Computation of the Nonlinear Nyquist Robust Stability Margin for Structured Uncertainties

B. J. Remark and O. D. Crisalle

Department of Chemical Engineering
University of Florida, Gainesville, FL 32611-6005

crisalle@che.ufl.edu

ABSTRACT

The critical direction theory for analyzing the robustness of uncertain feedback systems is extended to include nonlinear elements in the closed loop, making the approach applicable to a wider class of systems. A redefinition of the critical perturbation radius is introduced, preserving key properties of the original theory. Necessary and sufficient conditions for robust stability are developed, and the case of real affine parametric uncertainties is treated in detail. An example is presented to illustrate the theory.

1.0 INTRODUCTION

Much attention has been given to the problem of computing a robust stability margin for systems with frequency-domain uncertainty descriptions, including the structured singular value μ (Doyle, 1982) and the multivariable stability margin k_m (Safonov, 1982). The *critical direction theory* developed by Latchman and Crisalle (1995) and by Latchman *et al*. (1997, 2001) introduced the *Nyquist robust stability margin* k_N, a scalar measure of robustness that has been effectively computed for single-input/single-output (SISO) systems with both convex (Latchman *et al*., 1997) and non-convex (Baab *et al*., 2001) critical value sets. This paper extends the critical direction theory to a class of closed-loop nonlinear systems whose stability properties can be analyzed using the *describing function* approach.

2.0 GENERALIZATION OF THE CRITICAL DIRECTION THEORY

2.1 Preliminaries

Consider the SISO linear time-invariant system

$$g(s) = g_o(s) + \delta(s) \tag{1}$$

where $g_o(s)$ is a known nominal transfer function and $\delta(s) \in \Delta$ is an unknown perturbation belonging to an unstructured uncertainty family

$$\Delta := \left\{ \delta(s) \ : \ |\delta(j\omega)| \le r(\omega), r(\omega) > 0 \right\}$$

characterized through a known radius function $r(\omega)$.

The uncertain system (1) is arranged in a negative feedback configuration that features a nonlinear element $f(e)$, as shown in Fig. 1. For simplicity of exposition, the figure introduces a slight abuse of notation given that the map $f(e): \Re \to \Re$ is a time-domain operator, while $g(s)$ is a Laplace-domain operator.

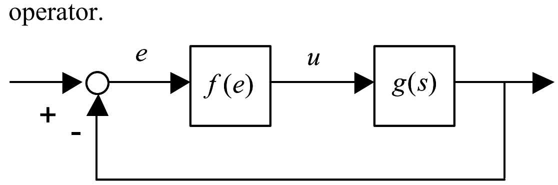

Figure 1. Closed-loop configuration under negative feedback of a nonlinear operator $f(e)$ and an uncertain linear plant $g(s)$.

The robust stability analysis proposed here involves three assumptions: (*i*) the nominal transfer function is stable under unity negative feedback, (*ii*) the nominal transfer function $g_o(s)$ and the uncertain system $g(s)$ have the same number of unstable poles, and (*iii*) the

0-7803-8335-4/04/$17.00

higher harmonics of the nonlinear element $f(e)$ can be neglected.

2.2 Brief review of the critical direction theory

The critical direction theory was initially developed for linear systems, namely, for the special case of Fig. 1 with $f(e) = 1$. Figure 2 shows a typical Nyquist diagram showing the nominal frequency response loci for $g_o(j\omega)$. The figure also shows the point representing the nominal frequency response $g_o(j\omega_1)$ at a specific frequency ω_1, along with its associated circular uncertainty value set $V(\omega_1)$ of radius $r(\omega_1)$. For the specific case of circular value sets, the critical perturbation radius $\rho_c(\omega)$ is defined as the distance from the nominal point $g_o(j\omega)$ to the boundary of the uncertain circular value set, *i.e.*

$$\rho_c(\omega) = r(\omega) \tag{2}$$

The main result of the critical direction theory for linear systems is summarized in Theorem 1.

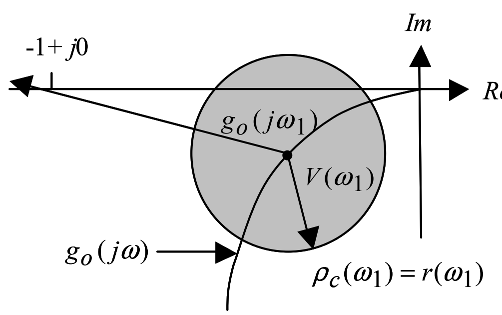

Figure 2. Critical direction theory illustrating the location on the Nyquist plane of nominal point $g_o(j\omega_1)$ along its associated circular value set $V(\omega_1)$ (shaded area) and critical perturbation radius $\rho_c(\omega_1)$ at a frequency ω_1.

Theorem 1. *Let $g_o(j\omega)$ be a nominal SISO system with an associated circular uncertainty set Δ. Then the closed loop system given in Fig.* 1 *with $f(e) = 1$ is robustly stable with respect to uncertainties $\delta(s) \in \Delta$ if and only if*

$$\frac{\rho_c(\omega)}{|1 + g_o(j\omega)|} < 1 \quad \forall \omega$$

The proof is omitted for brevity. Details are given in Latchman *et al.* (1997).

The *Nyquist robust-stability margin* is then defined as

$$k_N(\omega) := \frac{\rho_c(\omega)}{|1 + g_o(j\omega)|} \tag{3}$$

Let $k_N := \max k_N(\omega)$, where the maximization is carried out over all frequencies ω. From Theorem 1 it is then obvious that a necessary and sufficient condition for robust stability is given by the inequality

$$k_N < 1 \tag{4}$$

The calculation of the critical perturbation radius is often the most challenging problem to address when applying the critical direction theory to extract numerical values for the Nyquist robust stability margin (3); however, for the case of circular value sets, the task is trivial given the relationship (2).

2.3 Brief review of the describing function method

Consider now the nonlinear feedback loop shown in Fig. 1 with $f(e) \neq 1$. The analysis of closed loop stability via the *describing-function method* hinges on determining whether the limiting-stability condition of sustained signal oscillation can occur (Khalil, 1992). In that limiting case, the feedback error $e(t)$ is sinusoidal, and it can be written as $e(t) = a \sin \omega t$, where $a > 0$ is the signal amplitude. Using a Fourier series expansion, the output from the nonlinear function $f(e)$ can be written in the form

$$f(a \sin \omega t) = a_o' + \sum_{k=1}^{\infty} (a_k' \cos k\omega t + b_k' \sin k\omega t)$$

which features a mean level a_o', a set of fundamental components of amplitudes a_1' and b_1', and higher harmonic components of amplitudes a_k' and b_k' at frequencies $k\omega$, $k = 2, 3, \cdots, \infty$ (Khalil, 1992).

Assuming that all the higher harmonics can be neglected, the nonlinear function $f(e)$ can be approximated by its *describing function*

$$n(a) := \frac{a_1 + j\, b_1}{a} \tag{5}$$

also referred to as the *equivalent gain* ,where

$$a_1(\omega) = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x \, dx$$

and

$$b_1(\omega) = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin x \, dx$$

Hence, the describing function is a complex number

.

that depends on the amplitude of the sinusoid, and it can be represented as a loci of points on the complex plane. It is useful to note that $|n(a)| \neq 0$ for all amplitudes $a > 0$, which implies that the complex number $1/n(a)$ is well defined at those amplitudes.

The validity of the method relies on the assumption that the higher harmonics produced by the nonlinear function $f(e)$ can be discarded (Leigh, 1983). This is ensured in all cases where the transfer function $g(s)$ behaves as a low-pass filter.

3.0 CRITICAL DIRECTION THEORY FOR NONLINEAR SYSTEMS

In this section the critical direction theory is extended to nonlinear systems. The concepts of critical direction, critical perturbation radius, and critical point are appropriately redefined to take into account the nonlinearity of the system.

We restrict the analysis to cases where the higher harmonics are negligible, so that the describing function method is valid. After substituting the nonlinear element $f(e)$ by its describing function $n(a)$, the characteristic equation for the control configuration of Fig. 1 is $1 + n(a)g(s) = 0$. Rearranging terms yields the equivalent equality

$$g(s) = -\frac{1}{n(a)} \tag{6}$$

In the suite we refer to the Nyquist map of $-1/n(a)$ for $a > 0$ as the *critical loci*. From the above equation it can be concluded that the robust stability of the closed loop is ensured if the value sets of $g(j\omega)$ exclude all points belonging to the critical loci. Under the assumptions established earlier, this *critical loci exclusion principle* is in fact a necessary and sufficient condition for robust closed loop stability. The following definitions introduce key concepts that are useful for formulating a nonlinear critical direction theory for systems with a circular uncertainty description (*cf.* Fig. 3 for a graphical interpretation):

(1) The *critical directions*

$$d_c(j\omega, a) = -\frac{g_o(j\omega) + \dfrac{1}{n(a)}}{\left|g_o(j\omega) + \dfrac{1}{n(a)}\right|} \tag{7}$$

are interpreted as a set of unit vectors originating at $g_o(j\omega)$ and pointing towards the critical loci for all scalars $a > 0$.

.

(2) The *critical perturbation radii*

$$\rho_c(\omega, a) = \max_{\alpha \in \Re_+} \{ \alpha : z = g_o(j\omega) + \alpha d_c(\omega, a) \in V(\omega) \} \tag{8}$$

Equation (8) states that the critical perturbation radius for a convex value set $V(\omega)$ is the distance from the nominal point $g_o(\omega)$ to the value-set boundary $\partial V(\omega)$ along the critical direction $d_c(\omega, a)$. At each finite amplitude *a* let the *critical value-set* $V_c(\omega, a)$ be defined as the subset of points in $V(\omega)$ that lie along the ray $z = g_o(j\omega) + \alpha d_c(\omega, a)$, $\alpha > 0$. Equation (8), however, is not suitable for non-convex critical value sets $V_c(\omega, a)$. Define the set of critical boundary intersections $\beta_c(\omega, a) := \{V_c(\omega, a) \cap \partial V(\omega)\} - \{g_o(j\omega)\}$ as the subset of points in $V_c(\omega, a)$ that are also elements of the value-set boundary $\partial V(\omega)$, excluding the nominal point $g_o(j\omega)$. The following generalized definition of the critical perturbation radius, applicable to both the convex and non-convex cases, is proposed in a fashion analogous to that presented in Baab *et al.* (2001):

$$\rho_c(\omega, a) := \begin{cases} \left|\dfrac{1}{n(a)} + g_o(j\omega)\right| - \xi(\omega) & \text{if } -\dfrac{1}{n(a)} \notin V(\omega) \\ \left|\dfrac{1}{n(a)} + g_o(j\omega)\right| + \xi(\omega) & \text{otherwise} \end{cases} \tag{9}$$

where

$$\xi(\omega) = \min_{z \in \beta_c(\omega, a)} \left| \frac{1}{n(a)} + z \right| \tag{10}$$

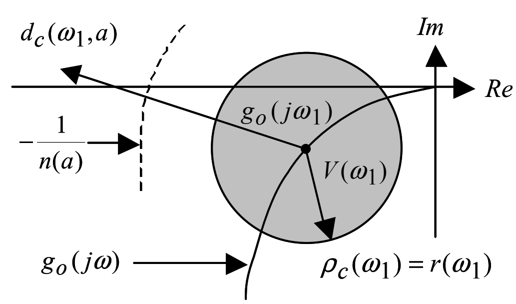

Figure 3. A circular value set at a frequency ω_1 (shaded region), the critical perturbation radius $\rho_c(\omega_1)$, and the critical loci (dashed line).

Equation (10) represents the closest distance from the critical loci $-1/n(a)$ to a point in $\beta_c(\omega, a)$. The upper statement in Eq. (9) states that when the critical loci is not an element of the uncertainty value set $V(\omega)$, the critical perturbation radius $\rho_c(\omega,a)$ is defined as the difference between two distances, namely, the distance from the critical loci to the nominal point (represented by $|(1/n(a))+g_o(j\omega)|$) and the distance from the critical loci to the closest critical-boundary intersection (represented by $\xi(\omega)$). Alternatively, when the critical loci is an element of the uncertainty value set, the lower statement in Eq. (9) states that the critical perturbation is the sum of the two respective distances. Note that when the critical value set is convex, Eq. (9) reduces to Eq. (8) because in that case there is only one critical boundary intersection. To calculate the critical perturbation radius (9) it is necessary to have full knowledge of the critical boundary intersections $\beta_c(\omega,a)$ and to evaluate whether the set membership condition $-1/n(a)\in V(\omega)$ holds. It should be noted that $\rho_c(\omega,a)\geq 0$ for all frequencies.

At a given frequency ω_1, the *set of critical-lines* is defined as the set of oriented lines with origin at the nominal point $g_o(j\omega_1)$, and that pass through any point in the critical loci $-1/n(a)$. The critical perturbation radius $\rho_c(\omega_1,a)$ is interpreted as the distance from the nominal point to the point where the critical line defined for an amplitude a intersects the uncertain value set. Finally, the *Nyquist robust stability margin* for nonlinear systems is defined as

$$k_N(\omega) := \max_{a>0} \frac{\rho_c(\omega,a)}{\left|\frac{1}{n(a)}+g_o(j\omega)\right|} \qquad (11)$$

The ensuing robust-stability theorems for nonlinear systems use the assumptions (*i*)-(*iii*) invoked previously.

Theorem 2. *Let $g_o(s)$ be a nominal system with an associated uncertainty set Δ. Then the closed loop system given in Figure* 1 *is robustly stable with respect to all uncertainties $\delta(s)\in\Delta$ if and only if*

$$k_N(\omega)<1 \ \ \forall\omega\geq 0. \qquad (12)$$

A complete proof is found in Latchman and Crisalle (1997) for the case where $V_c(\omega)$ is convex. The proof given here is for the non-convex case, and it takes advantage of the general definition (9). From the critical-loci exclusion principle it follows that the uncertain closed loop is stable if and only if $-1/n(a)\notin V(\omega)\ \ \forall\omega$. Therefore, to prove that (12) is sufficient for robust stability, it must be shown that $k_N(\omega)<1\ \ \forall\omega$ implies that $-1/n(a)\notin V(\omega)\ \ \forall\omega$. To prove this by contradiction first assume that $k_N(\omega)<1\ \ \forall\omega$ and that $\exists\omega$ such that $-1/n(a)\in V(\omega)$. Applying definitions (9) and (11) for a frequency at which $-1/n(a)\in V(\omega)$ yields

$$k_N(\omega)=\frac{\rho_c(\omega)}{\left|\frac{1}{n(a)}+g_o(j\omega)\right|}=1+\frac{\xi(\omega)}{\left|\frac{1}{n(a)}+g_o(j\omega)\right|}$$

where from Eq. (10) the real scalar $\xi(\omega)$ is non-negative. Therefore $k_N(\omega)\geq 1$ for at least one frequency, which contradicts the assumption. Thus, if $k_N(\omega)<1\ \ \forall\omega$ it must follow that $-1/n(a)\notin V(\omega)\ \ \forall\omega$. To prove that (12) is necessary for robust stability it must be shown that $-1/n(a)\notin V(\omega)\ \ \forall\omega$ implies that $k_N(\omega)<1\ \ \forall\omega$. Applying definitions (9) and (11) when $-1/n(a)\notin V(\omega)$ yields

$$k_N(\omega)=\frac{\rho_c(\omega)}{\left|\frac{1}{n(a)}+g_o(j\omega)\right|}=1-\frac{\xi(\omega)}{\left|\frac{1}{n(a)}+g_o(j\omega)\right|} \quad \forall\omega$$

where $\xi(\omega)$ is given by (10). Because $-1/n(a)\notin V(\omega)$ by assumption, it follows that $-1/n(a)\notin\beta_c(\omega,a)$, and thus $\xi(\omega)$ must be strictly positive. This fact is now used in the above equation to lead to the conclusion that $k_N(\omega)<1\ \ \forall\omega$ and that Eq. (12) is a necessary and sufficient condition for robust stability. *Q.E.D.*

It should be noted that the Nyquist robust stability margin must be less than unity at all nonnegative frequencies and positive amplitudes to guarantee stability; therefore, $k_N(\omega)$ must be maximized over all frequencies in order to assess robust stability. Let

$$k_N := \max_{\omega\geq 0} k_N(\omega) \qquad (13)$$

Then, from Theorem 2 it follows that a necessary and sufficient condition for the robust stability of the uncertain nonlinear closed loop is that $k_N<1$.

Note that the computation of $k_N(\omega)$ requires knowledge of the critical perturbation radius (9). The challenging task in this problem is to calculate the critical perturbation radius.

Also note that when $V_c(\omega,a)$ is convex, Eq. (8) indicates that $\rho_c(\omega,a)$ is associated with the unique point where the critical line intersects the boundary of the uncertain value set $V(\omega)$. However, when $V_c(\omega,a)$ is non-convex, there are multiple points where the critical line intersects $V(\omega)$. Then Eq. (9) states that $\rho_c(\omega)$ is a function of the distance between $g_o(j\omega)$ and the boundary-intersection point that is closest to the critical loci.

.

4.0 REAL AFFINE PARAMETRIC UNCERTAINTY

The generalized critical direction theory developed in Section 3 is specialized to systems with real affine parametric uncertainties of the form

$$g(s,\mathbf{q}) = \frac{n_0(s) + \sum_{i=1}^{p} q_i\ n_i(s)}{d_0(s) + \sum_{i=1}^{p} q_i\ d_i(s)}, \quad \mathbf{q} \in Q \qquad (14)$$

where $n_0(s) := \sum_{k=0}^{\ell} n_{0k}\ s^k$ and $d_0(s) := \sum_{k=0}^{m} d_{0k}\ s^k$ are known nominal polynomials;

$$n_i(s) = \sum_{k=0}^{\ell} n_{ik}\ s^k$$

and

$$d_i(s) = \sum_{k=0}^{m} d_{ik}\ s^k$$

are known perturbation polynomials; and $\mathbf{q} = [q_1\ q_2\ \ldots\ q_p]^{\mathrm{T}} \in R^p$ is a vector of real perturbation parameters belonging to the bounded rectangular polytope

$$Q = \left\{ \mathbf{q} \in R^p : q_i^- \le q_i \le q_i^+, i = 1, 2, \ldots, p \right\} \quad (15)$$

where q_i^- and q_i^+, i = 1, 2, …, p, are finite real bounds. Equations (14) and (15) define a class of finite dimensional, linear, time-invariant, real plants with affine uncertainties.

The computation of the Nyquist robust stability margin $k_N(\omega)$ for the affine system (14)-(15) requires that the critical perturbation radius $\rho_c(\omega,a)$ be calculated first. As illustrated by (9) and (10), this in turn requires determining the set $\beta_c(\omega,a)$. This task is easily accomplished because, for the case of affine-uncertain systems of the form given by (14), Baab *et al.* (2001) have developed a two-step strategy that effectively determines all the elements of the set $\beta_c(\omega,a)$. Details are omitted here for brevity

5.0 EXAMPLE

Consider the system with affine uncertainty structure

$$g(s,\mathbf{q}) = \frac{n(s)}{d(s)}$$

where

$$n(s) = (0.6s^3 + 2.9s^2 + 7.5s + 15) + (0.12s^2 + 0.7s + 1)q_1 + (0.06s^2 + 0.2s)q_2 + (-0.3s - 1)q_3$$

$$d(s) = (2s^4 + 11s^3 + 30s^2 + 20s + 0.2) + (0.5s^3 + 2s^2 - s)q_1 + (-0.5s^3 + s^2)q_2 + (0.5s^3 + s)q_3$$

and where the parameter uncertainty vector belongs to the rectangular polytope

$$Q = \{ \mathbf{q} \in R^3 : -10 \le q_1 \le 10, \quad -0.3 \le q_2 \le 0.3, \quad -0.3 \le q_3 \le 0.3 \}$$

For illustrative purposes, the describing function considered in this example is

$$n(a) = 7 + j\frac{4}{\pi a}$$

The critical loci $-1/n(a)$ is plotted in Fig. 4. The figure also shows a value set $V(\omega)$ for the uncertain affine system at frequency $\omega = 1.6$. The set of arcs of circles and straight-line segments shown belong to the value set, and the value-set boundary $\partial V(\omega)$ is readily determined from a subset of those segments. It is straightforward to verify that the system has a non-convex critical uncertainty value set $V_c(\omega,a)$ at the frequency $\omega = 1.6$ and at an amplitude a = 0.11, as can be determined by inspection: the straight-line segment joining the nominal point with the point $-1/n(a)$ at a = 0.11 defines two disjoint segments as it intersects the value set. The set $V_c(\omega,a)$ at the amplitude in question is the union of those two disjoint segments.

The critical perturbation radius at $\omega = 1.6$ can be calculated from the upper statement of Eq. (9) because from an inspection of Figure 4 it can be readily determined that $-1/n(a) \notin V(\omega)$ for all possible amplitudes a. Such graphical analysis, however, is not practical when carrying out extensive calculations at all frequencies of interest.. It is possible, however, to avoid such graphical inspections using a procedure proposed by Baab *et al.* (2001) where the problem of determining whether $-1/n(a) \notin V(\omega)$ is shown to be equivalent to solving a linear feasibility problem that can be resolved in a numerically efficient fashion..

The critical perturbation radius $\rho_c(\omega)$ and Nyquist robust stability margin $k_N(\omega)$ can now be calculated for any frequency using the method described in Section 4 and the techniques proposed in Baab *et al.* (2001). Invoking Eq. (9) it is readily determined that at the frequency $\omega = 1.6$ the Nyquist robust stability margin has the value

$$k_N(1.6) = 0.7698$$

The result that $k_N(1.6) < 1$ is consistent with the

.

graphical observation that the value set does not include the critical loci at this frequency.

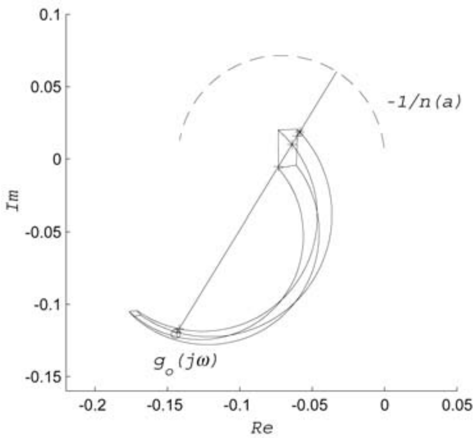

Figure 4. Critical loci $-1/n(a)$ and value set for the example system at the frequency $\omega = 1.6$. A straight-line segment is shown joining the nominal point $g_o(j\omega)$ with the critical loci point $-1/n(a)$ defined at the amplitude $a = 0.11$.

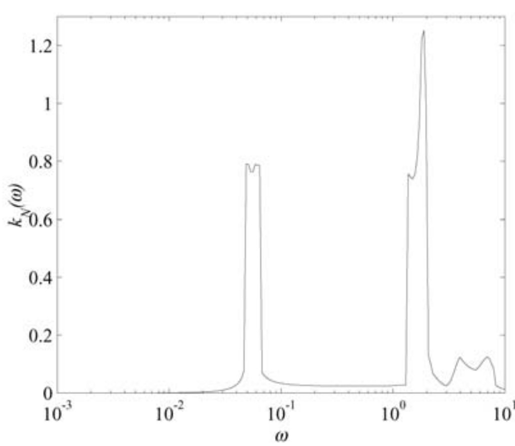

Figure 5. Nyquist robust stability margin versus the frequency for real, affine parametric uncertainties

Figure 5 illustrates the values of $k_N(\omega)$ calculated for a sequence of 200 frequency points equally spaced in a logarithmic scale in the range [0.001, 10]. From Fig. 5 it is readily concluded that at some frequencies $k_N(\omega) > 1$, making the closed loop unstable with respect to the given uncertainty by virtue of Theorem 2.

6.0 CONCLUSIONS

The theory presented for analyzing the robust stability of a certain class of nonlinear closed loops affected with unstructured uncertainty is complete, systematic, and numerically tractable. Extensions of the approach to other kinds of uncertainty descriptions are possible; however the usefulness of the results depends on whether the critical perturbation radius can be computed precisely and efficiently.

7.0 REFERENCES

Baab, C. T., Cockburn, J. C., Latchman, H. A., and Crisalle, O. D. (2001). "Extension of the Nyquist robust stability margin to systems with nonconvex values sets," *Proceedings of the American Control Conference*, Washington D.C, pp. 1414-1419.

Bhattacharya, S., L.H. Keel, and S.P. Bhattacharya (1993). "Robust stabilizer synthesis for interval plants using *H*-infinity methods," *Proceedings of Conference on Decision and Control*, San Antonio, Texas, pp. 3003-3008.

Doyle, J.C. (1982). "Analysis of feedback systems with structured uncertainties," *IEE Proceedings Part D*, vol. 129, pp. 242-250.

Doyle, J.C., B.A. Francis and A.R. Tannenbaum (1992). *Feedback Control Theory*, Macmillan Publishing Company, N.Y.

Guzzella, L,, O.D. Crisalle, F. Kraus and D. Bonvin (1991). "Necessary and sufficient conditions for the robust stabilizing control of linear plants with ellipsoidal parametric uncertainties," *Proceeding of Conference on Decision and Control*, Brighton, U.K., pp. 2948-2953.

Khalil, H.K. (1992). *Nonlinear Systems*, Macmillan Publishing Company, New York.

Latchman, H. A. and Crisalle, O. D. (1995). "Exact robustness analysis for highly structured frequency-domain uncertainties," *Proceedings of the American Control Conference*, Seattle, WA, pp. 3982-3987.

Latchman, H. A., Crisalle, O. D., and Basker, V. R. (1997). "The Nyquist robust stability margin – A new metric for robust stability," *International Journal of Robust and Nonlinear Control*, Vol. 7, pp. 211-226.

Leigh, J.R. (1983). *Essentials of Nonlinear Control Theory*, Peter Peregrinus Ltd., London, U.K.

Mohler, R.R. (1991), *Nonlinear Systems Volume I: Dynamics and Control*, Prentice Hall, Englewood Cliffs, New Jersey.

Safonov, M. G. (1982). "Stability margins of diagonally perturbed multivariable feedback systems," *IEE Proceedings Part D*, 129, pp. 251-256.

.