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Stabilization of Switched Symmetric Systems

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***Abstract*— This paper studies the stabilization control synthesis of switched symmetric systems. The main contribution includes two parts. First, a necessary and sufficient condition for existence of asymptotic stable output feedback controller of symmetric switched linear systems under arbitrary switching signal is established. It is also pointed that if the condition holds, constant controller can be used as well. Next, a sufficient condition for existence of output feedback controller for quadratic stabilizability of switched symmetric systems is established. If there are only two subsystems, this condition is also necessary.**

I. INTRODUCTION

As for switched symmetric systems which appear quite often in many engineering disciplines(e.g. electrical and power networks, structural systems, viscoelastic materials and chemical reaction), [5] first investigated the stability and $\mathcal{L}_2$ gain analysis under arbitrary switching signals. It is proved that when all the subsystems are stable, the switched system is exponentially stable under arbitrary switching. Motivated by these analysis work, we try to investigate the stabilization problem via output feedback for switched symmetric systems. Two problem are considered. One is the output feedback controller synthesis for switched symmetric systems under arbitrary switching signals, the other is the output feedback controller synthesis for quadratic stabilizability of switched symmetric systems.

II. PRELIMINARY

Consider a switched linear system given by

$$\begin{cases} \dot{x}(t) = A_{r(t)}x(t) + B_{r(t)}u(t) \\ y(t) = C_{r(t)}x(t) \end{cases} \tag{1}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^p$ is the vector of control inputs, $y(t) \in \Re^p$ is the vector of measured outputs, the right continuous function $r(t) : \Re^+ \rightarrow \{1, 2, \cdots, N\}$ is the switching signal to be designed or created by some unknown or nondeterministic function. Moreover, $r(t) = i$ implies that the subsystem (A_i, B_i, C_i) is activated, $i = 1, 2, \cdots, N$.

System (1) is called *(state-space) symmetric*, if for any $i = 1, \cdots, N$,

$$A_i = A_i^T, \quad C_i = B_i^T. \tag{2}$$

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Moreover, if $B_1 = \cdots = B_N = 0$, we get the autonomous symmetric switched linear system

$$\dot{x}(t) = A_{r(t)}x(t) \tag{3}$$

The notations to be used in this paper are as follows: given a real matrix M, the orthogonal complement $M^\perp$ is defined as the (possibly non-unique) matrix with maximum row rank that satisfied $M^\perp M = 0$ and $M^\perp M^{\perp T} > 0$. Hence, $M^\perp$ can be computed from the singular value decomposition of M as follows: $M^\perp = TU_2^T$ where T is an arbitrary nonsingular matrix and U_2 is defined from the singular value decomposition of M

$$M = [U_1, U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

Given a real matrix $M \in \Re^{n\times p}$ has full column rank($n \geq p$), the pseudo-inverse $M^+ \in \Re^{p\times n}$ is defined as

$$M^+ M = I_p$$

The standard notation $>$ $(<)$ is used to denote the positive (negative) definite ordering of symmetric matrices.

Let $N = 1$, system (1)-(2) reduces to a common symmetric LIT system. K. Tan and K. M. Grigoriadis gave the following basic lemma for the stabilization of symmetric LTI systems [2].

Lemma 1: [2] Given a symmetric LTI system (A, B, C), there exists a symmetric output feedback matrix $G = G^T$ such that the closed-loop system is asymptotically stable, i.e. the matrix $A + BGC$ is Hurwitz stable, if and only if

$$B^\perp A B^{\perp T} < 0. \tag{4}$$

If this condition is satisfied, all stabilizing symmetric output feedback gains G satisfy

$$G < B^+[AB^{\perp T}(B^\perp AB^{\perp T})^{-1}B^\perp A - A]B^{+T}. \tag{5}$$

III. MAIN RESULTS

A. Stabilization Under Arbitrary Switching Signal

First, we discuss the stabilization problem under arbitrary switching signal.

Definition 1: [5] The system (1) is said to be exponentially stable if $||x(t)|| \leq c\,e^{-\alpha t}||x_0(t)||$ with $c > 0, \alpha > 0$ holds for any $t > 0$ and any initial state x_0.

Lemma 2: [5] For system (3), given any switching signal $r(t)$, it is exponentially stable if and only if each subsystem of system (3) is asymptotically stable.

Theorem 1: For system (1)-(2), given any switching signal $r(t)$, there exist symmetric output feedbacks $u = G_i y$, $i = 1, \cdots, N$ such that the closed-loop system

$$\dot{x}(t) = (A_{r(t)} + B_{r(t)} G_{r(t)} C_{r(t)}) x(t) \tag{6}$$

is exponentially stable if and only if

$$B_i^{\perp} A_i B_i^{\perp T} < 0, \forall\, i = 1, \cdots, N. \tag{7}$$

If this condition is satisfied, all stabilizing symmetric output feedback gains G_i satisfy $G_i < H_i$, where

$$H_i = B_i^{+}[A_i B_i^{\perp T}(B_i^{\perp} A_i B_i^{\perp T})^{-1} B_i^{\perp} A_i - A_i] B_i^{+T} \tag{8}$$

Proof:
(Sufficiency) If the inequalities (7) hold, by Lemma 1, then we can select symmetric output feedback control laws $u = G_i y$ such that the systems $\dot{x}(t) = (A_i + B_i G_i C_i) x(t)$ are asymptotically stable, respectively. Thus, by Lemma 2, the closed-loop system is exponentially stable.
(Necessary) It is obvious. ■

Definition 2: Denote a matrix $A = [a_{ij}] \in \Re^{n\times n}$, A is said to be strictly diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \forall\, i = 1, \cdots, n \tag{9}$$

Lemma 3: [1] Let $A = A^T \in \Re^{n\times n}$ be strictly diagonally dominant, if all main diagonal entries of A are positive, then all the eigenvalues of A are real and positive, i.e. $A > 0$.

Corollary 1: For system (1)-(2), if the inequalities (7) hold, we can found constant symmetric output feedback $u = Gy$ such that the closed-loop system

$$\dot{x}(t) = (A_{r(t)} + B_{r(t)} G C_{r(t)}) x(t) \tag{10}$$

is exponentially stable.

Proof: Since the inequalities (7) hold, we can get G_i for each subsystem and $G_i < H_i$. Denote $H_i = H_i^T = [h_{kl}^i] \in \Re^{p\times p}$. We can select $G = diag(g_1, \cdots, g_p)$ such that

$$g_k < \min_{i=1,\cdots,N} \{\, h_{kk}^i - \sum_{\substack{l=1 \\ l\neq k}}^{p} |h_{kl}^i|\, \}, \forall\, k = 1, \cdots, p \tag{11}$$

Then we have $h_{kk}^i - g_k > \sum_{\substack{l=1 \\ l\neq k}}^{p} |h_{kl}^i|, \forall\, k = 1, \cdots, p$. That is, the real symmetric matrix $H_i - G$ is strictly diagonally dominant and all its main diagonal entries are positive. By Lemma 3, it follows that $H_i - G > 0$, i.e. $G < H_i$. Hence the closed-loop system (10) is exponentially stable when we select the constant output feedback control law $u = Gy$. ■

Remark 1: Theorem 1 and Corollary 1 can be extended to a class of switched symmetric systems with time-delay described by equations of the form

$$\begin{cases} \dot{x}(t) = A_{r(t)} x(t) + \hat{A}_{r(t)} x(t-\tau) + B_{r(t)} u(t) \\ x(t) = \phi(t), \forall\, t \in [-\tau, 0] \\ y(t) = C_{r(t)} x(t) \end{cases} \tag{12}$$

where τ is the time delay in the state, $\phi(t)$ is the initial condition, $\hat{A}_i = \delta_i I_n (\delta_i$ are known scalars), and all the other notations are the same as in (1).

B. Construction of Stable Switching Signal

Now, we discuss the problem of construction of stable switching signal for switched symmetric systems.

Definition 3: [4] The system (3) is said to be quadratic stabilizable if and only if there exists a positive-definite function $V(x) = x^T P x$, a positive number ε and a switching signal $r(t)$ such that $\frac{d}{dt}V(x) < \varepsilon x^T x$ for all trajectories of the system (3).

Lemma 4: [4] For system (3), if there exist $\alpha_i > 0$, $i = 1, \cdots, N$, $\sum_{i=1}^{N} \alpha_1 = 1$, such that the LTI system $\dot{x} = \bar{A}x$ is asymptotically stable, where $\bar{A} = \sum_{i=1}^{N} \alpha_i A_i$, then there exists a switching signal $r(t) = r(x(t))$, only depended on system state, such that the system $\dot{x}(t) = A_{r(x(t))} x(t)$ is quadratically stable. Moreover, if $N = 2$, the condition is also necessary.

Theorem 2: For system (1)-(2), if there exist $\alpha_i > 0$, $i = 1, \cdots, N$, $\sum_{i=1}^{N} \alpha_1 = 1$, such that the symmetric LTI system $(\bar{A}, \bar{B}, \bar{C})$ satisfied

$$\bar{B}^{\perp} \bar{A} \bar{B}^{\perp T} < 0 \tag{13}$$

where $\bar{A} = \sum_{i=1}^{N} \alpha_i A_i$, $\bar{B} = \sum_{i=1}^{N} \alpha_i B_i$, and $\bar{C} = \sum_{i=1}^{N} \alpha_i C_i$, then there exist a symmetric matrix G and a switching signal $r(t) = r(x(t))$, only depended on system state, such that the closed-loop system

$$\dot{x}(t) = A_{r(x(t))} + B_{r(x(t))} G C_{r(x(t))} x(t) \tag{14}$$

is quadratically stable. Moreover, if $N = 2$, the condition is also necessary.

Proof: For system $(\bar{A}, \bar{B}, \bar{C})$, the condition $\bar{B}^{\perp} \bar{A} \bar{B}^{\perp T} < 0$ hold, by Lemma 1, there exists a symmetric output feedback matrix $G = G^T$ such that the closed-loop system $\dot{x}(t) = (\bar{A} + \bar{B} G \bar{C}) x(t)$ is asymptotically stable.Then by Lemma 3, there exists a switching signal $r(t) = r(x(t))$, only depended on system state, such that the system (14) is quadratically stable. ■

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