrix} \begin{pmatrix} x(k) \\ z(k) \end{pmatrix} \\ &=: A_c \begin{pmatrix} x(k) \\ z(k) \end{pmatrix} && (35) \end{aligned}$$

Proposition 3: The determinant of $I - A_c$ satisfies

$$\det(I - A_c) = \det(I - A)\det(P) \quad (36)$$

Proof: See [9].

Corollary 4: If the open-loop matrix A has an eigenvalue of 1 (i.e., if $(I - A)$ is singular), this eigenvalue cannot be shifted using this type of dynamic feedback.
Proof: Follows from Proposition 3.

Lemma 8: Let the number of unstable eigenvalues of A that are real and greater than 1 be odd (i.e., $\det(I-A) < 0$). Then, for the closed-loop state dynamics matrix A_c to be Schur stable, $I - P$ must also have an odd number of real eigenvalues greater than 1 in value.
Proof: See [9].

We will show that there exist a P, G such that A_c is Schur stable. Recall that eigenvalues are preserved under similarity transformations.

Let $T_1 = \begin{pmatrix} I & 0 \\ 0 & P^{-1} \end{pmatrix}$. Then we have

$$\begin{aligned} A_{c1} &:= T_1 A_c T_1^{-1} \\ &= \begin{pmatrix} A+BG & -BGP \\ I & I-P \end{pmatrix}. \end{aligned}$$

Next, let $T_2 = \begin{pmatrix} I & M \\ 0 & I \end{pmatrix}$ implying $T_2^{-1} = \begin{pmatrix} I & -M \\ 0 & I \end{pmatrix}$. Applying the transformation T_2 to A_{c1} gives

$$\begin{aligned} A_{c2} &:= T_2 A_{c1} T_2^{-1} \\ &= \begin{pmatrix} A+BG+M & A_{c2}(1,2) \\ I & -M+I-P \end{pmatrix} \end{aligned}$$

where

$$A_{c2}(1,2) = -AM - BGM - M^2 - BGP + M - MP. \quad (37)$$

Consider the block term $A_{c2}(1,2)$. Suppose $P = \epsilon P_1$ and $M = M_0 + \epsilon M_1 + O(\epsilon^2)$ with $\epsilon > 0$ and sufficiently small. It is straightforward to show, after setting the $A_{c2}(1,2)$ to zero and collecting terms with same power in ϵ that the $O(1)$ terms yield

$$(A + BG - I + M_0)M_0 = 0 \quad (38)$$

This holds if $M_0 = 0$ or $M_0 = -A - BG + I$. Taking $M_0 = -A - BG + I$ and finding the ϵ^1 terms gives

$$M_1(A + BG - I) + (A - I)P_1 = 0 \quad (39)$$

Since $A + BG - I$ can be guaranteed nonsingular (by restricting K so that $1 \notin \sigma(A + BG)$), we find that $M_1 = -(A - I)P_1(A + BG - I)^{-1}$. Since M_1 can be determined uniquely through matrix inversion, it is clear that the Implicit Function Theorem implies that (37) has a locally unique solution $M(\epsilon) = M_0 + \epsilon M_1 + O(\epsilon^2)$ near M_0. Therefore, $M = M_0 + \epsilon M_1 + O(\epsilon^2) = -A - BG + I - \epsilon(A - I)P_1(A + BG - I)^{-1} + O(\epsilon^2)$. Substituting M and P in A_{c2} yields

$$A_{c2} = \begin{pmatrix} I - \epsilon(A - I)P_1(A + BG - I)^{-1} + O(\epsilon^2) & 0 \\ I & A_c(2,2) \end{pmatrix}$$

where

$$\begin{aligned} A_c(2,2) &= A + BG + \epsilon((A - I)P_1(A + BG - I)^{-1} - P_1) \\ &\quad + O(\epsilon^2). \end{aligned}$$

Assume that $I - A$ is nonsingular (i.e., $1 \notin \sigma(A)$). To make A_{c2} Schur stable, we need to choose P_1 such that $I - \epsilon(A-I)P_1(A+BG-I)^{-1}$ is Schur stable. Clearly such a P_1 exists. Also we need to choose G such that $A + BG$ is Schur stable with eigenvalues away from 1. Such a G is guaranteed to exist since the pair (A, B) is assumed to be stabilizable.

Proposition 4: Consider the closed-loop system (35). Suppose that the matrix $I - A$ is nonsingular. Suppose also that the pair (A, B) is stabilizable. Then there exists a nonsingular $P \in R^{n\times n}$, a $G \in R^{m\times n}$ and an $\bar{\epsilon} > 0$ such that $\forall \epsilon \in (0, \bar{\epsilon}]$, $(x_o^T, x_o^T)^T$ is an asymptotically stable fixed point of (35).

Example 1: Consider the two-dimensional map [20]

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} 1.9 & 1 \\ 0.5 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} - \begin{pmatrix} x_1^3(k) \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k) \quad (40)$$

The uncontrolled system (40) has three fixed points $x_{o1} = (0,0)$, $x_{o2} = (\sqrt{1.4}, \sqrt{1.4}/2)$ and $x_{o3} = -x_{o2}$, and indeed displays chaotic motion (see [20]). The fixed point x_{o1} is unstable: the eigenvalues of the linearization at the origin are $\lambda_1 = 2.1343$ and $\lambda_2 = -0.2343$. Since $\lambda_2 > 1$, the origin cannot be stabilized using DFC nor using stable washout filters. It is straightforward to show that the origin of (40) can be stabilized using one unstable washout filter with $d = -0.05$ and control gain $\gamma = -1.8$ (see [9] for details).

Next, we show that the origin can be stabilized using the generalized washout filter design calculations. We choose the gain vector G so that $A + bG$ is Schur stable. A stabilizing control gain vector is $G = [-1.6343 \;\; -0.7657]$. Choosing

$$\begin{aligned} P &= 0.1(A-I)^{-1}(A+BG-I) \\ &= \begin{pmatrix} -0.1674 & -0.5469 \\ -0.5837 & 0.7265 \end{pmatrix}, \end{aligned}$$

yields

$$\begin{aligned} A_c &= \begin{pmatrix} A+BG & -BG \\ P & I-P \end{pmatrix} \\ &= \begin{pmatrix} 0.2657 & 0.2343 & 1.6343 & 0.7657 \\ 0.5000 & 0 & 0 & 0 \\ -0.0167 & -0.0547 & 1.0167 & 0.0547 \\ -0.0584 & 0.0727 & 0.0584 & 0.9273 \end{pmatrix}. \end{aligned}$$

The eigenvalues of A_c are $\{-0.2343,\ 0.7277,\ 0.8164,\ 0.9000\}$. Thus, the closed-loop system is asymptotically stable. Figure 1 demonstrates the effectiveness of the controller. Note that the control input vanishes after stabilization of the origin is achieved.

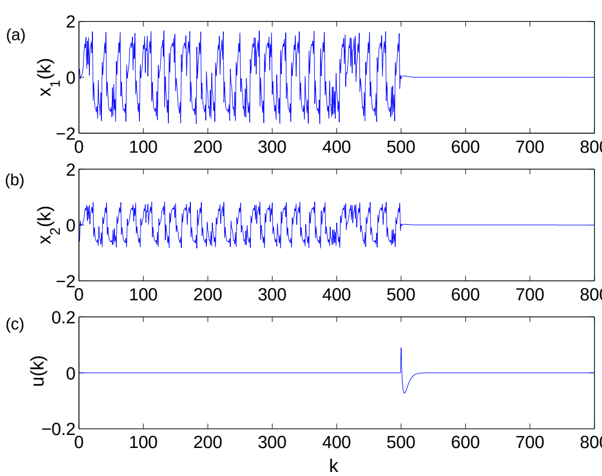

Fig. 1. Time series (with initial condition (0.3,-0.6)) of (a) x_1 (b) x_2 and (c) control input u. The control is applied when the trajectory of the open-loop system enters the neighborhood $\{x = (x_1, x_2) \in \Re^2 : \|x\| < 0.15\}$ of the origin.

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