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A New Delay-dependent Stability Condition and H_∞ Control for Jump Time-delay Systems

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***Abstract*— This paper studies the problem of the stochastic stability and H_∞ control for the continuous-time time-delay systems with randomly Markovian jumping parameters. A new delay-dependent condition on the stochastic stability is proposed using a new stochastic Lyapunov-Krasovskii functional. The condition is formulated as a set of coupled linear matrix inequalities. Then a new stable H_∞ control design method is developed. At last a numerical example illustrates the effectiveness of the proposed approach, and the conservativeness is compared with the existing results.**

I. Introduction

A class of linear stochastic systems, introduced by Krasovskii and Lidskii in 1961 [13], has been received great attention in the past decades. This family of systems is modelled by a set of linear systems with the transitions between the models determined by a Markov chain taking values in a finite set. Applications of this class of systems may be found in processes including those in production systems and economic problems. Many fundamental properties and control-theoretic problems are reported, see e.g. [14], [12], [9], [8], [1], [7], [2], [3].

The presence of time-delay is very common in practical dynamical systems. Extensive research has been conducted with their stability and stabilization using linear static and dynamic feedback, see, e.g., [10], [5] and the references therein. Since the late 1980s, fairly complete solutions for the H_∞ control of linear time-invariant systems have been obtained. The problem of H_∞ control of linear uncertain systems with time-delay has also gathered much attention of researchers and some sufficient conditions have been presented [15], [11], [6]. When time-delay control systems are subject to the influence of the Markovian jumping parameters, the behavior of the system becomes stochastic. Standard techniques for the design of robust controller of deterministic systems mentioned above are no longer applicable.

In this paper, the stochastic stability and H_∞ disturbance attenuation problem of the systems with time-delay and Markovian jumping parameters are studied by a new stochastic Lyapunov-Krasovskii functional approach. In [4], a delay-independent condition on the stochastic stability was presented for this class of jump time-delay systems. A delay-dependent stability condition is also presented in [3]. In this paper, we will use a new Lyapunov-Krasovskii functional to present a less conservative delay-dependent condition.

The paper is organized as follows. Definitions and preliminary results are given in Section 2 for state-delay systems with Markovian jumping parameters. Sufficient conditions on stochastic stability and the H_∞ disturbance attenuation problem are developed in Sections 3 and 4 by a new Lyapunov-Krasovskii functional involving the derivative variable of the state. A design method on the state feedback controller is also developed by the LMI optimization based approach. A numerical example is presented in Section 5 to show the effectiveness of the result and the comparison with the known results. The paper is concluded in Section 6.

Notations: The following notations will be used throughout the paper. $\mathcal{R}$ denotes the set of real numbers, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, and $\mathcal{R}^{m\times n}$ denotes the set of all $m\times n$ real matrices. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). $\mathcal{C}_{n,\tau} = \mathcal{C}([-\tau, 0], \mathcal{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathcal{R}^n$ with the topology of uniform convergence. The following norms will be used: $||\cdot||$ refers to either the Euclidean vector norm or the induced matrix 2-norm; $||\psi||_c = \sup_{-\tau\leq t\leq 0} ||\psi(t)||$ stands for the norm of a function $\psi \in \mathcal{C}_{n,\tau}$. Moreover, we denote by $\mathcal{C}^v_{n,\tau}$ the set defined by $\mathcal{C}^v_{n,\tau} = \{\psi \in \mathcal{C}_{n,\tau} : ||\psi||_c < v\}$, where v is a positive real number. $\mathbf{E}[\cdot]$ stands for the mathematical expectation.

II. Problem Statement

Consider the class of linear state-delay systems with Markovian jumping parameters

$$\begin{aligned} \dot{x}(t) &= A_1(r(t))x(t) + A_2(r(t))x(t-\tau) \\ &\quad +B_1(r(t))w(t) + B_2(r(t))u(t), \quad (1) \\ z(t) &= C_1(r(t))x(t) + C_2(r(t))x(t-\tau) \\ &\quad +D_1(r(t))w(t) + D_2(r(t))u(t), \quad (2) \\ x(t) &= \psi(t), \quad t \in [-\tau, 0], \quad r(0) = r_0, \quad (3) \end{aligned}$$

where $x(t) \in \mathcal{R}^n$ is the system state, $w(t) \in \mathcal{R}^q$ the exogenous disturbance input which belongs to $\mathcal{L}_2[0,\infty)$, $u(t) \in \mathcal{R}^m$ the control input and $z(t) \in \mathcal{R}^p$ the output to be controlled. $A_1(r(t))$, $A_2(r(t))$, $B_1(r(t))$, $B_2(r(t))$, $C_1(r(t))$, $C_2(r(t))$, $D_1(r(t))$ and $D_2(r(t))$ are matrix functions of

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the random jumping process $\{r(t)\}$. $r(t)$ is a finite state Markov jump process representing the system mode, that is, $r(t)$ takes discrete values in a given finite set $\mathcal{S} = \{1, 2, \dots, s\}$. Let $\Pi = [\pi_{ij}]$, where $i, j = 1, 2, \dots, s$, denote the transition probability matrix with

$$\Pr\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \pi_{ij}\Delta + o(\Delta) & i \neq j \\ 1 + \pi_{ii}\Delta + o(\Delta) & i = j \end{cases}, \tag{4}$$

where $\Delta > 0$, $\pi_{ij} \geq 0$ for $i \neq j$, with

$$\sum_{j=1, j\neq i}^{s} \pi_{ij} = -\pi_{ii}, \tag{5}$$

for each mode $i \in \mathcal{S}$, and $o(\Delta)/\Delta \to 0$ as $\Delta \to 0$. τ is the constant delay time of the state in the system. $\psi(t)$ is a vector-valued initial continuous function defined on the interval $[-\tau, 0]$, and $r_0 \in \mathcal{S}$ are the initial conditions of the continuous state and the mode respectively.

When the system operates in the i-th mode ($r(t) = i$), $A_1(r(t))$, $A_2(r(t))$, $B_1(r(t))$, $B_2(r(t))$, $C_1(r(t))$, $C_2(r(t))$, $D_1(r(t))$ and $D_2(r(t))$ are denoted as A_{1i}, A_{2i}, B_{1i}, B_{2i}, C_{1i}, C_{2i}, D_{1i} and D_{2i} respectively. In the following, we will use $x(t)$ to represent the solution of system (1)–(4) at time t corresponding to the initial conditions $\psi(t)$ and r_0, and x_0 represents $x(t)$ at $t = 0$. First we introduce the definitions of stochastic stability and H_∞ disturbance attenuation performance of the jump time-delay systems [4] with $u(t) \equiv 0$.

Definition 1: The autonomous jump time-delay system (1)–(4) is said to be *stochastically stable* if, when $w(t) \equiv 0$, for all finite $\psi(t) \in \mathcal{R}^n$ defined on $[-\tau, 0]$ and initial mode $r_0 \in \mathcal{S}$, there exists a $\tilde{M} > 0$ satisfying

$$\lim_{T\to\infty} \mathbf{E}\left\{ \int_0^T x^T(t)x(t)dt \,\middle|\, \psi, r_0 \right\} \leq x_0^T \tilde{M} x_0.$$

Definition 2: For a real number $\gamma > 0$, the autonomous jump time-delay system (1)–(4) is said to possess the γ-*disturbance attenuation* property if for all $w \in \mathcal{L}_2[0, \infty)$, $w \neq 0$, the system (1)–(4) is stochastically stable and the response $z : [0, \infty) \to \mathcal{R}^p$ under zero initial condition, *i.e.*, $\psi = 0$, satisfies

$$\mathbf{E}\left\{\int_0^\infty z^T(t)z(t)dt\right\} \leq \gamma^2 \int_0^\infty w^T(t)w(t)dt. \tag{6}$$

Let

$$\begin{aligned} ||w||_2 &\triangleq \left\{\int_0^\infty w^T(t)w(t)dt\right\}^{1/2}, \\ ||z||_2 &\triangleq \left\{\mathbf{E}\int_0^\infty z^T(t)z(t)dt\right\}^{1/2}, \end{aligned}$$

and T_{zw} denote the system from the exogenous input $w(t)$ to the controlled output $z(t)$, then the H_∞-norm of T_{zw} is

$$||T_{zw}||_\infty = \sup_{w(t)\in\mathcal{L}_2(0,\infty)} \frac{||z||_2}{||w||_2}.$$

Hence, (6) implies $||T_{zw}||_\infty \leq \gamma$. In other words, γ-disturbance attenuation implies γ-suboptimal H_∞ control.

III. Stochastic Stability Analysis

In this section, we will establish a delay-dependent stability condition for the autonomous jump time-delay system with $w(t) \equiv 0$ by applying a new Lyapunov-Krasovskii functional. In [4], a delay-independent condition on the stochastic stability was presented for this class of jump time-delay systems. It is well known that the delay-independent result may be very conservative in practice, especially in situations where delays are small. When studying the stability of an industrial process, it is almost always necessary to use a criterion which is delay-dependent.

A. Stability analysis

Theorem 1: The autonomous jump time-delay system (1)–(4) is stochastically stable for $\tau \leq \tau_0$ if, for each mode $i \in \mathcal{S}$, there exist matrices $P_i > 0, Q > 0, S > 0, H_i > 0, N_i$ satisfying the following LMIs

$$\begin{bmatrix} M_{0i} + \tau_0(A_{1i}^T S A_{1i} + H_i) & * \\ A_{2i}^T P_i - N_i^T + \tau_0 A_{2i}^T S A_{1i} & -Q + \tau_0 A_{2i}^T S A_{2i} \end{bmatrix} < 0, \tag{7}$$

$$\begin{bmatrix} H_i & N_i \\ N_i^T & S \end{bmatrix} \geq 0, \tag{8}$$

for $i = 1, \dots, s$, where $*$ represents blocks that are readily inferred by symmetry and

$$M_{0i} \triangleq A_{1i}^T P_i + P_i A_{1i} + \sum_{j=1}^{s} \pi_{ij} P_j + Q + N_i + N_i^T.$$

Proof: Let the mode at time t be i, that is $r(t) = i \in \mathcal{S}$. Then the autonomous jump time-delay system is

$$\begin{aligned} \dot{x}(t) &= A_{1i}x(t) + A_{2i}x(t-\tau), \\ x(t) &= \psi(t), \quad t \in [-\tau, 0], \quad r(0) = r_0. \end{aligned} \tag{9}$$

To simplify the notation in the proof, $x(t)$ is used to denote the solution $x(t, \psi, r_0)$ under the initial condition $\psi(t)$ and r_0.

Since $x(t)$ is continuously differentiable for $t \geq 0$, Leibniz-Newton formula gives

$$x(t-\tau) = x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds$$

for $t \geq \tau$. Then the system is equivalent to:

$$\dot{x}(t) = A_i x(t) - A_{2i}\int_{t-\tau}^{t} \dot{x}(s)ds,$$

where $A_i = A_{1i} + A_{2i}$. Hence, the global uniform asymptotic stability of the above system will ensure the global uniform asymptotic stability of the original time-delay system [10].

In the following, we will use x_t, $\forall t > 0$, to denote the restriction of x to the interval $[t-\tau, t]$ translated to $[-\tau, 0]$, that is, $x_t(\theta) = x(t+\theta), \forall\theta \in [-\tau, 0]$. Noticing that $r(t)$ is a Markov process, it is easy to find that with known initial condition $x_0 = \psi(t)$, $(x_t, r(t))$ is also a Markov process.

Take the stochastic Lyapunov-Krasovskii functional $V(\cdot,\cdot):\mathbb{R}^n\times\mathcal{S}\rightarrow\mathbb{R}_+$ to be

$$V(x_t,i) \triangleq V_1(x_t,i)+V_2(x_t,i), \quad (10)$$
$$V_1(x_t,i) \triangleq x^T(t)P_ix(t), \quad (11)$$
$$V_2(x_t,i) \triangleq \int_{t-\tau}^{t} x^T(s)Qx(s)ds + \int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)S\dot{x}(s)dsd\theta. \quad (12)$$

The weak infinitesimal operator $\mathcal{A}$ of the stochastic process $\{r(t),x(t)\}$, $t\geq 0$, is given by

$$\mathcal{A}V(x_t,r(t)) = \lim_{\Delta\rightarrow 0}\frac{1}{\Delta} \cdot\left[\mathbf{E}\{V(x(t+\Delta),r(t+\Delta))|x_t,r(t)\}-V(x_t,r(t))\right].$$

We have

$$\begin{aligned}\mathcal{A}V_1 &= x^T(t)(A_i^TP_i+P_iA_i+\sum_{j=1}^{s}\pi_{ij}P_j)x(t)\\ &\quad -2x^T(t)P_iA_{2i}\int_{t-\tau}^{t}\dot{x}(s)ds,\\ \mathcal{A}V_2 &= x^T(t)Qx(t)-x^T(t-\tau)Qx(t-\tau)\\ &\quad +\tau\dot{x}^T(t)S\dot{x}(t)-\int_{-\tau}^{0}\dot{x}^T(t+\theta)S\dot{x}(t+\theta)d\theta,\end{aligned}$$

Note that [16]

$$\begin{aligned}&-2x^T(t)P_iA_{2i}\int_{t-\tau}^{t}\dot{x}(s)ds\\ \leq\ &\tau x^T(t)H_ix(t)+2x^T(t)(N_i-P_iA_{2i})\int_{t-\tau}^{t}\dot{x}(s)ds\\ &+\int_{t-\tau}^{t}\dot{x}^T(\theta)S\dot{x}(\theta)d\theta,\end{aligned}$$

for any matrices H_i, N_i and S satisfying (8). Hence

$$\begin{aligned}\mathcal{A}V(x_t,i) &= \mathcal{A}V_1(x_t,i)+\mathcal{A}V_2(x_t,i)\\ &\leq x_e^T(t)M_ix_e(t),\end{aligned}$$

where $x_e^T=\left[\begin{array}{cc} x^T(t) & x^T(t-\tau)\end{array}\right]$ and

$$M_i=\left[\begin{array}{cc} M_{i0}+\tau(A_{1i}^TSA_{1i}+H_i) & * \\ A_{2i}^TP_i-N_i^T+\tau A_{2i}^TSA_{1i} & -Q+\tau A_{2i}^TSA_{2i}\end{array}\right].$$

It is easy to see that $M_i<0$ if LMI (7) holds for any $\tau\leq\tau_0$.

Note that $M_i<0$, we have

$$\mathcal{A}V(x_t,i)\leq-\beta_1x_e^T(t)x_e(t)$$

where $\beta_1=\min_{i\in\mathcal{S}}(\lambda_{\min}(-M_i))>0$. By Dynkin's formula, we have for each $r(t)=i\in\mathcal{S},t>0$

$$\mathbf{E}\left\{V(x_t,i)\right\}-V(\psi,r_0)=\mathbf{E}\left\{\int_0^t\mathcal{A}V(x_s,r(s))ds\right\} \leq-\beta_1\int_0^t\mathbf{E}\left\{x_e^T(s)x_e(s)\right\}ds$$

On the other hand, for each $r(t)=i$, we can show that

$$\begin{aligned}\mathbf{E}\left\{V(x_t,i)\right\} &= \mathbf{E}\left\{V_1(x_t,i)\right\}+\mathbf{E}\left\{V_2(x_t,i)\right\}\\ &\geq \beta_2\mathbf{E}\left\{x_e^T(t)x_e(t)\right\}\end{aligned}$$

where $\beta_2=\min_{i\in\mathcal{S}}(\lambda_{\min}(P_i))>0$. The above two inequalities imply that

$$\begin{aligned}\mathbf{E}\left\{x_e^T(t)x_e(t)\right\} &\leq -\lambda_1\int_0^t\mathbf{E}\left\{x_e^T(s)x_e(s)\right\}ds\\ &\quad +\lambda_2V(x_0,r_0)\end{aligned}$$

where $\lambda_1=\beta_1\beta_2^{-1},\lambda_2=\beta_2^{-1}$. Then, we obtain

$$\mathbf{E}\left\{x_e^T(t)x_e(t)\right\}\leq\lambda_2\exp(-\lambda_1t)V(x_0,r_0)$$

Therefore,

$$\mathbf{E}\left\{\int_0^t x_e^T(s)x_e(s)ds|\psi,r_0\right\} \leq\lambda_1^{-1}\lambda_2[1-\exp(-\lambda_1t)]V(x_0,r_0)$$

Taking limit as $t\rightarrow\infty$, we have

$$\lim_{t\rightarrow\infty}\mathbf{E}\left\{\int_0^t x_e^T(s)x_e(s)ds|\psi,r_0\right\}\leq\lambda_1^{-1}\lambda_2V(x_0,r_0)$$

Noting that there always exists a scalar $c>0$, such that $\lambda_1^{-1}\lambda_2V(x_0,r_0)\leq c\sup_{\tau_0\leq s\leq 0}|\psi(s)|^2$, it then follows that the system is stochastically stable. ■

Based on Theorem 1, one can determine an upper bound of time-delay τ such that the time-delay jump system is stochastic stable by solving a following optimization problem.

It should note that the Lyapunov functional (10) is different to that of [3]. In (10), the derivative variable of the state $\dot{x}$ is involved. We will show that our result is less conservative than that of [3] by a numerical example.

B. State feedback design

In this subsection, we consider the following state feedback controller design

$$u(r(t))=F_1(r(t))x(t)+F_2(r(t))x(t-\tau). \quad (13)$$

The above control law is the general formula of the state feedback. When $F_2=0$, it is just the instantaneous state feedback, while it is the delayed state feedback when $F_1=0$. The closed-loop system can then be written as

$$\begin{aligned}\dot{x}(t)&=\hat{A}_1(r)x(t)+\hat{A}_2(r)x(t-\tau)+B_1(r)w(t),\\ z(t)&=\hat{C}_1(r)x(t)+\hat{C}_2(r)x(t-\tau)+D1(r)w(t),\end{aligned}$$

where

$$\begin{aligned}\hat{A}_j(r(t)) &= A_j(r(t))+B_2(r(t))F_j(r(t)),\\ \hat{C}_j(r(t)) &= C_j(r(t))+D_2(r(t))F_j(r(t)),\ j=1,2.\end{aligned}$$

By Theorem 1, we know that the closed-loop system is stochastically stable for $\tau\leq\tau_0$ if, for each mode $i\in\mathcal{S}$, there exist matrices $P_i > 0,\ Q > 0,\ S > 0,\ H_i > 0,\ N_i$ satisfying the following LMIs

$$\begin{bmatrix} M_{i0} + \tau(\hat{A}_{1i}^T S \hat{A}_{1i} + H_i) & * \\ \hat{A}_{2i}^T P_i - N_i^T + \tau \hat{A}_{2i}^T S \hat{A}_{1i} & -Q + \tau \hat{A}_{2i}^T S \hat{A}_{2i} \end{bmatrix} < 0, \quad (14)$$

$$\begin{bmatrix} H_i & N_i \\ N_i^T & S \end{bmatrix} \geq 0, \quad (15)$$

where

$$M_{0i} \triangleq \hat{A}_{1i}^T P_i + P_i \hat{A}_{1i} + \sum_{j=1}^{s} \pi_{ij} P_j + Q + N_i + N_i^T.$$

It is easy to see that (14) is equivalent to

$$\begin{bmatrix} M_{0i} + \tau H_i & P_i \hat{A}_{2i} - N_i & \tau \hat{A}_{1i}^T \\ \hat{A}_{2i}^T P_i - N_i^T & -Q & \tau \hat{A}_{2i}^T \\ \tau \hat{A}_{1i} & \tau \hat{A}_{2i} & S^{-1} \end{bmatrix} < 0. \quad (16)$$

Define

$$\Delta_i \triangleq X_i A_{1i}^T + A_{1i} X_i + Y_{1i}^T B_{2i}^T + B_{2i} Y_{1i} + \pi_{ii} X_i \quad (17)$$

$$\Xi_i \triangleq \begin{bmatrix} \sqrt{\pi_{i,1}} X_i & \cdots & \sqrt{\pi_{i,i-1}} X_i & \sqrt{\pi_{i,i+1}} X_i & \cdots & \sqrt{\pi_{i,s}} X_i \end{bmatrix} \quad (18)$$

$$\Upsilon_i \triangleq \operatorname{diag} \begin{bmatrix} X_1 & \cdots & X_{i-1} & X_{i+1} & \cdots & X_s \end{bmatrix} \quad (19)$$

and let

$$Y_{1i} = F_{1i} X_i,\ Y_{2i} = F_{2i} X_i,\ \hat{Q} = Q^{-1},$$
$$\hat{N}_i = X_i N_i X_i,\ \hat{H}_i = X_i H_i X_i, \hat{S} = S^{-1}.$$

It is easy to see that the inequalities (14) and (15) are equivalent to

$$\begin{bmatrix} \Delta_i + \hat{N}_i + \hat{N}_i^T + \tau \hat{H}_i & * & * & * & \\ X_i A_{2i}^T + Y_{2i}^T B_{2i}^T - \hat{N}_i^T & T_{22} & * & 0 & 0 \\ \tau(A_{1i} X_i + B_{2i} Y_{1i}) & T_{32} & -\hat{S} & 0 & 0 \\ \Xi_i^T & 0 & 0 & -\Upsilon_i & 0 \\ X_i & 0 & 0 & 0 & -\hat{Q} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \hat{H}_i & \hat{N}_i \\ \hat{N}_i^T & X_i \hat{S}^{-1} X_i \end{bmatrix} \geq 0,$$

respectively, where

$$T_{22} = -X_i \hat{Q}^{-1} X_i,$$
$$T_{32} = \tau(A_{2i} X_i + B_{2i} Y_{2i}).$$

Note that

$$X_i \hat{S}^{-1} X_i \geq 2X_i - \hat{S}, \quad X_i \hat{Q}^{-1} X_i \geq 2X_i - \hat{Q}.$$

Then we have the following theorem.

Theorem 2: There exists a state feedback control (13) such that the jump time-delay system (1)–(4) is stochastically stabilized for $\tau \leq \tau_0$ if, for each mode $i = 1, \ldots, s$, there exist matrices $X_i > 0, \hat{Q} > 0, \hat{S} > 0, \hat{H}_i > 0, \hat{N}_i$ and $Y_{1i},\ Y_{2i}$ satisfying the following LMIs

$$\begin{bmatrix} Z_{11} & * & * & * & * \\ Z_{21} & Z_{22} & * & 0 & 0 \\ Z_{31} & Z_{32} & -\hat{S} & 0 & 0 \\ \Xi_i^T & 0 & 0 & -\Upsilon_i & 0 \\ X_i & 0 & 0 & 0 & -\hat{Q} \end{bmatrix} < 0, \quad (20)$$

$$\begin{bmatrix} \hat{H}_i & \hat{N}_i \\ \hat{N}_i^T & 2X_i - \hat{S} \end{bmatrix} \geq 0, \quad (21)$$

where

$$Z_{11} = \Delta_i + \hat{N}_i + \hat{N}_i^T + \tau_0 \hat{H}_i,$$
$$Z_{21} = X_i A_{2i}^T + Y_{2i}^T B_{2i}^T - \hat{N}_i^T,$$
$$Z_{22} = -\hat{Q} + 2X_i,$$
$$Z_{31} = \tau_0(A_{1i} X_i + B_{2i} Y_{1i}),$$
$$Z_{32} = \tau_0(A_{2i} X_i + B_{2i} Y_{2i}).$$

Then the feedback control law can be obtained by $F_{1i} = Y_{1i} X_i^{-1}$ and $F_{2i} = Y_{2i} X_i^{-1}$.

IV. H_∞ Disturbance Attenuation Analysis

In this section, we analyze the H_∞ disturbance attenuation performance of the jumped time-delay systems.

Theorem 3: For the autonomous jump time-delay system (1)–(4) and a given disturbance attenuation level γ, it possesses the γ-disturbance attenuation property for $0 \leq \tau \leq \tau_0$, for all $w \in \mathcal{L}_2[0, \infty),\ w \neq 0$, if for each mode $i \in \mathcal{S}$, there exist matrices $P_i > 0, Q > 0, S > 0, H_i > 0$, and N_i satisfying the coupled LMIs (8) and

$$\Theta_i \triangleq \begin{bmatrix} M_{i0} + T_{11} & * & * \\ T_{21} & T_{22} & * \\ B_i^T P_i + D_{1i}^T C_{1i} & D_{1i}^T C_{2i} & T_{33} \end{bmatrix} < 0, \quad (22)$$

for $i = 1, \ldots, s$, where

$$T_{11} = \tau_0(A_{1i}^T S A_{1i} + H_i) + C_{1i}^T C_{1i},$$
$$T_{21} = A_{2i}^T P_i - N_i^T + \tau_0 A_{2i}^T S A_{1i} + C_{2i}^T C_{1i},$$
$$T_{22} = -Q + \tau_0 A_{2i}^T S A_{2i} + C_{2i}^T C_{2i},$$
$$T_{33} = -\gamma^2 I + D_{1i}^T D_{1i}.$$

Proof: Let the mode at time t be i, that is $r(t) = i \in \mathcal{S}$. The the autonomous system can be rewritten as

$$\dot{x}(t) = A_{1i} x(t) + A_{2i} x(t - \tau) + B_{1i} w(t), \quad (23)$$
$$z(t) = C_{1i} x(t) + C_{2i} x(t - \tau) + D_{1i} w(t). \quad (24)$$

It is equivalent to the following system:

$$\dot{x}(t) = A_i x(t) + B_{1i} w(t) - A_{2i} \int_{t-\tau}^{t} \dot{x}(s) ds,$$
$$z(t) = C_{1i} x(t) + C_{2i} x(t - \tau) + D_{1i} w(t),$$
$$x(t) = 0, \quad t \leq 0.$$

It can be easily seen that inequality (22) implies inequality (7). Therefore it follows from Theorem 1 that the autonomous jump system with $w(t) = 0$ is stochastically stable.

Choose the stochastic Lyapunov functional $\hat{V}(\cdot,\cdot) : \mathcal{R}^n \times \mathcal{S} \to \mathcal{R}_+$ as in (10), (11) and (12). Then we have

$$\mathcal{A}V(x_t, i) \le x_e^T(t) M_i x_e(t) + 2x^T(t) P_i B_i w(t),$$

In the following, we assume zero initial condition, $x(t) = 0$ for $t \in [-\tau, 0]$, and define

$$J_T \triangleq \mathbf{E}\left\{\int_0^\infty [z^T(t)z(t) - \gamma^2 w^T(t)w(t)]dt\right\}.$$

From Dynkin's formula [14] and the fact that $x_{t=0} = 0$, we have

$$\mathbf{E}\{\hat{V}(x_{t=\infty}, r(\infty))\} = \mathbf{E}\left\{\int_0^\infty \mathcal{A}\hat{V}(x_s, r(s))ds\right\},$$

since $V(x_{t=0}, r_0) = 0$. On the other hand, since $w(t) \in \mathcal{L}_2[0,\ \infty)$, the stability of the system for $0 \le \tau \le \tau_0$, implies the boundedness of $||z||_2$ and that $x(t)$ tends to zero as $t \to \infty$. Then, for any nonzero $w(t) \in \mathcal{L}_2[0,\ \infty)$,

$$J_T = \mathbf{E}\{\int_0^\infty [z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \mathcal{A}\hat{V}(x_t, r]dt\} \\ -\mathbf{E}\{\hat{V}(x_{t=\infty}, r(\infty))\}.$$

So

$$J_T \le \mathbf{E}\{\int_0^\infty [z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \mathcal{A}\hat{V}(x_t, r)]dt\} \\ = \int_0^\infty [\sigma^T(t)\Theta_i\sigma(t)]dt,$$

where

$$\sigma(t) \triangleq \begin{bmatrix} x^T(t) & x^T(t-\tau) & w^T(t) \end{bmatrix}^T.$$

Note that $\Theta_i < 0$ if LMI (22) holds. Therefore, $J_T < 0$ and the dissipativity inequality (6) holds for $t > 0$ if LMI (22) and (8) hold. In other words, we have $z \in \mathcal{L}_2[0, \infty)$, for any nonzero $w \in \mathcal{L}_2[0, \infty)$, and $||z||_2 < \gamma||w||_2$. ■

Remark 1: Similarly, we can determine an upper bound for τ such that the closed-loop system possesses a given γ-disturbance attenuation property by solving a corresponding optimization problem when the disturbance attenuation level γ is given. Also, if an upper bound τ_0 of τ is known, we can determine a lower bound of the disturbance attenuation level by solving the following optimization problem

$$\begin{array}{rl} \textbf{Op:} & \min \gamma^2 \\ \text{subject to} & X_i > 0, Q > 0, S > 0, H_i > 0, \\ & \text{and LMIs } (22), (8). \end{array}$$

In [4], the authors presented delay-independent conditions on the stochastic stabilizability and H_∞ γ-disturbance attenuation. The results there are applicable to those situations where *a priori* knowledge of delay time is not available. However, they may be very conservative in practice, especially in situations where delays are small. The above theorems can be easily extended to γ-suboptimal H_∞ control of jump linear systems without delay. In [8], the suboptimal H_∞ control problem was addressed based on a set of coupled algebraic Riccati equations for a special class of jump linear systems without delay. However, no solution method for these coupled equations was presented. In [3], by a similar Lyapunov-Krasovskii functional as in [6], a delay-dependent stochastic stability condition is proposed. We will show in the next section that our new conditions are less conservative than that of [3].

Theorem 4: There exists a state feedback control (13) such that the jump time-delay system (1)–(4) possesses the γ-disturbance attenuation property for all $\tau \le \tau_0$ if, for each mode $i = 1, \ldots, s$, there exist matrices $X_i > 0, \hat{Q} > 0, \hat{S} > 0, \hat{H}_i > 0, \hat{N}_i$ and $Y_{1i},\ Y_{2i}$ satisfying LMIs (21) and

$$\begin{bmatrix} Z_{11} & * & * & \Xi_i & X_i & B_{1i} & * \\ Z_{21} & Z_{22} & * & 0 & 0 & 0 & * \\ Z_{31} & Z_{32} & -\hat{S} & 0 & 0 & 0 & 0 \\ \Xi_i^T & 0 & 0 & -\Upsilon_i & 0 & 0 & 0 \\ X_i & 0 & 0 & 0 & -\hat{Q} & 0 & 0 \\ B_{1i}^T & 0 & 0 & 0 & 0 & -\gamma^2 I & D_{1i}^T \\ Z_{61} & Z_{62} & 0 & 0 & 0 & D_{1i} & -I \end{bmatrix} < 0,$$

where

$$\begin{aligned} Z_{61} &= C_{1i}X_i + D_{2i}Y_{1i} \\ Z_{62} &= C_{2i}X_i + D_{2i}Y_{2i}. \end{aligned}$$

Then the feedback control law can be obtained by $F_{1i} = Y_{1i}X_i^{-1}$ and $F_{2i} = Y_{2i}X_i^{-1}$.

V. Numerical Example

In this section, we present a simple example to illustrate the usefulness of the proposed results. We borrow the example with two modes from [2]. The dynamics in each mode is described as follows

$$\begin{aligned} A_{11} &= \begin{bmatrix} 0.5 & -1 \\ 0 & -3 \end{bmatrix},\ A_{21} = \begin{bmatrix} 0.5 & -0.2 \\ 0.2 & 0.3 \end{bmatrix}, \\ A_{12} &= \begin{bmatrix} -5 & 1 \\ 1 & 0.2 \end{bmatrix},\ A_{22} = \begin{bmatrix} -0.3 & 0.5 \\ 0.4 & -0.5 \end{bmatrix}. \end{aligned}$$

The initial condition is assumed to be $x(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $r(t) = 1$ for $-\tau \le t \le 0$. The generator matrix of the stochastic process $r(t)$ is

$$\Pi = \begin{bmatrix} -\pi_1 & \pi_1 \\ \pi_2 & -\pi_2 \end{bmatrix}.$$

When $\pi_1 = 7$ and $\pi_2 \le 6$, the result of [2] cannot judge the stability. When $\pi_2 = 6$, the system is found to be delay-independent stable by the result of [4]. However, when $\pi_2 = 3$, the optimization algorithm of [4] is infeasible, which means the stability of the system cannot be judged. Fortunately, by Theorem 1, we can obtain a feasible solution when $\tau_0 \le 1.23$, and hence we can conclude on the stability of the system when $\tau_0 \le 1.23$. Note that the upper bound of time-delay which can be obtained by the theory of [3] is $\tau_0 \le 0.84$. This implies our result is less conservative than that of [3]. The simulation also shows the system is stable when $\tau = 1.23$. Figure 1 gives the state and mode trajectories.

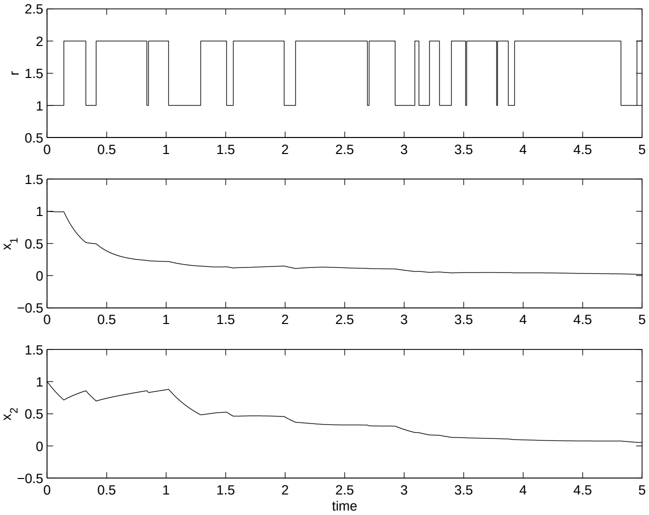

Fig. 1. Mode and state trajectories of the example.

VI. Conclusion

In this paper, we proposed a new Lyapunov-Krasovskii functional for the stochastic stability analysis and H_∞ control design problem of the jump systems with time-delay. A new delay-dependent sufficient condition on the stochastic stability and γ-disturbance attenuation was presented based on the stochastic Lyapunov-Krasovskii stability approach. A design method on the general state feedback controller involving instantaneous state feedback and delayed state feedback was also developed by the LMI optimization method. The problem of determining the upper bound of the time-delay such that the jump system is stochastically stable and with a given γ-disturbance attenuation performance is formulated to an LMI optimization problem. A numerical example is also presented to illustrate the effectiveness of the proposed method and compare the conservativeness with the known results.

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