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Reduced-Order $\mathcal{H}_\infty$ Filtering for Discrete-Time, Linear, Time-Varying Systems

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Abstract— The reduced-order, $\mathcal{H}_\infty$ filtering problem for discrete-time, linear, time-varying systems is considered. A solution is obtained by converting the reduced-order, filtering problem into a full-order, filtering problem with the same dimension as the reduced-order filter. The resulting filter is unbiased and has a unique realization for each set of design parameters. Furthermore, it is shown that the previously proposed solution is independent of the $\mathcal{H}_\infty$ design parameters and, therefore, does not represent a solution to the problem.

I. INTRODUCTION

The $\mathcal{H}_\infty$ filtering problem has been studied extensively due to the robustness of $\mathcal{H}_\infty$ filters to modeling error and disturbances with unknown statistics [1], [2]. In many filtering applications, estimates are generated in real-time from measurements and, as a result, there has been considerable interest in filtering algorithms that are computational efficient. One solution has been to use a reduced-order filter [3], [4], [5]. In this paper, the reduced-order $\mathcal{H}_\infty$ filtering problem is considered for discrete-time, linear, time-varying (LTV) systems on a finite horizon.

Much of the previous work on the reduced-order $\mathcal{H}_\infty$ filtering problem has focused on the solution for continuous-time, linear, time-invariant (LTI) systems [5], [6]. More recently, these results has been extended to stochastic systems [7] and a class of bilinear systems [8]. In [4], solutions for the finite and infinite horizon, reduced-order $\mathcal{H}_\infty$ filtering problems have been suggested for continuous-time, LTI systems using an approach developed to solve the reduced-order $\mathcal{H}_2$ filtering problem [3]. The nature of the solution admits a generalization to LTV systems and motivates the solution approach considered in this paper. For discrete-time, LTV systems, a solution of the reduced-order $\mathcal{H}_\infty$ filtering problem has been proposed in [9], [10], [11] where the state dimension of the reduced-order filter is minimized. However, it is shown in the appendix to this paper that the solution of the proposed filtering problem is independent of the design parameters used in the formulation of the $\mathcal{H}_\infty$ filtering problem. Therefore, the reduced-order $\mathcal{H}_\infty$ filtering problem for discrete-time, LTV systems remains an open question.

In this paper, the solution of the reduced-order $\mathcal{H}_\infty$ filtering problem for discrete-time, LTV systems on a finite horizon is presented. The state dimension of the reduced-order filter is treated as a design parameter and is bounded above by the number of states (or linear combination thereof) to be estimated and below by the plant state dimension minus the number of measurements. The difference between the current work and the results in [9], [10], [11] is that the state dimension of the reduced-order filter is assumed to be strictly greater than its minimum allowable value, the plant state dimension minus the number of measurements.

The solution strategy for the reduced-order $\mathcal{H}_\infty$ filtering problem is motivated by the results in [4]. In reduced-order filtering, an unbiased filter is important because it ensures that the estimation error is zero if the initial estimate is correct and there are no disturbances acting on the system [3], [4]. The unbiasedness requirement manifests itself as a set of constraints on the coefficient matrices of the filter realization. If the dimension of the reduced-order filter is greater than the minimum allowable value, the unbiasedness constraints do not have a unique solution. Using the extra degree of freedom in the solution of the unbiasedness constraints, the reduced-order $\mathcal{H}_\infty$ filtering problem is converted into a full-order $\mathcal{H}_\infty$ filtering problem with the same dimension as the reduced-order filter. The solution of the reduced-order $\mathcal{H}_\infty$ filtering problem is obtained by applying the full-order $\mathcal{H}_\infty$ filtering results from [12].

The remainder of the paper is organized as follows. Section II describes the details of the reduced-order $\mathcal{H}_\infty$ filtering problem. Section III contains a summary of relevant results from the solution of the full-order $\mathcal{H}_\infty$ filtering problem. The reduced-order filter and the unbiasedness constraints are presented in Section IV. The solution of the reduced-order $\mathcal{H}_\infty$ filtering problem is presented in Section V. Conclusions and directions for future research are described in Section VI. An examination of the reduced-order $\mathcal{H}_\infty$ filtering results

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from [9], [10], [11] is provided in the appendix.

II. PROBLEM STATEMENT

In this paper, a linear, time-varying discrete-time plant of the form

$$\begin{aligned} x(k+1) &= A(k)\,x(k) + B(k)\,w(k), \quad x(0) = x_0 \\ y(k) &= C(k)\,x(k) + D(k)\,w(k) \\ z(k) &= E_1(k)\,x(k) + E_2(k)\,w(k) \end{aligned} \tag{2.1}$$

is considered for $k = 0, \ldots, N$ where x is the $n \times 1$ vector of state variables, w is the $q \times 1$ vector of exogenous inputs, disturbances, and noise, y is the $p \times 1$ vector of measured quantities, and z is the $m \times 1$ vector of quantities to be estimated. The following assumptions are made about the realization in (2.1) for $k = 0, \ldots, N$.

1) All matrices are bounded.
2) $D(k)\,D^*(k) > 0$
3) rank $C(k) = p$
4) rank $E_1(k) = m$
5) rank $\begin{bmatrix} E_1(k) \\ C(k) \end{bmatrix} = n$

where $*$ indicates the conjugate transpose of a matrix. Note that Assumptions 1 and 2 are standard, Assumptions 3 and 4 are reasonable and simplify the solution of the problem, and Assumption 5 is necessary for a general solution to the problem. Furthermore, Assumption 5 implies that $m + p \geq n$.

Given a tolerance $\gamma > 0$ and an initial estimation error weight $R > 0$, the objective of the $\mathcal{H}_\infty$ filtering problem is to obtain a filter $\hat{\mathbf{z}} = \mathbf{F}\,\mathbf{y}$ that produces estimates satisfying

$$\sup_{[w\ e_s(0)] \neq 0} \frac{\|\hat{z}(k) - z(k)\|^2_{2,[0,N]}}{\|w\|^2_{2,[0,N]} + e_s^*(0)\,R\,e_s(0)} < \gamma^2 \tag{2.2}$$

where $e_s(0)$ is the difference between the initial state of the filter and true initial state and

$$\|f\|^2_{2,[0,N]} := \sum_{k=0}^{N} f^*(k)\,f(k) \tag{2.3}$$

Note that the formulation of the $\mathcal{H}_\infty$ problem (2.2) is independent of the order of the filter.

III. FULL-ORDER $\mathcal{H}_\infty$ FILTERING PROBLEM

The solution of the reduced-order $\mathcal{H}_\infty$ filtering problem for the plant in (2.1) will be obtained by applying a full-order $\mathcal{H}_\infty$ filtering results to a related plant with the same dimension as the reduced-order filter. Accordingly, selected full-order $\mathcal{H}_\infty$ filtering results are summarized below.

For the plant in (2.1), consider the full-order filter

$$\begin{aligned} \hat{x}(k+1) &= A(k)\,\hat{x}(k) + K(k)\,(y(k) - C(k)\,\hat{x}(k)) \\ \hat{z}(k) &= E_1(k)\,\hat{x}(k) + J(k)\,(y(k) - C(k)\,\hat{x}(k)) \\ \hat{x}(0) &= \hat{x}_0 \end{aligned} \tag{3.1}$$

where the gains $K(k)$ and $J(k)$ are determined from the solution of the $\mathcal{H}_\infty$ filtering problem (2.2). The solution of the full-order $\mathcal{H}_\infty$ filtering problem has been presented in [12] and is summarized in the following theorem.

Theorem 1: Given the plant in (2.1), a tolerance $\gamma > 0$, and an initial estimation error weight $R > 0$, the $\mathcal{H}_\infty$ filtering problem (2.2) has a solution if and only if

$$\begin{aligned} S_3(k) &> 0 \\ \left(S_1(k) - S_2(k)\,S_3(k)^{-1}\,S_2^*(k)\right)^{-1} &< 0 \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} S(k) &= \begin{bmatrix} S_1(k) & S_2(k) \\ S_2^*(k) & S_3(k) \end{bmatrix} \\ &= \begin{bmatrix} -\gamma^2\,I + E_2(k)\,E_2^*(k) + E_1(k)\,Q(k)\,E_1^*(k) & -E_2(k)\,D^*(k) - E_1(k)\,Q(k)\,C^*(k) \\ -D(k)\,E_2^*(k) - C(k)\,Q(k)\,E_1^*(k) & D(k)\,D^*(k) + C(k)\,Q(k)\,C^*(k) \end{bmatrix} \end{aligned} \tag{3.3}$$

and $Q(k)$ is the solution of the Riccati equation

$$\begin{aligned} Q(k+1) &= A(k)\,Q(k)\,A^*(k) + B(k)\,B^*(k) \\ &\quad -M(k)\,S^{-1}(k)\,M^*(k) \\ Q(0) &= R^{-1} \end{aligned} \tag{3.4}$$

with

$$\begin{aligned} M(k) &= \begin{bmatrix} M_1(k) & M_2(k) \end{bmatrix} \\ &= [-B(k)\,E_2^*(k) - A(k)\,Q(k)\,E_1^*(k) \\ &\quad B(k)\,D^*(k) + A(k)\,Q(k)\,C^*(k)] \end{aligned} \tag{3.5}$$

Furthermore, if a solution exists, one such filter has as a realization as in (3.1) with

$$\begin{aligned} K(k) &= M_2(k)\,S_3^{-1}(k) \\ J(k) &= -S_2(k)\,S_3^{-1}(k) \end{aligned} \tag{3.6}$$

In the sequel, a connection will be made between the estimation error equations for the reduced-order filter and those for a full-order filter. For the full-order filter, the equations governing the estimation error are

$$\begin{aligned} e_s(k+1) &= (A(k) - K(k)\,C(k))\,e_s(k) \\ &\quad +(K(k)\,D(k) - B(k))\,w(k) \\ e_z(k) &= (E_1(k) - J(k)\,C(k))\,e_s(k) \\ &\quad +(J(k)\,D(k) - E_2(k))\,w(k) \end{aligned} \tag{3.7}$$

where $e_s(k) := \hat{x}(k) - x(k)$ and $e_z(k) := \hat{z}(k) - z(k)$. For the comparison in Section V, it is convenient to write the error equations in the form

$$\begin{aligned} e_s(k+1) &= A(k)\,e_s(k) - B(k)\,w(k) \\ &\quad +K(k)\,(-C(k)\,e_s(k) + D(k)\,w(k)) \\ e_z(k) &= E_1(k)\,e_s(k) - E_2(k)\,w(k) \\ &\quad +J(k)\,(D(k)\,w(k) - C(k)\,e_s(k)) \end{aligned} \tag{3.8}$$

IV. REDUCED-ORDER FILTERING

A general reduced-order filter has a realization

$$\begin{aligned} \xi(k+1) &= F(k)\,\xi(k) + K(k)\,y(k), \quad \xi(0)=\xi_0 \\ \hat{z}(k) &= H(k)\,\xi(k) + J(k)\,y(k) \end{aligned} \tag{4.1}$$

where $\xi(k)$ is a $v \times 1$ estimate of the quantity $T(k)\,x(k)$ and v satisfies $m \geq v \geq n-p$. Note that if filter dimension is minimized (i.e., $v = n-p$), the filter in (4.1) coincides with the reduced-order filter considered in [9], [10], [11].

The matrix function $T(k)$ is specified as part of the filter design and the only requirements are

$$\begin{aligned} \operatorname{rank} T(k) &= v \\ \operatorname{rank} \begin{bmatrix} T(k) \\ C(k) \end{bmatrix} &= n \end{aligned} \tag{4.2}$$

If $v = m$, it is natural to choose $T(k) = E_1(k)$ for $k = 0, \ldots, N$. In this case, $\xi(k) = \hat{z}(k)$ for $k = 0, \ldots, N$ and the output equation in (4.1) is not needed.

For the reduced-order filter (4.1), the equations governing the estimation error are

$$\begin{aligned} e_s(k+1) &= F(k)\,e_s(k) \\ &\quad + (K(k)\,D(k) - T(k+1)\,B(k))\;w(k) \\ &\quad + \Xi_1(k)\,x(k) \\ e_z(k) &= H(k)\,e_s(k) + (J(k)\,D(k) - E_2(k))\;w(k) \\ &\quad + \Xi_2(k)\,x(k) \end{aligned} \tag{4.3}$$

where $e_s(k) := \xi(k) - T(k)\,x(k)$ and

$$\begin{aligned} \Xi_1(k) &:= F(k)\,T(k) + K(k)\,C(k) \\ &\quad - T(k+1)\,A(k) \\ \Xi_2(k) &:= H(k)\,T(k) + J(k)\,C(k) - E_1(k) \end{aligned} \tag{4.4}$$

To ensure the estimation error is identically zero if $e_s(0) = 0$ and $\|w\|_{2,[0,N]} = 0$ (i.e., the filter is unbiased), the terms in (4.3) involving the plant state $x(k)$ must be removed. The resulting unbiasedness constraints can be expressed as

$$\begin{bmatrix} \Xi_1(k) \\ \Xi_2(k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.5}$$

The definition of $T(k)$ in (4.2) ensures that (4.5) has a solution. If $v = n-p$, the unbiasedness constraints (4.5) have a unique solution for any $T(k)$ satisfying (4.2). In the appendix, it is shown that the realization of the reduced-order $\mathcal{H}_\infty$ filter in [9], [10], [11] is the unique solution of (4.5) for a given $T(k)$ and, therefore, are independent of the solution of the $\mathcal{H}_\infty$ problem (2.2). If $v = m = n-p$, the coefficient matrices of the filter are completely specified since $T(k) = E_1(k)$.

In the sequel, it is assumed that reduced-order filter dimension v satisfies $v > n-p$ and, in this case, there exist matrix functions $\Gamma(k)$ and $\Omega(k)$ with dimensions $r \times v$ and $r \times p$, respectively, such that

$$\begin{bmatrix} \Gamma(k) & \Omega(k) \end{bmatrix} \begin{bmatrix} T(k) \\ C(k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.6}$$

where $r := v + p - n$. Furthermore, the rank condition in (4.2) ensures that $\Gamma(k)$ and $\Omega(k)$ can be chosen so that

$$\operatorname{rank} \begin{bmatrix} \Gamma(k) & \Omega(k) \end{bmatrix} = r \tag{4.7}$$

The following lemma shows that $\Omega(k)$ has full row rank. Note that $r \leq p$ if $v \leq n$.

Lemma 2: Given a matrix function $T(k)$ satisfying (4.2), there exist matrix functions $\Gamma(k)$ and $\Omega(k)$ satisfying (4.6) such that

$$\operatorname{rank} \Omega(k) = r \tag{4.8}$$

for $k = 0, \ldots, N$.

Proof: To prove that $\operatorname{rank} \Omega(k) = r$, a contradiction argument is used. Suppose that $\operatorname{rank}\Omega(k_0) < r$ for some k_0. It follows that there exists a $r \times 1$ vector $u_0 \neq 0$ such that $u_0^*\,\Omega(k_0) = 0$. In this case, the expression in (4.6) can be used to write

$$u_0^*\,\Omega(k_0)\,C(k_0) = -u_0^*\,\Gamma(k_0)\,T(k_0) = 0 \tag{4.9}$$

From the rank assumption in (4.7), $u_0^*\,\Gamma(k_0) \neq 0$. In this case, the expression in (4.9) contradicts the rank assumption on $T(k)$ in (4.2) and, therefore, $\operatorname{rank}\Omega(k) = r$ for $k = 0, \ldots, N$. ■

Given the results of Lemma 2, the matrix function $\Omega(k)$ is chosen such that

$$\Omega(k)\,\Omega^*(k) = I_r \tag{4.10}$$

where I_r is the $r \times r$ identity matrix. Furthermore, define a matrix function $\Omega^\perp(k)$ such that

$$\begin{bmatrix} \Omega(k) \\ \Omega^\perp(k) \end{bmatrix}$$

is unitary for all $k = 0, \ldots, N$ or, equivalently,

$$\begin{aligned} \Omega(k)\,\left(\Omega^\perp(k)\right)^* &= 0 \\ \Omega^\perp(k)\,\left(\Omega^\perp(k)\right)^* &= I_{p-r} \\ \Omega^*(k)\,\Omega(k) + \left(\Omega^\perp(k)\right)^*\,\Omega^\perp(k) &= I_p \end{aligned} \tag{4.11}$$

V. MAIN RESULT

For the $\mathcal{H}_\infty$ filtering problem in (2.2), the admissible set of reduced-order filters have realizations that satisfy the unbiasedness constraints (4.5). If $m + p > n$, the unbiasedness constraints do not have a unique solution. Using this extra degree of freedom, the reduced-order $\mathcal{H}_\infty$ filtering problem is converted into a full-order $\mathcal{H}_\infty$ filtering problem with the same dimension as the reduced-order filter. The solution of the reduced-order $\mathcal{H}_\infty$ filtering problem is obtained by applying a full-order $\mathcal{H}_\infty$ filtering results [12] to a related plant with the same dimension as the reduced-order filter.

If $m+p>n$, a family of admissible filter realizations in the form of (4.1) are obtained.

Lemma 3: Given $T(k)$ satisfying (4.2), suppose that the matrix functions $F_o(k)$, $K_o(k)$, $H_o(k)$, and $J_o(k)$ represent any solution of (4.5). The following matrix functions also satisfy the unbiasedness constraints

$$\begin{bmatrix} F(k) & K(k) \\ H(k) & J(k) \end{bmatrix} =$$

$$\begin{bmatrix} F_o(k) - K_o(k)\,\Omega^*(k)\,\Gamma(k) + \Delta(k)\,\Gamma(k) \\ H_o(k) - J_o(k)\,\Omega^*(k)\,\Gamma(k) + \Pi(k)\,\Gamma(k) \end{bmatrix.$$
$$\begin{matrix} K_o(k)\,\left(\Omega^\perp(k)\right)^*\,\Omega^\perp(k) + \Delta(k)\,\Omega(k) \\ J_o(k)\,\left(\Omega^\perp(k)\right)^*\,\Omega^\perp(k) + \Pi(k)\,\Omega(k) \end{matrix}\right] \tag{5.1}$$

where $\Delta(k)$ and $\Pi(k)$ are any bounded matrix functions.

Proof: Substituting the expressions for $F(k)$ and $K(k)$ in (5.1) into the left side of (4.5) and using the relationship $\left(\Omega^\perp(k)\right)^*\,\Omega^\perp(k) = I_p - \Omega^*(k)\,\Omega(k)$ from (4.11) yields

$$\begin{aligned} & F_o(k)\,T(k) + K_o(k)\,C(k) \\ + \quad & \left(-K_o(k)\,\Omega^*(k) + \Delta(k)\right)\,\left(\Gamma(k)\,T(k) + \Omega(k)\,C(k)\right) \end{aligned} \tag{5.2}$$

From the unbiasedness constraints (4.5), $F_o(k)\,T(k) + K_o(k)\,C(k) = T(k+1)\,A(k)$. Furthermore, by the definition of $\Gamma(k)$ and $\Omega(k)$ in (4.6), $\Gamma(k)\,T(k)+\Omega(k)\,C(k) = 0$. As a result, $F(k)$ and $K(k)$ in (5.1) satisfy the first unbiasedness constraint. A similar argument can be used to show that $H(k)$ and $J(k)$ in (5.1) satisfy the second unbiasedness constraint. ■

Using the matrix functions in (5.1), the estimation error equations in (4.3) can be rewritten as

$$\begin{aligned} e_s(k+1) &= \left(F_o(k) - K_o(k)\,\Omega^*(k)\,\Gamma(k)\right)\,e_s(k) \\ &\quad + \left(K_o(k)\,\left(\Omega^\perp(k)\right)^*\,\Omega^\perp(k)\,D(k)\right. \\ &\quad \left. -T(k+1)\,B(k)\right)\,w(k) \\ &\quad +\Delta(k)\,\left(\Gamma(k)\,e_s(k) + \Omega(k)\,D(k)\,w(k)\right) \\ e_z(k) &= \left(H_o(k) - J_o(k)\,\Omega^*(k)\,\Gamma(k)\right)\,e_s(k) \\ &\quad +J_o(k)\,\left(\Omega^\perp(k)\right)^*\,\Omega^\perp(k)\,D(k)\,w(k) \\ &\quad +\Pi(k)\,\left(\Gamma(k)\,e_s(k) + \Omega(k)\,D(k)\,w(k)\right) \end{aligned} \tag{5.3}$$

Comparing the above error equations with error equations for the full-order filter in (3.8), it follows that the error equations in (5.3) are equivalent to those obtained if a full-order filter with a realization of the form in (3.1) where $\tilde{K}(k) = \Delta(k)$ and $\tilde{J}(k) = \Pi(k)$ were applied to the v-dimensional plant

$$\begin{bmatrix} \tilde{A}(k) & \tilde{B}(k) \\ \tilde{C}(k) & \tilde{D}(k) \\ \tilde{E}_1(k) & \tilde{E}_2(k) \end{bmatrix}$$

$$= \begin{bmatrix} F_o(k) - K_o(k)\,\Omega^*(k)\,\Gamma(k) & \tilde{B}(k) \\ -\Gamma(k) & \Omega(k)\,D(k) \\ H_o(k) - J_o(k)\,\Omega^*(k)\,\Gamma(k) & -J_o(k)\,\left(\Omega^\perp(k)\right)^*\,\Omega^\perp(k)\,D(k) \end{bmatrix} \tag{5.4}$$

where

$$\begin{aligned} \tilde{B}(k) &= -K_o(k)\,\left(\Omega^\perp(k)\right)^*\,\Omega^\perp(k)\,D(k) \\ &\quad +T(k+1)\,B(k) \end{aligned} \tag{5.5}$$

From the above analysis, the reduced-order $\mathcal{H}_\infty$ filtering problem has a solution if and only if the v-dimensional full-order $\mathcal{H}_\infty$ filtering problem for (5.4) has a solution. Therefore, the main result is obtained by applying the results of Theorem 1 to the plant (5.4).

Theorem 4: Given the plant in (2.1) with $E_2(k) = 0$ for $k = 0, \ldots, N$, the tolerance $\gamma > 0$, the initial error weight $R > 0$, and a matrix function $T(k)$ satisfying (4.2), the $\mathcal{H}_\infty$ filtering problem (2.2) can be solved by a v-dimensional, reduced-order filter where $m \geq v > n-p$ if and only if

$$\begin{aligned} \tilde{S}_3(k) &> 0 \\ \left(\tilde{S}_1(k) - \tilde{S}_2(k)\,\tilde{S}_3(k)^{-1}\,\tilde{S}_2^*(k)\right)^{-1} &< 0 \end{aligned} \tag{5.6}$$

where $\tilde{S}$ is defined as in (3.3)-(3.5) using the coefficient matrices in (5.4). Furthermore, if a solution exists, one such filter has a realization

$$\begin{aligned} \xi(k+1) &= \left(F_o(k) - K_o(k)\,\Omega^*(k)\,\Gamma(k)\right. \\ &\quad \left. +\tilde{M}_2(k)\,\tilde{S}_3^{-1}(k)\,\Gamma(k)\right)\,\xi(k) \\ &\quad + \left(K_o(k)\,\left(\Omega^\perp(k)\right)^*\,\Omega^\perp(k)\right. \\ &\quad \left. +\tilde{M}_2(k)\,\tilde{S}_3^{-1}(k)\,\Omega(k)\right)\,y(k), \quad \xi(0) = \xi_0 \\ \hat{z}(k) &= \left(H_o(k) - J_o(k)\,\Omega^*(k)\,\Gamma(k)\right. \\ &\quad \left. -\tilde{S}_2(k)\,\tilde{S}_3^{-1}(k)\,\Gamma(k)\right)\,\xi(k) \\ &\quad + \left(J_o(k)\,\left(\Omega^\perp(k)\right)^*\,\Omega^\perp(k)\right. \\ &\quad \left. -\tilde{S}_2(k)\,\tilde{S}_3^{-1}(k)\,\Omega(k)\right)\,y(k) \end{aligned} \tag{5.7}$$

where $\tilde{M}$ is defined as in (3.5) using the coefficient matrices in (5.4) and $F_o(k)$, $K_o(k)$, $H_o(k)$, and $J_o(k)$ represent any solution of the unbiasedness constraints (4.5).

Proof: Since the results of Theorem 1 are being applied to the plant (5.4), it only remains to show that $\tilde{D}(k)\,\tilde{D}^*(k) > 0$. Using (5.4),

$$\tilde{D}(k)\,\tilde{D}^*(k) = \Omega(k)\,D(k)\,D^*(k)\,\Omega^*(k)$$

where $\tilde{D}(k)\,\tilde{D}^*(k)$ is a $r \times r$ matrix function. From Lemma 2, $\operatorname{rank}\Omega(k) = r$ for all $k = 0, \ldots, N$. Therefore, $\tilde{D}(k)\,\tilde{D}^*(k) > 0$ since $D(k)\,D^*(k) > 0$ by assumption. ■

As stated, the result in Theorem 4 depends on the particular solution ($F_0(k)$, $K_0(k)$, $H_0(k)$, and $J_0(k)$) of the unbiasedness constraints (4.5). The following corollary shows that the realization is unique up to the choice of γ, R, and the matrix function $T(k)$; that is, the coefficient matrices in (5.7) are independent of the choice of the matrix functions $F_o(k)$, $K_o(k)$, $H_o(k)$, and $J_o(k)$.

Corollary 5: Given the plant in (2.1) with $E_2(k) = 0$ for $k = 0, \dots, N$, the tolerance $\gamma > 0$, the initial estimation error weight $R > 0$, and a matrix function $T(k)$ satisfying (4.2), the realization of the reduced-order filter (5.7) in Theorem 4 is unique.

Proof: The first step in the proof is to show that v-dimensional plant realization (5.4) is independent of the choice of the matrix functions $F_o(k)$, $K_o(k)$, $H_o(k)$, and $J_o(k)$. Suppose that the matrix functions $F_1(k)$, $K_1(k)$, $H_1(k)$, and $J_1(k)$ represent another solution to the unbiasedness constraints (4.5). It follows that

$$\begin{bmatrix} F_\delta(k) & K_\delta(k) \\ H_\delta(k) & J_\delta(k) \end{bmatrix} \begin{bmatrix} T(k) \\ C(k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.8}$$

where

$$\begin{bmatrix} F_\delta(k) & K_\delta(k) \\ H_\delta(k) & J_\delta(k) \end{bmatrix} := \begin{bmatrix} F_1(k) - F_o(k) & K_1(k) - K_o(k) \\ H_1(k) - H_o(k) & J_1(k) - J_o(k) \end{bmatrix} \tag{5.9}$$

From the definition of the left null space of $\begin{bmatrix} T^*(k) & C^*(k) \end{bmatrix}^*$ in (4.6), there exist bounded, matrix functions $\Theta(k)$ and $\Xi(k)$ such that

$$\begin{bmatrix} F_\delta(k) & K_\delta(k) \\ H_\delta(k) & J_\delta(k) \end{bmatrix} = \begin{bmatrix} \Theta(k) \\ \Xi(k) \end{bmatrix} \begin{bmatrix} \Gamma(k) & \Omega(k) \end{bmatrix} \tag{5.10}$$

Using the definitions in (5.10), the difference

$$\begin{aligned} \tilde{A}_\delta(k) \; &:= \; F_1(k) - K_1(k)\,\Omega^*(k)\,\Gamma(k) \\ &\quad - \left(F_o(k) - K_o(k)\,\Omega^*(k)\,\Gamma(k)\right) \end{aligned}$$

can be rewritten as

$$\begin{aligned} \tilde{A}_\delta(k) &= F_\delta(k) - K_\delta(k)\,\Omega^*(k)\,\Gamma(k) \\ &= \Theta(k)\,\Gamma(k) - \Theta(k)\,\Omega(k)\,\Omega^*(k)\,\Gamma(k) \\ &= 0, \quad \forall k = 0, \dots, N \end{aligned} \tag{5.11}$$

because $\Omega(k)\,\Omega^*(k) = I_r$ for $k = 0, \dots, N$ by definition (4.10). Furthermore, the relationship in (5.10) yields that the difference $\tilde{B}_\delta(k) := -K_1(k)\,\left(\Omega^\perp(k)\right)^*\Omega^\perp(k)\,D(k) + T(k+1)\,B(k) - \left(-K_o(k)\,\left(\Omega^\perp(k)\right)^*\Omega^\perp(k)\,D(k) + T(k+1)\,B(k)\right)$ can be rewritten as

$$\begin{aligned} \tilde{B}_\delta(k) &= -K_\delta(k)\,\left(\Omega^\perp(k)\right)^*\Omega^\perp(k)\,D(k) \\ &= -\Theta(k)\,\Omega(k)\,\left(\Omega^\perp(k)\right)^*\Omega^\perp(k)\,D(k) \\ &= 0, \quad \forall k = 0, \dots, N \end{aligned} \tag{5.12}$$

because $\Omega(k)\,\left(\Omega^\perp(k)\right)^* = 0$ for $k = 0, \dots, N$ from (4.11). A similar process can be used to show that $H_1(k) - J_1(k)\,\Omega^*(k)\,\Gamma(k) = H_o(k) - J_o(k)\,\Omega^*(k)\,\Gamma(k)$ and $J_1(k)\,\left(\Omega^\perp(k)\right)^*\Omega^\perp(k) = J_o(k)\,\left(\Omega^\perp(k)\right)^*\Omega^\perp(k)$ for $k = 0, \dots, N$.

From the above analysis, the realization (5.4) is independent of the choice of the matrix functions $F_o(k)$, $K_o(k)$, $H_o(k)$, and $J_o(k)$. Furthermore, the solution of the Riccati equation is independent of $F_o(k)$, $K_o(k)$, $H_o(k)$, and $J_o(k)$ because the initial condition is independent of the solution of the unbiasedness constraints. Therefore, the realization of the reduced-order filter (5.7) is independent of the choice of the matrix functions $F_o(k)$, $K_o(k)$, $H_o(k)$, and $J_o(k)$ because it is computed from the v-dimensional plant realization (5.4) and the solution of the Riccati equation. ■

VI. Conclusion

In this paper, the reduced-order $\mathcal{H}_\infty$ filtering problem is solved for discrete-time, linear, time-varying (LTV) systems on a finite horizon. The resulting filter is unbiased and has a unique realization for each set of design parameters. The state dimension of the reduced-order filter is treated as a design parameter and is less than or equal to the number of states (or linear combination thereof) to be estimated and greater than the plant state dimension minus the number of measurements. In this case, the unbiasedness constraints do not have a unique solution and this extra degree of freedom is used to convert the reduced-order $\mathcal{H}_\infty$ filtering problem into a full-order $\mathcal{H}_\infty$ filtering problem with the same dimension as the reduced-order filter. The solution of the reduced-order $\mathcal{H}_\infty$ filtering problem is obtained by applying the full-order $\mathcal{H}_\infty$ filtering results.

In future work, the results of this paper will be extended to the infinite horizon. Furthermore, the effect of the filter order and the choice of states to be estimated in the reduced-order filter on the performance of the filter will be examined.

Appendix

In [9], [10], [11], a reduced-order $\mathcal{H}_\infty$ filter with minimum dimension (i.e. $v = n - p$) was derived. The analysis in Section IV shows that the realization is uniquely determined from the state estimation matrix function $T(k)$. In this appendix, it is shown that the realization in [9], [10], [11] satisfies this property and, therefore, is independent of the design parameters used in the formulation of the $\mathcal{H}_\infty$ filtering problem (2.2).

The reduced-order filter proposed in [9], [10], [11] has a realization

$$\begin{aligned} \xi(k+1) &= F(k)\,\xi(k) + K(k)\,y(k) + G(k)\,\hat{z}(k) \\ \xi(0) &= \xi_0 \\ \hat{z}(k) &= H(k)\,\xi(k) + J(k)\,y(k) \end{aligned} \tag{A.1}$$

This realization can be put in the form of (4.1) by substituting the expression for $\hat{z}(k)$ into the filter state equation yielding

$$\begin{aligned}\xi(k+1) &= (F(k)+G(k)\,H(k))\,\xi(k) \\ &\quad +(K(k)+G(k)\,J(k))\,y(k), \quad \xi(0)=\xi_0 \\ \hat{z}(k) &= H(k)\,\xi(k)+J(k)\,y(k) \qquad \text{(A.2)}\end{aligned}$$

As a result, the term $F(k)+G(k)\,H(k)$ takes the place of $F(k)$ in the unbiasedness constraint (4.5) and $K(k)+G(k)\,J(k)$ takes the place of $K(k)$.

If the reduced-order filter dimension is minimized (i.e., $v=n-p$), the $n\times n$ matrix function

$$\begin{bmatrix} T(k) \\ C(k) \end{bmatrix} \qquad \text{(A.3)}$$

is invertible for $k=0,\ldots,N$ because the definition of $T(k)$ in (4.2) ensures that the matrix in (A.3) has full rank. Using the definition of the coefficient matrices in (A.2) in [9], [10], [11], the following result shows that $F(k)+G(k)\,H(k)$, $K(k)+G(k)\,J(k)$, $H(k)$, and $J(k)$ are independent of the design parameters used in the formulation of the $\mathcal{H}_\infty$ filtering problem (2.2).

Lemma A.1: Suppose that the $n\times n$ matrix function in (A.3) is invertible for $k=0,\ldots,N$. Then, the coefficient matrix functions $F(k)+G(k)\,H(k)$, $K(k)+G(k)\,J(k)$, $H(k)$, and $J(k)$ defined in [9], [10], [11] are given by

$$\begin{aligned}F(k)+G(k)\,H(k) &= T(k+1)\,A(k)\,\Gamma_1(k) \\ K(k)+G(k)\,J(k) &= T(k+1)\,A(k)\,\Gamma_2(k) \\ H(k) &= E_1(k)\,\Gamma_1(k) \\ J(k) &= E_1(k)\,\Gamma_2(k) \qquad \text{(A.4)}\end{aligned}$$

for $k=0,\ldots,N$ where $\Gamma_1(k)$ and $\Gamma_2(k)$ are defined as

$$\begin{bmatrix} \Gamma_1(k) & \Gamma_2(k) \end{bmatrix} := \begin{bmatrix} T(k) \\ C(k) \end{bmatrix}^{-1} \qquad \text{(A.5)}$$

Proof: The proof utilizes the unbiasedness constraints (4.5) and the definitions of $\Gamma_1(k)$ and $\Gamma_2(k)$. ■

From the results of Lemma A.1, the realization of the filter in [9], [10], [11] is uniquely determined by the plant data in (2.1) and the matrix function $T(k)$. Therefore, the realization of the filter is independent of the design parameters used in the formulation of the $\mathcal{H}_\infty$ problem (2.2). Finally, if $m=n-p$, the filter realization is completely determined because $T(k)=E_1(k)$ for $k=0,\ldots,N$.

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