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Output Feedback Control Using High-Gain Observers in the Presence of Measurement Noise

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***Abstract*— Output feedback control using high-gain observers in the presence of measurement noise is considered for a class of nonlinear systems. We show that the closed-loop system with high-gain observer converges to a closed-loop system with ideal differentiation as the gain of the observer is increased. This ideal differentiation system is developed to reference the convergence properties. Analytical and simulation results are presented.**

I. Introduction

In the design of output feedback control, high-gain observers have gained in popularity due to their ability to accurately estimate the unmeasured states of a system while rejecting disturbances. An observer design to estimate the derivatives of the output together with a globally bounded state feedback control were introduced in [5]. They were used to prove a separation principle for the stabilization of a class of nonlinear systems in [2]. Recently, high-gain observers and their asymptotic properties have been studied when actuator and sensor dynamics, different design techniques, and varying discretization methods have been considered [7], [3], [4]. It is in this spirit, that we continue this study by investigating the convergence properties of the high-gain observer in the presence of measurement noise. Because feedback and state estimation based on the measurement of the output of a system are inherently noisy, it is important to have an understanding of the behavior of the output feedback controller with respect to the noise as the observer parameters are varied. In this note we consider a class of nonlinear systems and investigate the limiting effect of increasing the observer gain on the trajectories of the closed-loop system in the presence of measurement noise. We begin by motivating the discussion using linear systems and transfer function analysis. We then introduce the nonlinear system structure and develop a reference system based on ideal differentiation of the measured output from which we can compare the convergence properties of the high-gain observer. Analysis of the convergence of the trajectories of the closed loop system under high-gain observer feedback to the trajectories of the system under ideal differentiation feedback is presented. This is done by showing that the state trajectories of the two systems are close. Finally, we provide some simulation results to illustrate the analysis.

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II. Motivation

Consider the system

$$\dot{x}_1 = x_2 \quad (1)$$

$$\dot{x}_2 = a_1 x_1 + a_2 x_2 + u \quad (2)$$

$$y = x_1 + v \quad (3)$$

The system can be stabilized by the state feedback control

$$u = -b_1 x_1 - b_2 x_2 \quad (4)$$

where b_1 and b_2 are chosen such that the roots of

$$s^2 + (b_2 - a_2)s + (b_1 - a_1) \quad (5)$$

have negative real parts. In output feedback we use the linear high-gain observer

$$\dot{\hat{x}}_1 = \hat{x}_2 + \frac{\alpha_1}{\varepsilon}(y - \hat{x}_1) \quad (6)$$

$$\dot{\hat{x}}_2 = \frac{\alpha_2}{\varepsilon^2}(y - \hat{x}_1) \quad (7)$$

and implement the control as

$$u = -b_1 \hat{x}_1 - b_2 \hat{x}_2 \quad (8)$$

In the observer (6)-(7), the measurement noise v is multiplied by a gain of the order $O(1/\varepsilon^2)$. This gives the impression that as we decrease ε the effect of noise will blow up. This is not the case, however, as the following calculations show. The transfer function from the measurement noise v to the state x is given by

$$\frac{Num_1}{Den_1} \begin{bmatrix} -1 \\ -s \end{bmatrix} \quad (9)$$

$$Num_1 = [b_1(\varepsilon\alpha_1 s + \alpha_2) + b_2\alpha_2 s]$$

$$Den_1 = (s^2 - a_2 s - a_1)[(\varepsilon s)^2 + \alpha_1 \varepsilon s + \alpha_2] + [b_1(\varepsilon\alpha_1 s + \alpha_2) + \alpha_2 b_2 s]$$

which is a proper transfer function that approaches

$$\frac{(b_2 s + b_1)}{s^2 + (b_2 - a_2)s + (b_1 - a_1)} \begin{bmatrix} -1 \\ -s \end{bmatrix} \quad (10)$$

as $\varepsilon \rightarrow 0$. The transfer function (10) is nothing but the transfer function from v to x under the feedback control

$$u = -b_1 y - b_2 \dot{y} \quad (11)$$

We call the closed-loop system under (11) the ideal differentiation system. It determines the limiting behavior of the closed-loop system under the high-gain observer as $\varepsilon \rightarrow 0$.

The foregoing analysis does not take into consideration that, in the high-gain observer system, the control input is saturated to protect the plant from peaking. The saturation function should be effective only during peaking. This is ensured by choosing the control saturation level higher than $max|-b_1 x_1 - b_2 x_2|$ where the maximum is calculated over a compact set of interest for the closed-loop system under (noise-free) state feedback. How will the presence of noise affect the control saturation? To answer this question, let us consider the transfer function from the noise v to the control u, which is given by

$$\frac{Num_2}{Den_2} \quad (12)$$

$$Num_2 = -(s^2 - a_2 s - a_1)[b_1(\varepsilon\alpha_1 s + \alpha_2) + \alpha_2 b_2 s]$$

$$Den_2 = (s^2 - a_2 s - a_1)[(\varepsilon s)^2 + \alpha_1 \varepsilon s + \alpha_2] + b_1(\varepsilon\alpha_1 s + \alpha_2) + \alpha_2 b_2 s$$

for the observer-based system and by

$$\frac{-(s^2 - a_2 s - a_1)(b_2 s + b_1)}{s^2 + (b_2 - a_2)s + (b_1 - a_1)} \quad (13)$$

for the ideal differentiation system. It is clear that (12) approaches (13) as $\varepsilon \rightarrow 0$. However, (13) is an improper transfer function. To ensure that u does not saturate we must limit the class of signals v. We are interested in investigating wide-band noise. So, suppose v is the output of a linear system whose transfer function is

$$\frac{K}{(\mu s)^2 + \beta_1 \mu s + \beta_2} \quad (14)$$

and whose input is a bounded signal w. The constants β_1, β_2 and μ are positive and the bandwidth of v will be of the order of $1/\mu$ for a white (or almost white) input signal w. The transfer function

$$\left(\frac{-(s^2 - a_2 s - a_1)(b_2 s + b_1)}{(s^2 + (b_2 - a_2)s + (b_1 - a_1))}\right)\left(\frac{K}{(\mu s)^2 + \beta_2 \mu s + \beta_1}\right) \quad (15)$$

from w to u has an impulse response of the order of $1/\mu^2$. For the problem to be well formulated, we take $K = \mu^2$. When the plant has relative degree n, the foregoing discussion shows that the appropriate noise model is

$$\frac{\mu^n}{(\mu s)^n + \cdots + \beta_{n-1}(\mu s) + \beta_n} \quad (16)$$

III. System Description

We consider the SISO nonlinear system

$$\dot{x} = Ax + B[a(x) + b(x)u] \quad (17)$$

$$y = Cx + v_1 \quad (18)$$

where u is the control input, $x \in R^n$ is the state, and v_1 is the measurement noise. The $n \times n$ matrix A, the $n \times 1$ matrix B, and the $1 \times n$ matrix C are given by

$$A = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & \cdots 0 \end{bmatrix}$$

For convenience, we do not include zero dynamics in the model (17)-(18), but it is expected that zero dynamics will not alter our conclusions. It can easily be seen that when $v_1 = 0$ the states of the system are successive derivates of the output y. The output feedback controller is designed by first considering a state feedback controller

$$u = \gamma(x) \quad (19)$$

that meets the performance objectives. The output feedback controller is implemented using the observer

$$\dot{\hat{x}} = A\hat{x} + B[\hat{a}(\hat{x}) + \hat{b}(\hat{x})u] + H(y - C\hat{x}) \quad (20)$$

where the observer gain is designed as

$$H^T = \begin{bmatrix} \frac{\alpha_1}{\varepsilon} & \frac{\alpha_2}{\varepsilon^2} & \cdots & \frac{\alpha_n}{\varepsilon^n} \end{bmatrix}$$

ε is a small positive parameter, $\hat{a}(\hat{x})$ and $\hat{b}(\hat{x})$ are the nominal models of $a(x)$ and $b(x)$, and the positive constants α_i are chosen such that the roots of the polynomial

$$s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n = 0 \quad (21)$$

have negative real parts. It is shown in [5] that the state estimation error will reduce to an $O(\varepsilon)$ quantity after a short transient period. During this period the estimate may experience a peaking phenomenon described in [5]. To overcome peaking, the control is saturated outside a compact region of interest so that $u = \gamma(x)$ is a globally bounded function of x. To facilitate the study of high gain observers in the presence of measurement noise we introduce a reference system based on ideal differentiation of the measured output. The control of such a system takes the form

$$u = \gamma(y, \dot{y}, \ddot{y}, \cdots, y^{(n-1)}) \quad (22)$$

Suppose the noise signal v_1 is the output of a linear system driven by a bounded input w. The system is

$$v_1 = \frac{\mu^n}{(\mu s)^n + \beta_1(\mu s)^{n-1} + \cdots + \beta_{n-1}\mu s + \beta_n} w \quad (23)$$

where μ is a positive constant and the roots of

$$(\mu s)^n + \beta_1(\mu s)^{n-1} + \cdots + \beta_{n-1}\mu s + \beta_n \qquad (24)$$

have negative real parts. The constant μ parametrizes the bandwidth of the noise, which is of the order of $1/\mu$. For wide band noise, μ will be small. In the ideal differentiation system, the control u depends on derivatives of v_1 up to the $(n-1)th$ derivative. To ensure that u will be bounded as $\mu \to 0$, we take the numerator of (23) to be μ^n.

Let

$$v = [v_1, \dot{v}_1, \ddot{v}_1, \cdots, v_1^{(n-1)}] \stackrel{\text{def}}{=} [v_1 \cdots v_n] \qquad (25)$$

be a vector with elements consisting of v_1 and its successive derivatives. The closed-loop system under ideal differentiation is

$$\dot{x} = Ax + B[a(x) + b(x)\gamma(x+v)] \stackrel{\text{def}}{=} f(x,v) \qquad (26)$$

The ideal differentiation system provides us with a reference system that has some intuitive appeal, and that we can use to study the convergence properties of the output feedback high-gain observer as the parameter ε is decreased.

IV. Trajectory Convergence

We start by stating our assumptions.

Assumption 1: The measurement noise is generated by the exogenous system (23), where w is a bounded input.

Let

$$\begin{aligned} \mu\dot{\bar{v}}_1 &= \bar{v}_2 \\ \mu\dot{\bar{v}}_2 &= \bar{v}_3 \\ &\vdots \\ \mu\dot{\bar{v}}_n &= \bar{v}_{n+1} \\ v_1 &= \mu^{n-1}\bar{v}_1 \end{aligned}$$

be a state-space realization of (23), where $\bar{v}_{n+1} = -\beta_n\bar{v}_1 - \beta_{n-1}\bar{v}_2 - \cdots - \beta_1\bar{v}_n + \mu w$. Written in a more compact form we have

$$\mu\dot{\bar{v}} = A_1\bar{v} + \mu B w \qquad (27)$$

where A_1 is a Hurwitz matrix. We take $\bar{v}(0) = 0$.

The elements of the vectors v and $\bar{v}$ are related through

$$\bar{v}_i = \frac{v_1^{(i-1)}}{\mu^{n-i}} \quad \text{for} \quad i = 1 \cdots n \qquad (28)$$

It can be shown that $\|\bar{v}\| \leq \mu K_v$ for some $K_v > 0$.

Assumption 2: The function $f(x,y)$ and its first partial derivatives with respect to x are locally lipschitz in their arguments and the function $\gamma(x)$ is globally bounded in x.

Assumption 3: For every $\mu \in (0, \mu^*]$ the ideal differentiator system has a steady-state solution $\bar{x}_r(t,\mu)$ which has following properties uniformly in μ for all $\mu \in (0, \mu^*]$:

1. $\bar{x}_r$ is bounded
2. for $z = x_r - \bar{x}_r$, the origin $z = 0$ is a uniformly asymptotically stable equilibrium point of the system

$$\dot{z} = f(z + \bar{x}_r, v) - f(\bar{x}_r, v) \stackrel{\text{def}}{=} F(z, \bar{x}_r, v) \qquad (29)$$

with a region of attraction that includes a domain $R_z \subset R^n$ Furthermore, $z = 0$ is locally exponentially stable.

Under the above assumptions we can show that the trajectories of the closed-loop system under high-gain observer feedback approach the trajectories of the ideal differentiation system as ε tends to zero. The idea behind the analysis is to represent the system in the singularly perturbed form by rescaling the error between the observer states and the derivatives of the output, much as in the proof given in [6]. These rescaled errors are given by

$$\xi_i = \frac{x_i - \hat{x}_i + v_i}{\varepsilon^{n-i}} \quad , \quad \text{for} \quad i = 1 \cdots n \qquad (30)$$

Hence,

$$x - \hat{x} + v = D\xi \qquad (31)$$

where $D = \text{diag}[1, \varepsilon, \cdots, \varepsilon^{n-1}]$. The rescaled system appears as

$$\begin{aligned} \varepsilon\dot{\xi}_1 &= -\alpha_1\xi_1 + \xi_2 \\ \varepsilon\dot{\xi}_2 &= -\alpha_2\xi_1 + \xi_3 \\ &\vdots \\ \varepsilon\dot{\xi}_n &= -\alpha_n\xi_1 + \varepsilon\delta(x,\hat{x}) + \frac{\varepsilon}{\mu}\bar{v}_{n+1} \end{aligned}$$

where

$$\delta(x,\hat{x}) = a(x) - \hat{a}(\hat{x}) + \left[b(x) - \hat{b}(\hat{x})\right]\gamma(\hat{x}) \qquad (32)$$

In view of these equations the closed-loop system under the observer is given by

$$\dot{x} = f(x, v - D\xi) \qquad (33)$$

$$\varepsilon\dot{\xi} = A_0\xi + \varepsilon B\delta(x,\hat{x}) + \frac{\varepsilon}{\mu}B\bar{v}_{n+1} \qquad (34)$$

where

$$A_0 = \begin{bmatrix} -\alpha_1 & 1 & \cdots & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ -\alpha_{n-1} & \cdots & \cdots & 0 & 1 \\ -\alpha_n & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

We can see that setting ε equal to zero results in the closed-loop system under ideal differentiation. This result is stated formally in the following theorem.

Theorem: Consider the closed-loop system under the high-gain observer (33)-(34) *and the ideal differentiation system* (26) *and denote their solutions by x and x_r, respectively. Suppose Assumptions* 1-3 *hold and $x(t_0) = x_r(t_0)$. Let $Q \subset R^n$ and $\Omega \subset R_z$ be compact sets. Given $\nu > 0$ there exists ε^*, dependent on μ^* but independent of μ, such that for all $0 < \varepsilon \leq \varepsilon^*$, $0 < \mu \leq \mu^*$, $z(t_0) \in \Omega$, and $\hat{x}(t_0) \in Q$*

$$\|x(t) - x_r(t)\| \leq \nu \qquad (35)$$

for all $t \in [t_0, \infty)$.

Proof:

Let $\eta = x - \bar{x}_r$ and rewrite the equations (33)-(34) as

$$\dot{\eta} = F(\eta, \bar{x}_r, v) + f(x, v - D\xi) - f(x, v) \quad (36)$$

$$\varepsilon\dot{\xi} = A_0\xi + \varepsilon B\delta(x, \hat{x}) + \frac{\varepsilon}{\mu}B\bar{v}_{n+1} \quad (37)$$

where $\eta(t_0) = x(t_0) - \bar{x}_r(t_0) = x_r(t_0) - \bar{x}_r(t_0) = z(t_0) \in \Omega$. Note that

$$\|\bar{v}_{n+1}\| \leq \beta_1\|\bar{v}_n\| + \cdots + \beta_n\|\bar{v}_1\| + \mu\|w\| \quad (38)$$

Since w is bounded and from (27) $\bar{v}$ is $O(\mu)$, $\|\bar{v}_{n+1}\| = O(\mu)$. Therefore, $(1/\mu)\bar{v}_{n+1}$ on the right-hand side of (37) is bounded, uniformly in μ. Consider the system $\dot{z} = F(z, \bar{x}_r, v)$ and let $\mathcal{R}$ be any connected subset of its region of attraction. By Corollary (6.1) of [1], there is a Lyapunov function $V(z)$, dependent on μ^* but independent of μ, such that

$$\alpha_1(w_0(z)) \leq V(z) \leq \alpha_2(w_0(z)) \quad (39)$$

$$\frac{\partial V}{\partial z}F(z, \bar{x}_r, v) \leq -\alpha_3(w_0(z)) \quad (40)$$

where α_1,α_2 are class $\mathcal{K}_\infty$ functions, α_3 is a continuous positive definite function, $w_0(z)$ is positive definite, and $w_0(z) \to \infty$ as $z \to \partial\mathcal{R}$.

Choose $c > 0$ such that the set $\Omega_c = \{V(\eta) \leq c\}$ satisfies $\Omega \subset \Omega_c \subset \mathcal{R}$. Repeating arguments used in [2], it can be shown that there exist ρ and ε_1 such that for all $0 < \varepsilon \leq \varepsilon_1$, the set $\Omega_c \times \{W(\xi) \leq \rho\varepsilon^2\}$ is positively invariant and trajectories of (36)-(37) enter this set within a time interval $[t_0, T(\varepsilon) + t_0]$, where $lim_{\varepsilon\to 0}T(\varepsilon) = 0$.

Due to the global boundedness of $\gamma(\hat{x})$, $\dot{x}$ is uniformly bounded. Hence, over the period $[t_0, T(\varepsilon) + t_0]$

$$\|x(t) - x_r(t)\| \leq kT(\varepsilon) \quad (41)$$

for some $k > 0$. For $t \geq t_0 + T(\varepsilon)$,

$$\dot{x} = f(x, v - D\xi) = f(x, v) + O(\varepsilon) \quad (42)$$

while

$$\dot{x_r} = f(x_r, v) \quad (43)$$

Hence by continuous dependence of the solution on parameters, we see that for any finite $T_1 > t_0$

$$\|x(t) - x_r(t)\| \leq \delta(\varepsilon) \quad (44)$$

for all $t \in [t_0, T_1]$, where $\delta(\varepsilon)$ is a continuous function with $lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$. It remains now to deal with the interval $[T_1, \infty)$. For that we employ the local exponential stability of $\dot{z} = F(z, \bar{x}_r, v)$, which implies the existence of a Lyapunov function $\bar{V}(t, z)$ that satisfies

$$c_1\|z\|^2 \leq \bar{V}(t, z) \leq c_2\|z\|^2 \quad (45)$$

$$\frac{\partial\bar{V}}{\partial t} + \frac{\partial\bar{V}}{\partial z}F(z, \bar{x}_r, v) \leq -c_3\|z\|^2 \quad (46)$$

$$\left\|\frac{\partial\bar{V}}{\partial z}\right\| \leq c_4\|z\| \quad (47)$$

in some neighborhood of $z = 0$, for some positive constants c_i. The error $e(t) = x(t) - x_r(t)$ satisfies

$$\dot{e} = F(e(t), \bar{x}_r(t), v(t)) + \Delta_1 + \Delta_2 \quad (48)$$

where

$$\Delta_1 = f(e + x_r, v - D\xi) - f(e + x_r, v) \quad (49)$$

$$\Delta_2 = f(\bar{x}_r, v) - f(e + \bar{x}_r, v) - f(x_r, v) + f(e + x_r, v) \quad (50)$$

The error terms Δ_1 and Δ_2 satisfy the bounds

$$\|\Delta_1\| \leq K_1\|\xi\| \quad (51)$$

$$\|\Delta_2\| \leq K_2\|e\|^2 + K_3\|z\|\|e\| \quad (52)$$

for some positive constants K_i. Equation (48) is viewed as a perturbation of the exponentially stable system $\dot{e} = F(e, \bar{x}_r, v)$. The Lyapunov function $\bar{V}(t, e)$ satisfies

$$\dot{\bar{V}} = \frac{\partial\bar{V}}{\partial t} + \frac{\partial\bar{V}}{\partial e}F(e, \bar{x}_r, v) + \frac{\partial\bar{V}}{\partial e}(\Delta_1 + \Delta_2) \quad (53)$$

$$\dot{\bar{V}} \leq -c_3\|e\|^2 + c_4\|e\|[K_1\|\xi\| + K_2\|e\|^2 + K_3\|z\|\|e\|] \quad (54)$$

Using the fact that after some time $t \geq T_1$, $\|\xi\|$ is $O(\varepsilon)$ and $\|z(t)\|$ satisfies

$$\|z(t)\| \leq K_z\|z(T_1)\|e^{-\gamma(t-T_1)} \quad , \quad t \geq T_1 \quad (55)$$

for some positive constants K_z and γ, independent of μ, it can be shown that, given $\nu_1 > 0$ there exits ε_2 such that

$$\|x(t) - x_r(t)\| \leq \nu_1 \quad \forall \quad t \geq T_1 \quad (56)$$

$\forall\ 0 < \varepsilon \leq \varepsilon_2$. From equations (41), (44), and (56) we can see that given $\nu > 0$ there exists ε^* such that for all $0 < \varepsilon < \varepsilon^*$,

$$\|e(t)\| \leq \nu \quad (57)$$

V. Simulation Results

Example: To illustrate the above analytical results, we consider the following example of the pendulum system

$$\dot{\theta} = \omega \quad (58)$$

$$\dot{\omega} = -9.3429sin(\theta) - 0.1333\omega + 6.0469u \quad (59)$$

We simulated the observer based and ideal differentiation systems for comparison. The high-gain observer for this system takes the form

$$\dot{\hat{\theta}} = \hat{\omega} + \frac{2}{\varepsilon}(\theta - \hat{\theta} + v) \quad (60)$$

$$\dot{\hat{\omega}} = \frac{1}{\varepsilon^2}(\theta - \hat{\theta} + v) \quad (61)$$

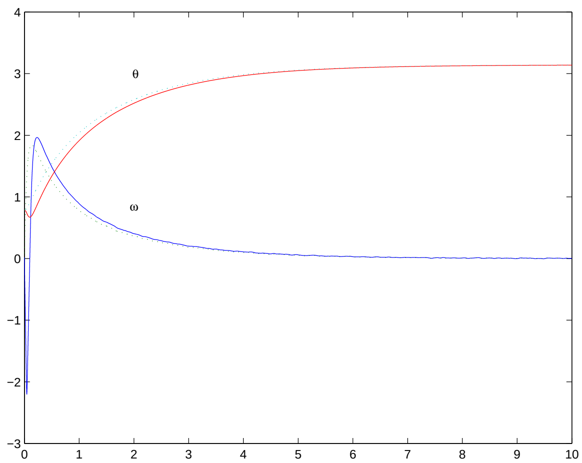

Fig. 1. Plot of the pendulum trajectories(θ,ω) (solid) and (θ_r,ω_r) (dotted) vs. time with $\varepsilon = 0.01$, $\mu = 0.05$

The control used is

$$u = -4sat((\theta - \pi + v_1) + \omega + v_2)) \qquad (62)$$

This controller is used to stabilize the system at ($\theta = \pi, \omega = 0$). The parameter values for the noise system are $\beta_1 = 2$, $\beta_2 = 3$. The noise input w is taken to be a uniformly distributed noise with values in $[-10, 10]$. The observer parameters are $\alpha_1 = 2$ and $\alpha_2 = 1$. The initial conditions are $\theta_r(0) = \theta(0) = \frac{\pi}{4}$, $\omega_r(0) = \omega(0) = \hat{\theta}(0) = \hat{\omega}(0) = 0$. The trajectories for the observer system and ideal differentiation system are shown in Figure 1 for $\varepsilon = 0.01$ and $\mu = 0.05$. The trajectories θ and θ_r approach π while ω and ω_r approach 0. Figure 2 provides a closer look at the steady-state trajectory error by plotting $\theta - \theta_r$ for $\varepsilon = 0.05$, 0.005 and $\mu = 0.05$. Figure 3 displays the corresponding errors $\omega - \omega_r$. These plots show a reduction in the error as ε is decreased. Figure 4 shows a plot of the error of the system states $\|x - x_r\|_2$ at steady-state versus the parameter ε for $\mu = 1, 0.1, 0.01$, and 0.001. We choose to examine the steady-state error only and ignore the effect of peaking to illustrate the result. Of course, the error during peaking also decreases as ε decreases. The general trend in this figure indicates that the error decreases as ε decreases and as μ decreases.

VI. CONCLUSIONS

We have studied the use of high-gain observers in output feedback control of nonlinear systems in the presence of measurement noise. The technical challenge in our analysis was to prove the trajectory convergence property as the observer parameter ε tends to zero uniformly in the parameter μ that parametrizes the noise bandwidth. It is true that in our problem formulation we had to limit the noise amplitude to $O(\mu^n)$, but we argued that such limitation is needed in the ideal differentiation system to have a well-behaved control signal. We have seen that as the observer gain is increased the trajectories of the closed-loop system approach those

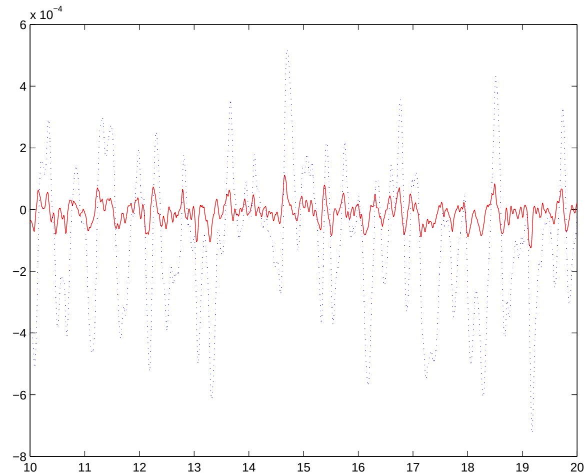

Fig. 2. Plot of the error ($\theta - \theta_r$) vs. time with $\mu = 0.05$, $\varepsilon = 0.05$ (dotted) and 0.005 (solid)

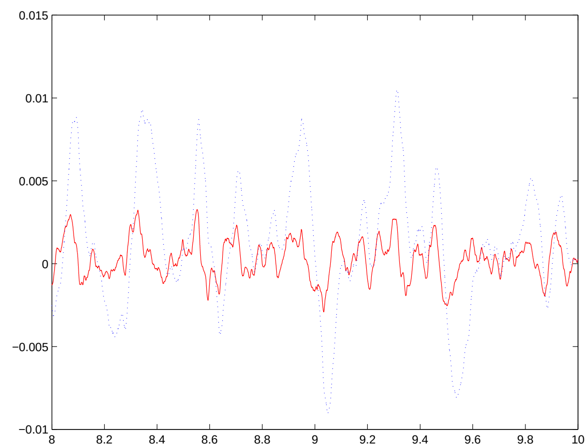

Fig. 3. Plot of the error ($\omega - \omega_r$) vs. time with $\mu = 0.05$, $\varepsilon = 0.05$ (dotted) and $\varepsilon = 0.005$ (solid)

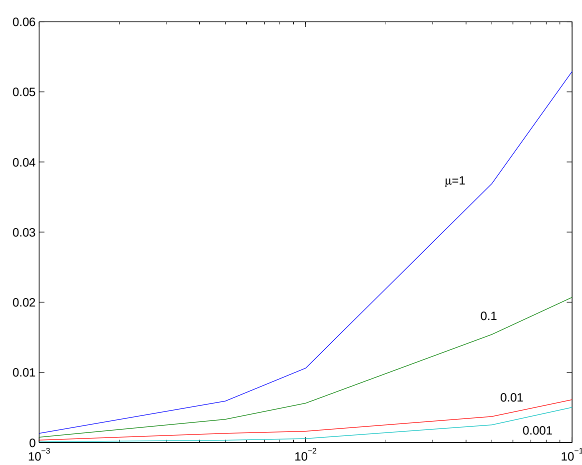

Fig. 4. Plot of the error ($\|x - x_r\|$) vs. ε with $\mu = 1, 0.1, 0.01$, and 0.001

of the ideal differentiation system in which the derivatives of the output are fed back. Obviously, measurement noise and differentiation of such noise is undesirable in feedback systems, but the results provide information about the limiting case as $\varepsilon \to 0$. We note that the performance of the high-gain observer system will be better for finite $\varepsilon > 0$ because in that case the observer acts as a low pass filter, but the point of the paper is to argue that as we push ε smaller we will not do worse than the ideal differentiation case.

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