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Global Adaptive Control of Feedforward Systems Using Dynamic High Gain Scaling †

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Abstract

In this paper, we propose an adaptive control design technique for feedforward systems based on our recent results on dynamic high-gain scaling techniques for controller design for strict-feedback-type systems. Both the state-feedback and the output-feedback cases are considered. The system is allowed to contain uncertain functions of all the states even in the output-feedback case. Unknown parameters are allowed in the bounds assumed on the uncertain functions appearing in the dynamics. The designed controllers have a very simple structure being essentially a linear feedback with state-dependent dynamic gains and do not involve any saturations or recursive computations. The observer in the output-feedback case is similar to a Luenberger observer with dynamic observer gains. The Lyapunov functions are quadratic in the states and the parameter estimation errors (and the observer errors in the case of output-feedback). The stability analysis is based on our recent results on uniform solvability of coupled state-dependent Lyapunov equations. The controller design provides strong robustness properties both with respect to uncertain parameters in the system model and additive disturbances. This robustness is the key to the output-feedback controller design.

I. Introduction

We consider the class of systems given by

$$\begin{aligned}
\dot{x}_n &= \phi_{(n,n-1)}(x)x_{n-1} + \phi_n(\theta(t), x_1, \ldots, x_{n-2}, u) \\
\dot{x}_{n-1} &= \phi_{(n-1,n-2)}(x)x_{n-2} + \phi_{n-1}(\theta(t), x_1, \ldots, x_{n-3}, u) \\
&\vdots \\
\dot{x}_3 &= \phi_{(3,2)}(x)x_2 + \phi_3(\theta(t), x_1, u) \\
\dot{x}_2 &= \phi_{(2,1)}(x)x_1 + \phi_2(\theta(t), u) \\
\dot{x}_1 &= \mu_1(x)u
\end{aligned} \tag{1}$$

where $x = [x_1, \ldots, x_n]^T$ is the state and u the input. μ_1, $\phi_{(i,i-1)}, i = 2, \ldots, n$, and $\phi_i, i = 2, \ldots, n$, are continuous uncertain functions of their arguments. $\theta(t)$ is a vector of unknown time-varying parameters. The form commonly referred to as feedforward in the literature is obtained from (1) by restricting $\phi_{(i,i-1)}(x)$ to be a function of only $x_1, \ldots, x_{i-2}$ (in fact, most results on feedforward systems assume that $\phi_{(i,i-1)}$ are constants).

Available controller design techniques for feedforward systems in the literature include saturation-based designs [1, 2, 3, 4] and forwarding [5, 6]. Nested saturation designs rely on the use of small inputs and require the ϕ_i functions to be quadratic or higher powers in their arguments. Since the saturation levels are restricted to be sufficiently small, the scheme is sensitive to additive disturbances. Forwarding is a recursive passivation scheme which proceeds by adding one integrator at a time through the design of cross terms. However, forwarding is computationally complicated and the cross terms often need to be approximated numerically. Adaptive stabilization of feedforward systems was considered in [7]. A combination of forwarding and nested saturation was proposed in [8] to obtain weaker growth conditions. Scaling-based design with scaling governed by switching logic was considered in [9] to yield adaptive state-feedback stabilization. However, due to lack of robustness to additive disturbances in these designs, they can not be extended to the output-feedback case.

Recently, in [10], we proposed a new control design for feedforward systems using a dynamic high-gain scaling technique. The design is inspired by recent developments on dynamic high-gain scaling based control for strict-feedback-type systems [11, 12]. A dynamic high-gain scaling with the high gain parameter satisfying a scalar Riccati differential equation was proposed in [11] in the context of observer design for strict-feedback systems and was subsequently extended using duality considerations to design a dual high-gain observer/controller for strict-feedback systems in [12]. These designs differed from classical high-gain designs [13, 14] which provided only semiglobal results in that the high-gain scaling parameter was allowed to be a dynamic signal. The dual high-gain design in [12] utilized results on uniform solvability of coupled state-dependent Lyapunov equations [15, 16]. The dynamic high-gain scaling method essentially achieves an approximation of the system as a chain of nonlinear integrators. In [10], it was shown that the dynamic high-gain scaling technique can be used to design global state-feedback and output-feedback (with the output being either $[x_1, x_n]^T$ or x_n) controllers for feedforward systems of the form (1) under certain assumptions on the relative magnitudes of $\phi_{(i,i-1)}, i = 2, \ldots, n$, and the growth of the ϕ_i terms. Specifically, the $\phi_{(i,i-1)}$ terms that are closer to the input were required to be larger and the ϕ_i terms were assumed to be bounded up to $|\phi_{(n,n-1)}(x)|$ and a polynomially-bounded function of x_1 linearly in the states and the input. This assumption on the ϕ_i terms is, in a sense, complementary to the standard assumption in nested saturation designs that ϕ_i should involve only quadratic or higher powers. In this paper, we extend the results in [10] to weaken the assumptions needed on the uncertain functions ϕ_i. Firstly, the assumed bounds on ϕ_i are allowed to contain unknown parameters (with no available lower or upper bounds) and adaptations are incorporated for the unknown parameters. Secondly, we show that if the dependence of

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ϕ_i on u is bounded, the assumption in [10] that the function of x_1 occuring in the bound on ϕ_i must be polynomially bounded can be relaxed. The designed controllers, both in the state-feedback and in the output-feedback cases have a very simple structure being essentially a linear feedback with state-dependent dynamic gains and do not involve any saturations. The observer in the output-feedback case is similar to a Luenberger observer with dynamic observer gains. The Lyapunov functions are quadratic in the states (and, in the case of output-feedback, the observer errors). It is observed that a greater generality and complexity of bounds on ϕ_i does not increase the complexity of the control law, the observer, and the Lyapunov function, but is instead handled through the dynamics of the scaling parameter.

The control objective throughout this paper is to regulate the state x to the origin. This problem is addressed under the assumption of full state feedback in Section II. This assumption is relaxed in Section III where output feedback (with x_1 and x_n being the outputs) is considered. A particular case in which only x_n measurement is necessary is also identified. In Section IV, the assumption that the function of x_1 appearing in the bound on ϕ_i must be polynomially bounded is relaxed in the case that ϕ_i depends on u in a bounded manner.

II. Dynamic State Feedback

A. Assumptions

Assumption A1: $\phi_i, i = 2, \ldots, n$, can be bounded as

$$|\phi_i(\theta, x_1, \ldots, x_{i-2}, u)| \leq \overline{\theta}|\phi_{(n,n-1)}(x)|\gamma_1(x_1)\Big[\sum_{j=1}^{i-2}|x_j| + \Gamma_u(x)|u|\Big], i = 3, \ldots, n \quad (2)$$

$$|\phi_2(\theta, u)| \leq \overline{\theta}|\phi_{(n,n-1)}(x)|\gamma_1(x_1)\Gamma_u(x)|u| \quad (3)$$

where $\overline{\theta}$ is an unknown positive parameter and γ_1 and Γ_u are known continuous nonnegative functions. Furthermore, nonnegative constants p_1, p_2, and α_1 exist such that $\gamma_1(x_1) \leq p_1 + p_2|x_1|^{\alpha_1}$ for all $x_1 \in \mathcal{R}$.

Assumption A2: Positive constants σ, σ_u, and $\rho_i, i = 3, \ldots, n$, exist such that the functions $\phi_{(i,i-1)}(x)$ and $\mu_1(x)$ satisfy for all $x \in \mathcal{R}^n$

$$\begin{aligned} |\phi_{(i,i-1)}(x)| &\geq \sigma > 0\ ,\ i = 2, \ldots, n \\ |\phi_{(i,i-1)}(x)| &\leq \rho_i|\phi_{(i-1,i-2)}(x)|\ ,\ i = 3, \ldots, n \\ |\mu_1(x)| &\geq \sigma_u > 0. \end{aligned} \quad (4)$$

Assumption A3: A continuous nonnegative function $\Gamma_o(x_1)$ exists such that for all $x \in \mathcal{R}^n$

$$\frac{|\phi_{(2,1)}(x)|\Gamma_u(x)}{|\mu_1(x)|} \leq \Gamma_o(x_1). \quad (5)$$

Furthermore, nonnegative constants p_3, p_4, and α_2 exist such that $\Gamma_o(x_1) \leq p_3 + p_4|x_1|^{\alpha_2}$ for all $x_1 \in \mathcal{R}$.

Remark 1: The functions $\phi_i, i = 2, \ldots, n$, can depend on all the states x as long as the bounds in Assumption A1 hold. The function arguments are written as in (1) to emphasize the structure of the bounds.

Remark 2: Assumption A3 is satisfied in two important special cases: (1) if $\Gamma_u(x)$ and $\phi_{(2,1)}(x)$ are bounded by polynomial functions of x_1; (2) if $\Gamma_u(x) = 0$, i.e., $|\phi_i|$ are bounded by functions of $x_1, \ldots, x_i$. In case (2), Γ_o can be taken to be identically zero and it is shown in Section IV that the polynomial boundedness assumption on γ_1 can be relaxed.

B. Controller Design

The control input is designed as

$$u = \frac{\phi_{(1,0)}(x)\xi_0}{\mu_1(x)r^{1-b}} \quad (6)$$

where $\phi_{(1,0)}(x) = \rho_2\phi_{(2,1)}(x)$ with ρ_2 being any positive constant. ξ_0 and r are new state variables with the dynamics of ξ_0 given by

$$\dot{\xi}_0 = v - \frac{\dot{r}}{r}\xi_0. \quad (7)$$

The dynamics of r will be designed later. v is a new control input. $b > 0$ is a constant to be picked during the stability analysis. Further, define

$$\xi_i = \frac{x_i}{r^{i-1+b}}\ ,\ i = 1, \ldots, n. \quad (8)$$

The dynamics of $\xi_i, i = 1, \ldots, n$, are[1]

$$\begin{aligned} \dot{\xi}_i &= \frac{1}{r^{i-1+b}}\dot{x}_i - (i-1+b)\frac{\dot{r}}{r^{i+b}}x_i \\ &= \frac{1}{r}\phi_{(i,i-1)}\xi_{i-1} + \frac{1}{r^{i-1+b}}\phi_i - (i-1+b)\frac{\dot{r}}{r}\xi_i \end{aligned} \quad (9)$$

where we have introduced the dummy variable $\phi_1 \equiv 0$. The dynamics of ξ_0 are given in (7). The dynamics of $\xi = [\xi_0, \ldots, \xi_n]^T$ can be written in matrix form as

$$\dot{\xi} = \frac{1}{r}A\xi - \frac{\dot{r}}{r}D\xi + Bv + \Phi \quad (10)$$

$$A = \begin{bmatrix} 0 & 0 & 0 & \ldots & \ldots & 0 \\ \phi_{(1,0)} & 0 & 0 & \ldots & \ldots & 0 \\ 0 & \phi_{(2,1)} & 0 & \ldots & \ldots & 0 \\ 0 & 0 & \phi_{(3,2)} & 0 & \ldots & 0 \\ 0 & & & \ddots & & \\ 0 & \ldots & \ldots & 0 & \phi_{(n,n-1)} & 0 \end{bmatrix} \quad (11)$$

$$D = \text{diag}(1, b, 1+b, \ldots, n-1+b) \quad (12)$$

$$B = [1, 0, \ldots, 0]^T \quad (13)$$

$$\Phi = [0, 0, \frac{1}{r^{1+b}}\phi_2, \ldots, \frac{1}{r^{n-1+b}}\phi_n]^T. \quad (14)$$

The control input v is picked to be of the form

$$\begin{aligned} v &= \frac{1}{r}[k_0(x), k_1(x), k_2(x), \ldots, k_n(x)]\xi \\ &= [\frac{1}{r}k_0(x), \frac{1}{r^{1+b}}k_1(x), \frac{1}{r^{2+b}}k_2(x), \ldots, \frac{1}{r^{n+b}}k_n(x)]\begin{bmatrix} \xi_0 \\ x \end{bmatrix} \end{aligned} \quad (15)$$

where $k_0, \ldots, k_n$ are continuous functions of the state which will be picked to ensure uniform solvability of a pair of coupled state-dependent Lyapunov equations. Note that the control law (15) does not involve any saturation and is computationally very simple being essentially a linear feedback with state-dependent gains.

Using (15) in (10),

$$\dot{\xi} = \frac{1}{r}A_c\xi - \frac{\dot{r}}{r}D\xi + \Phi \quad (16)$$

$$A_c = \begin{bmatrix} k_0 & k_1 & k_2 & \ldots & \ldots & k_n \\ \phi_{(1,0)} & 0 & 0 & \ldots & \ldots & 0 \\ 0 & \phi_{(2,1)} & 0 & \ldots & \ldots & 0 \\ 0 & 0 & \phi_{(3,2)} & 0 & \ldots & 0 \\ 0 & & & \ddots & & \\ 0 & \ldots & \ldots & 0 & \phi_{(n,n-1)} & 0 \end{bmatrix}. \quad (17)$$

C. Stability Analysis

Theorem A1 in the Appendix implies that if Assumption A2 is satisfied, a symmetric positive-definite matrix P and functions $k_0(x), \ldots, k_n(x)$ can be found to satisfy

[1] For notational convenience, we drop the arguments of functions whenever no confusion will result.

the coupled Lyapunov equations

$$PA_c + A_c^T P \leq -\nu_1 |\phi_{(n,n-1)}(x)| I$$
$$\underline{\nu}_2 I \leq PD + DP \leq \overline{\nu}_2 I \quad (18)$$

with $\nu_1, \underline{\nu}_2$, and $\overline{\nu}_2$ being positive constants. Defining a Lyapunov function

$$V = \xi^T P \xi + \frac{1}{2}(\hat{\theta} - \overline{\theta})^2 \quad (19)$$

with $\hat{\theta}$ being a dynamic parameter estimate for $\overline{\theta}$, and differentiating along the trajectories of (16),

$$\begin{aligned}\dot{V} &= \frac{1}{r}\xi^T(PA_c + A_c^T P)\xi - \frac{\dot{r}}{r}\xi^T(PD+DP)\xi + 2\xi^T P\Phi \\ &\quad +(\hat{\theta}-\overline{\theta})\dot{\hat{\theta}} \\ &\leq -\frac{\nu_1}{r}|\phi_{(n,n-1)}(x)||\xi|^2 - \frac{\dot{r}}{r}\xi^T(PD+DP)\xi \\ &\quad +2\lambda_{max}(P)|\xi||\Phi| + (\hat{\theta}-\overline{\theta})\dot{\hat{\theta}} \qquad (20)\end{aligned}$$

where $\lambda_{max}(P)$ denotes the maximum eigenvalue of P. The dynamics of r are designed as

$$\dot{r} = |\phi_{(n,n-1)}(x)|\left[\frac{a_1}{r}\gamma_2(x_1,\hat{\theta}) - a_2\right] \quad (21)$$

where a_1 and a_2 are positive constants and γ_2 is a positive function of x_1 and $\hat{\theta}$ to be picked later. If γ_2 is picked such that $\gamma_2(x_1,\hat{\theta}) \geq \frac{a_2}{a_1}$ for all $x_1, \hat{\theta} \in \mathcal{R}$, then the dynamics of r in (21) implies that $\dot{r}|_{r=1} \geq 0$ so that if r is initialized greater than 1, $r(t)$ remains greater than 1 for all time t.

Remark 3: The choice of the form of the dynamics of r is a key step that differs from the case of strict-feedback systems [11, 12] where the dynamics of r are of the form of a scalar Riccati equation. The motivation for the form (21) can be seen from the Lyapunov analysis below.

Using (21) in (20), we have

$$\begin{aligned}\dot{V} &\leq -\frac{\nu_1}{r}|\phi_{(n,n-1)}(x)||\xi|^2 + \frac{a_2}{r}\overline{\nu}_2|\phi_{(n,n-1)}(x)||\xi|^2 \\ &\quad -\frac{a_1}{r^2}\underline{\nu}_2|\phi_{(n,n-1)}(x)|\gamma_2(x_1,\hat{\theta})|\xi|^2 + 2\lambda_{max}(P)|\xi||\Phi| \\ &\quad +(\hat{\theta}-\overline{\theta})\dot{\hat{\theta}}. \qquad (22)\end{aligned}$$

If $r \geq 1$, a bound for $\phi_i, i = 3,\ldots,n$, can be obtained using Assumption A1 as

$$\begin{aligned}|\phi_i| &\leq \overline{\theta}|\phi_{(n,n-1)}(x)|\gamma_1(x_1)\Big[r^b|\xi_1| + r^{1+b}|\xi_2| + \ldots + r^{i-3+b}|\xi_{i-2}| \\ &\quad + \Gamma_u(x)\frac{|\phi_{(1,0)}(x)||\xi_0|}{|\mu_1(x)|r^{1-b}}\Big] \\ &\leq \overline{\theta}|\phi_{(n,n-1)}(x)|\gamma_1(x_1)[1+\rho_2\Gamma_o(x_1)]n^{\frac{1}{2}}|\xi|r^{i-3+b}. \qquad (23)\end{aligned}$$

Also,

$$|\phi_2| \leq \overline{\theta}|\phi_{(n,n-1)}(x)|\rho_2\gamma_1(x_1)\Gamma_o(x_1)|\xi_0|r^{-1+b}. \quad (24)$$

Hence,

$$|\Phi| = \sqrt{\sum_{i=2}^{n}\left(\frac{\phi_i}{r^{i-1+b}}\right)^2} \leq \frac{\overline{\theta}}{r^2}|\phi_{(n,n-1)}(x)|\gamma(x_1)n|\xi| \quad (25)$$

where $\gamma(x_1) = \gamma_1(x_1)[1+\rho_2\Gamma_o(x_1)]$. a_1, a_2, and γ_2 are chosen to satisfy

$$a_1 \geq \frac{2\lambda_{max}(P)n}{\underline{\nu}_2} \quad (26)$$

$$a_2 \leq \frac{\nu_1}{2\overline{\nu}_2} \quad (27)$$

$$\gamma_2(x_1,\hat{\theta}) = \hat{\theta}\max\left(\frac{a_2}{a_1}, \gamma(x_1)\right). \quad (28)$$

The dynamics of $\hat{\theta}$ are chosen as

$$\dot{\hat{\theta}} = \frac{2}{r^2}n\lambda_{max}(P)|\phi_{(n,n-1)}(x)|\gamma(x_1)|\xi|^2. \quad (29)$$

$\hat{\theta}$ is initialized to be greater than 1. Then, $\hat{\theta}$ governed by (29) remains greater than 1. Substituting (26)-(28) in (22) yields

$$\dot{V} \leq -\frac{\sigma\nu_1}{2r}|\xi|^2. \quad (30)$$

Note that $\gamma_2(x_1,\hat{\theta})$ as defined in (28) is bounded below by $\frac{a_2}{a_1}$ implying that $r(t) \geq 1$ for all time by initializing r greater than 1. By (30), ξ and $\hat{\theta}$ are bounded on the maximal interval of existence of solutions. To ensure boundedness of r, we pick the design freedom b so that

$$0 \leq b < \frac{1}{\alpha} \ ; \ \alpha = \alpha_1 + \alpha_2. \quad (31)$$

Since

$$\gamma(x_1) \leq \{p_1 + p_2|x_1|^{\alpha_1}\}\{1+\rho_2[p_3+p_4|x_1|^{\alpha_2}]\} \leq \overline{p}_1 + \overline{p}_2|x_1|^{\alpha} \quad (32)$$

with $\overline{p}_1$ and $\overline{p}_2$ being positive constants, we have

$$\gamma_2(x_1,\hat{\theta}) = \hat{\theta}\max\left(\frac{a_2}{a_1}, \gamma(r^b\xi_1)\right) \leq \hat{\theta}\max\left(\frac{a_2}{a_1}, \overline{p}_1 + \overline{p}_2 r^{b\alpha}|\xi_1|^{\alpha}\right)$$

$$\Longrightarrow \frac{a_1}{r}\gamma_2(x_1,\hat{\theta}) \leq \hat{\theta}\max\left(\frac{a_2}{r}, \frac{a_1\overline{p}_1}{r} + a_1\overline{p}_2 r^{b\alpha-1}|\xi_1|^{\alpha}\right). \quad (33)$$

Hence, $\dot{r}$ is negative for large enough r for any fixed ξ_1 and $\hat{\theta}$ since $b\alpha - 1 < 0$ by (31). Thus, boundedness of r follows from boundedness of ξ_1 and $\hat{\theta}$. The boundedness of $x_i, i = 1,\ldots,n$, follows from boundedness of $\xi_i = \frac{x_i}{r^{i-1+b}}$ and r. Since r is bounded below by 1, the control input u is also bounded. Thus, all closed-loop signals remain bounded. Furthermore, from (30), $\xi(t)$ and hence $x(t)$ and $u(t)$ go to zero as $t \to \infty$.

III. Dynamic Output Feedback

In this section, we consider the output-feedback problem for system (1). The outputs of the system are taken to be x_1 and x_n, i.e., output vector $y = [x_1, x_n]^T$. A particular case in which only x_n measurement is needed is indicated in Remark 4. For the output-feedback design, we need to introduce Assumption A4 given below and also need to strengthen Assumption A1 as shown in Assumption A1'. The output-feedback design in this section is based on Assumptions A1', A2, A3, and A4.

Assumption A4: The functions $\phi_{(i,i-1)}(x), i = 2,\ldots,n$, and $\mu_1(x)$ depend only on x_1 and x_n, i.e., with a slight abuse of notation

$$\phi_{(i,i-1)}(x) \equiv \phi_{(i,i-1)}(y) \ , \ \mu_1(x) \equiv \mu_1(y). \quad (34)$$

Furthermore, $\phi_{(i,i-1)}(y)$ satisfy for all $x \in \mathcal{R}^n$

$$|\phi_{(i,i-1)}(y)| \geq \tilde{\rho}_i|\phi_{(i-1,i-2)}(y)| \ , \ i = 3,\ldots,n \quad (35)$$

with $\tilde{\rho}_i, i = 3,\ldots,n$, being positive constants.

Assumption A1': $\phi_i, i = 2,\ldots,n$, can be bounded as

$$|\phi_i(\theta,x_1,\ldots,x_{i-2},u)| \leq |\phi_{(n,n-1)}(y)|\gamma_1(x_1)\Big[\overline{\theta}|x_1| + \sum_{j=2}^{i-2}|x_j| + \overline{\theta}\Gamma_u(x)|u|\Big], i = 3,\ldots,n \quad (36)$$

$$|\phi_2(\theta,u)| \leq \overline{\theta}|\phi_{(n,n-1)}(y)|\gamma_1(x_1)\Gamma_u(x)|u| \quad (37)$$

where $\overline{\theta}$ is an unknown positive parameter and γ_1 and Γ_u are known continuous nonnegative functions. Furthermore, a number $p_1 > 0$ exists such that $\gamma_1(rx_1) \leq r^{p_1}\gamma_1(x_1)$ for all $r \geq 1$ and all $x_1 \in \mathcal{R}$.

A. Observer Design

A full-order observer to estimate the unmeasured states $x_2, \ldots, x_{n-1}$ is designed as

$$\dot{\hat{x}}_i = \phi_{(i,i-1)}(y)\hat{x}_{i-1} + r^{i-n-1}g_i(y)(\hat{x}_n - x_n), 2 \leq i \leq n$$
$$\dot{\hat{x}}_1 = \mu_1(x_1, x_n)u + r^{-n}g_1(y)(\hat{x}_n - x_n) \quad (38)$$

where $g_1(y), \ldots, g_n(y)$ are functions of x_1 and x_n to be designed later and r is a new state variable satisfying the dynamics given by (21). The observer errors are defined as

$$e_i = \hat{x}_i - x_i\ ,\ i = 1, \ldots, n \quad (39)$$

and the scaled observer errors are defined as

$$\epsilon_i = \frac{1}{r^{i-1+b}}e_i \quad (40)$$

where $b > 0$ is a design parameter. The dynamics of the scaled observer errors are given by, $i = 1, \ldots, n$,

$$\dot{\epsilon}_i = \frac{1}{r}\phi_{(i,i-1)}\epsilon_{i-1} - \frac{1}{r^{i-1+b}}\phi_i + \frac{1}{r}g_i\epsilon_n - (i-1+b)\frac{\dot{r}}{r}\epsilon_i \quad (41)$$

where $\phi_1 \equiv 0$ and $\epsilon_0 \equiv 0$ are dummy variables. Defining $\epsilon = [\epsilon_1, \ldots, \epsilon_n]^T$, the dynamics (41) can be written in matrix form as

$$\dot{\epsilon} = \frac{1}{r}A_o\epsilon - \frac{\dot{r}}{r}D_o\epsilon + \Phi \quad (42)$$

where

$$A_o = \begin{bmatrix} 0 & 0 & 0 & \ldots & \ldots & g_1 \\ \phi_{(2,1)} & 0 & 0 & \ldots & \ldots & g_2 \\ 0 & \phi_{(3,2)} & 0 & \ldots & \ldots & g_3 \\ 0 & 0 & \phi_{(4,3)} & 0 & \ldots & g_4 \\ & & & \ddots & & \\ 0 & & & & & \\ 0 & \ldots & \ldots & 0 & \phi_{(n,n-1)} & g_n \end{bmatrix} \quad (43)$$

$$D_o = \mathrm{diag}(b, 1+b, \ldots, n-1+b) \quad (44)$$

$$\Phi = [0, -\frac{1}{r^{1+b}}\phi_2, \ldots, -\frac{1}{r^{n-1+b}}\phi_n]^T. \quad (45)$$

B. Controller Design

The control input is designed as

$$u = \frac{\phi_{(1,0)}(y)\xi_0}{\mu_1(y)r^{1-b}} \quad (46)$$

where $\phi_{(1,0)}(y) = \rho_2\phi_{(2,1)}(y)$ with ρ_2 being a positive constant and ξ_0 is a new state variable with the dynamics

$$\dot{\xi}_0 = v - \frac{\dot{r}}{r}\xi_0. \quad (47)$$

v is a new control input. Defining

$$\xi_i = \frac{\hat{x}_i}{r^{i-1+b}}\ ,\ i = 1, \ldots, n, \quad (48)$$

the dynamics of ξ_i, $i = 1, \ldots, n$, are given by

$$\dot{\xi}_i = \frac{1}{r^{i-1+b}}\dot{\hat{x}}_i - (i-1+b)\frac{\dot{r}}{r^{i+b}}\hat{x}_i$$
$$= \frac{1}{r}\phi_{(i,i-1)}\xi_{i-1} - (i-1+b)\frac{\dot{r}}{r}\xi_i + \frac{1}{r}g_i\epsilon_n. \quad (49)$$

The dynamics of ξ_0 are given in (47). The control input v is picked to be

$$v = \frac{1}{r}[k_0(y), k_1(y), k_2(y), \ldots, k_n(y)]\xi \quad (50)$$

where $\xi = [\xi_0, \ldots, \xi_n]^T$. $k_0(y), \ldots, k_n(y)$ are continuous functions of x_1 and x_n. The dynamics of ξ can be written in matrix form as

$$\dot{\xi} = \frac{1}{r}A_c\xi - \frac{\dot{r}}{r}D_c\xi + \frac{1}{r}g\epsilon_n \quad (51)$$

where A_c is given in (17) and

$$D_c = \mathrm{diag}(1, b, 1+b, \ldots, n-1+b)$$
$$g = [0, g_1, \ldots, g_n]^T. \quad (52)$$

C. Stability Analysis

Using Theorem A1 in the Appendix, Assumptions A2 and A4 imply the existence of functions $g_1, \ldots, g_n$, $k_0, \ldots, k_n$, and symmetric positive-definite matrices P_o and P_c such that the Lyapunov inequalities

$$P_oA_o + A_o^TP_o \leq -\nu_{1o}|\phi_{(n,n-1)}|I$$
$$\underline{\nu}_{2o}I \leq P_oD_o + D_oP_o \leq \overline{\nu}_{2o}I$$
$$P_cA_c + A_c^TP_c \leq -\nu_{1c}|\phi_{(n,n-1)}|I$$
$$\underline{\nu}_{2c}I \leq P_cD_c + D_cP_c \leq \overline{\nu}_{2c}I \quad (53)$$

are satisfied with ν_{1o}, $\underline{\nu}_{2o}$, $\overline{\nu}_{2o}$, ν_{1c}, $\underline{\nu}_{2c}$, and $\overline{\nu}_{2c}$ being positive constants. Moreover, using Theorem A1 in the Appendix, the functions $g_1, \ldots, g_n$ are linear constant-coefficient combinations of $\phi_{(i,i-1)}, i = 2, \ldots, n$. Using Assumption A4, a positive constant $\overline{g}$ exists such that

$$|g| = \sqrt{g_1^2(y) + g_2^2(y) + \ldots + g_n^2(y)} \leq \overline{g}|\phi_{(n,n-1)}(y)|. \quad (54)$$

Consider an *observer Lyapunov function* V_o and a *controller Lyapunov function* V_c given by

$$V_o = \epsilon^TP_o\epsilon\ ,\ V_c = \xi^TP_c\xi. \quad (55)$$

The derivatives of these Lyapunov functions can be bounded as

$$\dot{V}_o \leq |\phi_{(n,n-1)}|\Big\{ - \frac{\nu_{1o}}{r}|\epsilon|^2 + \frac{a_2}{r}\overline{\nu}_{2o}|\epsilon|^2 - \frac{a_1}{r^2}\underline{\nu}_{2o}\gamma_2|\epsilon|^2\Big\}$$
$$+2\lambda_{max}(P_o)|\epsilon||\Phi| \quad (56)$$

$$\dot{V}_c \leq |\phi_{(n,n-1)}|\Big\{ - \frac{\nu_{1c}}{r}|\xi|^2 + \frac{a_2}{r}\overline{\nu}_{2c}|\xi|^2 - \frac{a_1}{r^2}\underline{\nu}_{2c}\gamma_2|\xi|^2$$
$$+\frac{2}{r}\lambda_{max}(P_c)\overline{g}|\xi||\epsilon|\Big\}. \quad (57)$$

$\phi_i, i = 3, \ldots, n$, can be bounded as

$$|\phi_i| \leq |\phi_{(n,n-1)}(y)|\gamma_1(x_1)[1+\rho_2\Gamma_o(x_1)]\{\overline{\theta}[|\xi_0|+|\xi_1|+|\epsilon_1|]$$
$$+n^{\frac{1}{2}}[|\xi| + |\epsilon|]\}r^{i-3+b}. \quad (58)$$

Also,

$$|\phi_2| \leq \overline{\theta}|\phi_{(n,n-1)}(y)|\rho_2\gamma_1(x_1)\Gamma_o(x_1)|\xi_0|r^{-1+b}. \quad (59)$$

Hence,

$$|\Phi| \leq \frac{1}{r^2}|\phi_{(n,n-1)}(y)|\gamma(x_1)\{n^{\frac{1}{2}}\overline{\theta}[|\xi_0| + |\xi_1| + |\epsilon_1|]$$
$$+n[|\xi| + |\epsilon|]\} \quad (60)$$

where $\gamma(x_1) = \gamma_1(x_1)[1 + \rho_2\Gamma_o(x_1)]$. Using (60) in (56),

$$\dot{V}_o \leq |\phi_{(n,n-1)}|\Big\{ - \frac{\nu_{1o}}{r}|\epsilon|^2 + \frac{a_2}{r}\overline{\nu}_{2o}|\epsilon|^2 - \frac{a_1}{r^2}\underline{\nu}_{2o}\gamma_2|\epsilon|^2$$
$$+\frac{4}{r^2}\lambda_{max}(P_o)\gamma(x_1)n[|\epsilon|^2 + |\xi|^2]$$
$$+\frac{3\overline{\theta}^2}{r^2}\lambda_{max}(P_o)\gamma(x_1)[\xi_0^2 + \xi_1^2 + \epsilon_1^2]\Big\}. \quad (61)$$

A composite Lyapunov function is defined as

$$V = cV_o + V_c + \frac{1}{2}(\hat{\theta} - \overline{\theta}^2)^2 \quad (62)$$

where c is a positive constant such that

$$c \geq \frac{8}{\nu_{1o}\nu_{1c}}\lambda_{max}^2(P_c)\overline{g}^2 \quad (63)$$

and $\hat{\theta}$ is a dynamic parameter estimate with dynamics

$$\dot{\hat{\theta}} = \frac{3c}{r^2}|\phi_{(n,n-1)}(y)|\lambda_{max}(P_o)\gamma(x_1)[\xi_0^2+\xi_1^2+\epsilon_1^2]. \quad (64)$$

$\hat{\theta}$ is initialized to be greater than 1. By (64), $\hat{\theta}$ is monotonically increasing and hence, $\hat{\theta}$ remains greater than 1. Using (57) and (61),

$$\dot{V} \le |\phi_{(n,n-1)}|\Big\{-\frac{c\nu_{1o}}{2r}|\epsilon|^2 - \frac{\nu_{1c}}{2r}|\xi|^2 + \frac{ca_2}{r}\overline{\nu}_{2o}|\epsilon|^2 + \frac{a_2}{r}\overline{\nu}_{2c}|\xi|^2 - \frac{ca_1}{r^2}\underline{\nu}_{2o}\gamma_2|\epsilon|^2 - \frac{a_1}{r^2}\underline{\nu}_{2c}\gamma_2|\xi|^2 + \frac{4c}{r^2}\lambda_{max}(P_o)\gamma(x_1)n[|\epsilon|^2+|\xi|^2] + \frac{3c\hat{\theta}}{r^2}\lambda_{max}(P_o)\gamma(x_1)[|\epsilon|^2+|\xi|^2]\Big\}. \quad (65)$$

Picking a_1, a_2, and γ_2 to satisfy

$$a_1 \ge \frac{(4n+3)c\lambda_{max}(P_o)}{\min(c\underline{\nu}_{2o}, \underline{\nu}_{2c})} \quad (66)$$

$$a_2 \le \min\left(\frac{\nu_{1o}}{2\overline{\nu}_{2o}} - \frac{a_2^*}{c\overline{\nu}_{2o}}, \frac{\nu_{1c}}{2\overline{\nu}_{2c}} - \frac{a_2^*}{\overline{\nu}_{2c}}\right) \quad (67)$$

$$\gamma_2(x_1) = \hat{\theta}\max(\frac{a_2}{a_1}, \gamma(x_1)) \quad (68)$$

where a_2^* is a positive constant such that

$$a_2^* < \min(\frac{c\nu_{1o}}{2}, \frac{\nu_{1c}}{2}), \quad (69)$$

(65) reduces to

$$\dot{V} \le -\frac{a_2^*\sigma}{r}[|\epsilon|^2+|\xi|^2]. \quad (70)$$

As in Section II, it is inferred from the dynamics of r that if $r(0) > 1$, then $r(t)$ remains greater than 1 for all time t. From (70), it is seen that V and hence ϵ, ξ, and $\hat{\theta}$ are bounded. Also, boundedness of r can be inferred from boundedness of $\hat{\theta}$ and $\frac{|x_1|}{r^b} \le |\xi_1| + |\epsilon_1|$ by choosing b to satisfy (31). Boundedness of $\hat{x}_i = r^{i-1+b}\xi_i$ and $e_i = r^{i-1+b}\epsilon_i, i = 1,\ldots,n$, follows. This implies boundedness of $x_i, i = 1,\ldots,n$. Boundedness of u is inferred from boundedness of ξ, x_1, and x_n and lower boundedness of r by 1. Thus, all closed-loop signals remain bounded. Furthermore, from (70), $\xi(t)$ and $\epsilon(t)$, and hence $x(t)$ and $e(t)$ go to zero as $t \to \infty$.

Remark 4: From the stability analysis in Section IIIC, it is seen that only x_n measurement is required (measurement of x_1 not necessary) in the special case in which $\phi_{(i,i-1)}, i = 2,\ldots,n$, and μ_1 depend only on x_n, γ_1 and Γ_o are bounded functions, and $\overline{\theta}$ is known. In this case, it is not necessary to build the estimator $\hat{\theta}$ and the term $\hat{\theta}$ in (68) can be replaced by $\overline{\theta}^2$.

IV. Relaxation of Polynomial Bound Assumption on γ_1 if $\Gamma_u \equiv 0$

The designs in Sections II and III utilized a polynomial bound assumption on the functions γ_1 and Γ_o that occur in the bounds in Assumptions A1, A1', and A3. This assumption can be relaxed if the uncertain functions ϕ_i vanish at the equilibrium point $x_i = 0, i = 1,\ldots,n$, and the dependence of ϕ_i on the control input u is uniformly bounded in the sense that the function Γ_u in Assumption A1 (A1' in the output-feedback case) can be taken to be zero, i.e., it should be possible to bound any dependence on u through the constant $\overline{\theta}$. This includes, for instance, the case when u appears as the argument of a bounded function, e.g., $\phi_i = \sin(u)x_i$. The design under the assumption $\Gamma_u \equiv 0$ (which implies that Γ_o can be taken to be zero) is presented in this section for the output-feedback problem. A similar design can be carried out in the state-feedback case.

A full-order observer is designed to estimate the unmeasured states $x_2,\ldots,x_{n-1}$ as

$$\dot{\hat{x}}_i = \phi_{(i,i-1)}(y)\hat{x}_{i-1} + r^{i-n-1}g_i(y)(\hat{x}_n - x_n), 2 \le i \le n$$

$$\dot{\hat{x}}_1 = \mu_1(y)u + r^{-n}g_1(y)(\hat{x}_n - x_n) - \frac{\dot{r}}{r}(\hat{x}_1 - x_1). \quad (71)$$

The observer errors are $e_i = \hat{x}_i - x_i, i = 1,\ldots,n$, and the scaled observer errors are defined as $\epsilon_i = \frac{1}{r^{i-1}}e_i$. The dynamics of $\epsilon = [\epsilon_1,\ldots,\epsilon_n]^T$ are as in (42) with

$$D_o = \mathrm{diag}(1,1,2,\ldots,n-1) \quad (72)$$

$$\Phi = [0, -\frac{\phi_2}{r}, \ldots, -\frac{\phi_n}{r^{n-1}}]^T \quad (73)$$

and A_o is defined in (43). The scaled observer estimates are defined as $\xi_i = \frac{1}{r^{i-1}}\hat{x}_i, i = 1,\ldots,n$. The control input is designed as

$$u = \frac{1}{\mu_1(y)}\Big\{\frac{1}{r}[k_1(y),k_2(y),\ldots,k_n(y)]\xi - \frac{\dot{r}}{r}x_1\Big\} \quad (74)$$

where $\xi = [\xi_1,\ldots,\xi_n]^T$. Note that in this case, unlike the design in Sections II and III, we do not need the dynamic extension ξ_0. Instead, the control input includes the term $\frac{1}{\mu_1}\frac{\dot{r}}{r}x_1$. The assumption $\Gamma_u \equiv 0$ is crucial for the introduction of this term which eliminates the need for a scaling in the definition of $\xi_1 = \hat{x}_1$. The removal of the scaling in the definition of ξ_1 is instrumental in the relaxation of the polynomial bound assumption on γ_1. The dynamics of ξ are as given in (51) with

$$A_c = \begin{bmatrix} k_1 & k_2 & k_3 & \ldots & \ldots & k_n \\ \phi_{(2,1)} & 0 & 0 & \ldots & \ldots & 0 \\ 0 & \phi_{(3,2)} & 0 & \ldots & \ldots & 0 \\ 0 & 0 & \phi_{(4,3)} & 0 & \ldots & 0 \\ 0 & & & \ddots & & \\ 0 & \ldots & \ldots & 0 & \phi_{(n,n-1)} & 0 \end{bmatrix} \quad (75)$$

$$D_c = \mathrm{diag}(1,1,2,\ldots,n-1)\ ,\ g = [g_1,\ldots,g_n]^T. \quad (76)$$

The dynamics of r are as given in (21).

By an application of Theorem A1, functions $g_1,\ldots,g_n,k_1,\ldots,k_n$, positive-definite matrices P_o and P_c, and positive constants ν_{1o}, $\underline{\nu}_{2o}$, $\overline{\nu}_{2o}$, ν_{1c}, $\underline{\nu}_{2c}$, and $\overline{\nu}_{2c}$ are found to satisfy the Lyapunov inequalities (53). Furthermore, a positive constant $\overline{g}$ exists to satisfy (54). As in Section III, stability can be demonstrated via the Lyapunov function

$$V = c\epsilon^TP_o\epsilon + \xi^TP_c\xi + \frac{1}{2}(\hat{\theta} - \overline{\theta}^2)^2 \quad (77)$$

where c is a positive constant satisfying (63). The dynamics of $\hat{\theta}$ are designed as

$$\dot{\hat{\theta}} = \frac{2c}{r^2}|\phi_{(n,n-1)}(y)|\lambda_{max}(P_o)\gamma(x_1)[\xi_1^2+\epsilon_1^2]. \quad (78)$$

a_1, a_2, and γ_2 are picked to satisfy (66), (67), and (68). The derivative of the Lyapunov function (77) satisfies (70). This implies that ξ, ϵ, and $\hat{\theta}$ are bounded. Hence, $\xi_1 = \hat{x}_1$ and $\epsilon_1 = e_1 = \hat{x}_1 - x_1$ are bounded implying that x_1 is bounded. The boundedness of r follows from boundedness of x_1 and $\hat{\theta}$. Hence, all closed-loop signals are bounded. Furthermore, from (70), $\xi(t)$ and $\epsilon(t)$, and hence $x(t)$ and $e(t)$, go to zero as $t \to \infty$.

V. Appendix

Theorem A1: Let $A(x)$ be an $m \times m$ matrix function of $x \in \mathcal{R}^n$ with given functions $\phi_{(i,i-1)}(x)$ on the lower diagonal, design freedoms $g_i(x)$ on the last column, and zeros everywhere else, i.e.,

$$\begin{aligned} A_{(i,m)}(x) &= g_i(x)\ ,\ i = 1, \ldots, m \\ A_{(i,i-1)}(x) &= \phi_{(i,i-1)}(x)\ ,\ i = 2, \ldots, m. \end{aligned} \quad (79)$$

Let positive constants σ and ρ_i exist such that

$$|\phi_{(i,i-1)}(x)| \geq \sigma\ ,\ i = 2, \ldots, m \quad (80)$$

$$|\phi_{(i,i-1)}(x)| \leq \rho_i |\phi_{(i+1,i)}(x)|\ ,\ i = 2, \ldots, m-1. \quad (81)$$

Let $D(x)$ be an $m \times m$ diagonal matrix function of x with diagonal elements $D_1(x), \ldots, D_m(x)$. Let positive constants $\overline{\rho}_D$ and $\underline{\rho}_D$ exist such that

$$\overline{\rho}_D \geq D_i(x) \geq \underline{\rho}_D, i = 1, \ldots, m \quad (82)$$

for all $x \in \mathcal{R}^n$. Then, functions $g_1(x), \ldots, g_m(x)$ with each g_i being a linear constant-coefficient combination of $\phi_{(2,1)}, \ldots, \phi_{(m,m-1)}$, a symmetric positive-definite $m \times m$ matrix P, and positive constants ν_1, $\underline{\nu}_2$, and $\overline{\nu}_2$ exist such that for all $x \in \mathcal{R}^n$,

$$PA(x) + A^T(x)P \leq -\nu_1 |\phi_{(2,1)}(x)| I \quad (83)$$

$$\underline{\nu}_2 I \leq PD(x) + D(x)P \leq \overline{\nu}_2 I. \quad (84)$$

Remark 5: Theorem A1 was proved in [15] utilizing techniques from [17]. The proof is omitted here for brevity. More general results on uniform solvability of coupled Lyapunov equations can be found in [16].

Remark 6: The case of A having, as elements, given functions $\phi_{(i,i-1)}$ on the lower diagonal and design freedoms k_i on the first row is a dual of the case considered in Theorem A1 where the condition (81) for solvability of the coupled Lyapunov equations is replaced by existence of positive constants $\overline{\rho}_i$ such that

$$|\phi_{(i,i-1)}(x)| \geq \overline{\rho}_i |\phi_{(i+1,i)}(x)|\ ,\ i = 2, \ldots, m-1. \quad (85)$$

VI. Conclusion

In this paper, we proposed robust adaptive state-feedback and output-feedback controller design techniques for feedforward systems based on dynamic high-gain scaling. The designed controllers have a very simple structure being essentially a linear feedback with state-dependent dynamic gains and do not involve any saturations or recursive computations. The observer in the output-feedback case is similar to a Luenberger observer with dynamic observer gains. The Lyapunov functions are quadratic in states (and observer errors in the case of output-feedback) and the parameter estimation error. The controller design provides robustness both with respect to uncertain parameters in the system model and with respect to additive disturbances. This robustness is the key to the output-feedback controller design. The controller and observer designs are strongly parallel to our recent designs in the case of strict-feedback systems. This suggests that the proposed technique could allow further extensions for feedforward systems along various lines that have been hitherto investigated only for feedback systems. It is also interesting to note that as in the case of feedback systems [12], a greater generality and complexity of bounds on uncertain functions ϕ_i does not increase the complexity of control law, observer, and Lyapunov function, but is instead handled through the dynamics of the scaling parameter.

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