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Nonholonomic Motion Planning: Steering Using Bang-Bang Control

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***Abstract* – This paper studied the motion planning of nonholonomic systems. And a method based on bang-bang controls is proposed for n-dimensional chained form systems with two inputs in this paper. The method operates by switching between two kinds of inputs at time sequences to steer the system from initial configuration to an arbitrary final configuration. An algorithm is proposed to calculate time sequences of switching. At last, a numerical example and simulation results show the effectiveness of the proposed method.**

I. INTRODUCTION

Nonholonomic mechanical system is a class of mechanical systems subject to either nonholonomic constraints or non-integral constraints. In robotics, wheel mobile robot and mobile robot with n-trailers are typical examples of such systems. In the past few years, there are many research areas about control of such systems. Nonholonomic motion planning problem (NMPP) is one of these areas. This problem mainly concerned with obtaining open loop control, which steer a nonholonomic system from the initial configuration to final configuration over a finite time interval [7]. In contrast to holonomic system, the NMPP is difficult, because there are some motion constraints and these constraints are non-integral and cannot be transform into the generalized coordinate constraints.

In recent years, the motion planning for nonholonomic mechanical systems has been studied widely by many researchers in the world, and many methods have been proposed for this problem, such as sinusoidal input (Murray and Sastry [2]), multirate digital control (Monaco and Normand-Cyrot [3]), polynomial input (Tilbury, Murray and Sastry [4]) etc. These methods often utilized tools from differential geometric control theory, and in practice, it requires that the designers have some knowledge about different geometric. At the same time, using these methods to get the result for problem, it is often needed to calculate the integral of configurations based on u(t), and it is very difficult when the dimension is higher.

In this paper, a novel method based on bang-bang controls is proposed. It is proved that this method can steer the n-dimensional chained form systems from an initial configuration to an arbitrary final configuration. And an algorithm to calculate the sequence of switching time is given. And it needn't to calculate the integral of chained form with the bang-bang controls.
This paper is organized as follows. In section 2, we describe the strategy of steering based on bang-bang controls and prove that this strategy can steer n-dimensional chained form to an arbitrary final configuration. In section 3, an numerical example that steering a 5-dimensional chained form system from a given initial configuration to origin (final configuration) is shown. And simulation results are obtained using MATLAB/SIMULINK. Section 4 gives some concluding remarks.

II. MAIN RESULTS

In the following, we consider a subfamily of n-dimensional chained form systems with two inputs described as (1):

$$\dot{z}_1 = v_1, \dot{z}_2 = v_2, \dot{z}_3 = z_2 v_1, \cdots, \dot{z}_n = z_{n-1} v_1 \qquad (1)$$

In fact, many nonholonomic systems can be transformed into chained form via feedback transformation, such as n-trailer vehicle etc. Murray and Sastry have given sufficient conditions under which real systems can be converted into chained canonical form [2].

The following notation are employed: Ti, (i=1,2,…,2(n-2)+1) denotes each switching time, T0 denotes the start time and Tf denotes the final time, $\left|\xi_i\right|$, (i=1,2,…,2(n-2)+1) denoted the i th time-intervals of switching , and Ti =Ti-1+
$\left|\xi_i\right|$, Zstart and Zgoal denote a pair of initial and final configuration. We use superscript to denote the sequence of switching time and subscripts to denote the vector element.

In (1), **V**=[v1 v2] is the control input and is restricted to the space of bang-bang controls, in other words, the ith element of V, $v_i \in \{-1 \;\; 0 \;\; 1\}$. To steer this system using such a control input, we make a strategy with the following algorithm.

Algorithm 1:
Step 1: Let $T^i = T^0$;
Step 2:At $t = T^i$, Let $V = [0 \;\; \pm 1]$ and steer (1) until $t = T^{i+1}$;
Step 3: At $t = T^{i+1}$, if the final configuration is arrived, stop steering. Otherwise, let $V = [\pm 1 \;\; 0]$ and steer (1) until $t = T^{i+2}$;
Step 4: At $t = T^{i+2}$, if the final configuration is arrived, stop steering. Otherwise, let $T^i = T^{i+2}$ and return to step 2.

Proposition 1: Algorithm 1 can steer System (1) to an arbitrary configurations through $2 \times (n-2)$ switches mostly.

The proof of Proposition 1 is constructive, and it gives the Algorithm 2 that calculates the sequence of switching time.

Proof: In order to prove the proposition, we suppose that the number of switching is $2\times(n-2)$ and there are $2\times(n-2)+1$ time-intervals denoted by $|\xi_i|$, (i=1,2,…,2(n-2)+1) between T^0 and T^f.
According to Algorithm 1:
Step 2: Using $V=[0\ \pm 1]$ steer System (1) until t = T^1, and the sign of ξ_1 is the same as that of v_2. We can calculate the configuration of System (1) at T^1 presented by (2.1).

$$\begin{cases} z_1^1 = z_1^{start} \\ z_2^1 = z_2^{start} + \xi_1 \\ z_3^1 = z_3^{start} \\ \quad\vdots \\ z_k^1 = z_k^{start} \\ \quad\vdots \\ z_n^1 = z_n^{start} \end{cases} \tag{2.1}$$

Step 3: Using $\mathbf{V}=[\pm 1\quad 0]$ steer System (1) until t = T^2, and the sign of ξ_2 is the same as that of v_1. We also can calculate the configuration of System (1) at T^2 presented by (2.2).

$$\begin{cases} z_1^2 = z_1^1 + \xi_2 \\ z_2^2 = z_2^1 \\ z_3^2 = z_3^1 + z_2^2\xi_2 \\ \quad\vdots \\ z_k^2 = z_k^1 + \sum\limits_{i=1}^{k-2} \dfrac{z_{k-i}^1(\xi_2)^i}{i!} \\ \quad\vdots \\ z_n^2 = z_n^1 + \sum\limits_{i=1}^{n-2} \dfrac{z_{n-i}^1(\xi_2)^i}{i!} \end{cases} \tag{2.2}$$

Repeat Step 2 and Step 3 of Algorithm 1 till t = $T^{2(n-2)+1}$. At last, we can calculate the configurations of System (1) at T^i presented by (2.i), (i = 1,2,…,2(n-2)+1).

$$\begin{cases} z_1^{goal} = z_1^{2(n-2)+1} = z_1^{2(n-2)} \\ z_2^{goal} = z_2^{2(n-2)+1} = z_2^{2(n-2)} + \xi_{2(n-2)+1} \\ z_3^{goal} = z_3^{2(n-2)+1} = z_3^{2(n-2)} \\ \quad\vdots \\ z_k^{goal} = z_k^{2(n-2)+1} = z_k^{2(n-2)} \\ \quad\vdots \\ z_n^{goal} = z_n^{2(n-2)+1} = z_n^{2(n-2)} \end{cases} \tag{2.2(n-2)+1}$$

If we substitute (2.k) into (2.k+1), (k =1,2,…,2(n-2)), we can obtain (3) :

$$\begin{cases} z_1^{goal} = \sum\limits_{i=1}^{n-2}\xi_{2i} + z_1^{start} \\ z_2^{goal} = \sum\limits_{i=1}^{n-1}\xi_{2i-1} + z_2^{start} \\ z_3^{goal} = \sum\limits_{i=1}^{n-2}\left(\sum\limits_{j=i}^{n-2}\xi_{2j}\right)\xi_{2i-1} + \sum\limits_{i=1}^{1}\left(\sum\limits_{j=1}^{n-2}\xi_{2j}\right) z_{i+1}^{start} + z_3^{start} \\ \vdots \\ z_n^{goal} = \sum\limits_{i=1}^{n-2}\dfrac{\left(\sum\limits_{j=i}^{n-2}\xi_{2j}\right)^{(n-2)}}{(n-2)!}\xi_{2i-1} + \sum\limits_{i=1}^{n-2}\dfrac{\left(\sum\limits_{j=1}^{n-2}\xi_{2j}\right)^{(n-i-1)}}{(n-i-1)!} z_{i+1}^{start} + z_n^{start} \end{cases} \tag{3}$$

In order to prove Proposition 1, we must show:

- There is $\boldsymbol{\xi} = \left[\xi_1, \xi_2, \cdots, \xi_{2(n-2)+1}\right]$ satisfying (3).

Now we can obtain one of ξ using Algorithm 2.
Algorithm 2:
Step 1: Separate $\boldsymbol{\xi} = \left[\xi_1, \xi_2, \cdots, \xi_{2(n-2)+1}\right]$ into three vectors:
$\boldsymbol{\xi_1} = \left[\xi_1, \xi_3, \cdots, \xi_{2(n-2)-1}\right]$, $\boldsymbol{\xi_2} = \left[\xi_2, \xi_4, \cdots, \xi_{2(n-2)}\right]$, $\boldsymbol{\xi_3} = \left[\xi_{2(n-2)+1}\right]$;
Step 2: Specify $\boldsymbol{\xi_2}$ by random and $\boldsymbol{\xi_2}$ must be satisfied with: $\sum\limits_{i=1}^{n-2}\xi_{2i} = Z_1^{goal} - Z_1^{start}$, $\prod\limits_{i=1}^{n-2}\xi_{2i} \neq 0$;
Step 3: Calculate $\boldsymbol{\xi_1}$ by $\boldsymbol{\xi_1} = \mathbf{A}^{-1}\cdot\mathbf{B}$ where

$$\mathbf{A} = \begin{bmatrix} \sum\limits_{i=1}^{n-2}\xi_{2i} & \sum\limits_{i=2}^{n-2}\xi_{2i} & \cdots & \sum\limits_{i=n-2}^{n-2}\xi_{2i} \\ \dfrac{\left(\sum\limits_{i=1}^{n-2}\xi_{2i}\right)^2}{2} & \dfrac{\left(\sum\limits_{i=2}^{n-2}\xi_{2i}\right)^2}{2} & \cdots & \dfrac{\left(\sum\limits_{i=n-2}^{n-2}\xi_{2i}\right)^2}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\left(\sum\limits_{i=1}^{n-2}\xi_{2i}\right)^{n-2}}{(n-2)!} & \dfrac{\left(\sum\limits_{i=2}^{n-2}\xi_{2i}\right)^{n-2}}{(n-2)!} & \cdots & \dfrac{\left(\sum\limits_{i=n-2}^{n-2}\xi_{2i}\right)^{n-2}}{(n-2)!} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} z_3^{goal} - z_3^{start} - \sum\limits_{i=1}^{1}\left(\sum\limits_{j=1}^{n-2}\xi_{2j}\right) z_{i+1}^{start} \\ z_4^{goal} - z_4^{start} - \sum\limits_{i=1}^{2}\dfrac{\left(\sum\limits_{j=1}^{n-2}\xi_{2j}\right)^{(4-i-1)}}{(4-i-1)!} z_{i+1}^{start} \\ \vdots \\ z_n^{goal} - z_n^{start} - \sum\limits_{i=1}^{n-2}\dfrac{\left(\sum\limits_{j=1}^{n-2}\xi_{2j}\right)^{(n-i-1)}}{(n-i-1)!} z_{i+1}^{start} \end{bmatrix}$$

And we know that the inverse of A exists if and only if $\xi_{2i} \neq 0, (i=1,2,(n-2))$.

Step 4: Calculate $\boldsymbol{\xi}_3$ by $\xi_{2(n-2)+1} = z_2^{goal} - z_2^{start} - \sum_{i=1}^{n-2} \xi_{2i-1}$.

Step 5: Calculate the sequence of switching time:

$$T^i = T^{i-1} + |\xi_i|, \ (i = 1,2,\ldots,2(n-2)+1)$$

According to Algorithm 2, we know that the solution of (3) exists if and only if the appropriate $\boldsymbol{\xi}_2$ was chosen. And if all the elements of $\boldsymbol{\xi}_1$ are unequal to zero, the number of switching is 2(n-2). If there are m elements of $\boldsymbol{\xi}_1$ equal to zero, the number of switching is 2(n-m-2).

III. EXAMPLE

In this section, we consider a 5-dimensional chained form system as follows:

$$\dot{z}_1 = v_1, \dot{z}_2 = v_2, \dot{z}_3 = z_2 v_1, \dot{z}_4 = z_3 v_1, \dot{z}_5 = z_4 v_1 \qquad (4)$$

In practice, this chained form system can be transformed from the mobile robot with two trailers, See Figure 1. [1]

Now our problem is how to find the control input: $v_1(t)$ and $v_2(t)$ steering (4) from the initial configuration: Z^{start}=[5 5 5 5 5] to the final configuration: Z^{goal}=[0 0 0 0 0].

Now, we use the method above proposed.

Step 1: Initial the interval $\xi_i (i = 1,2,\cdots,m)$. And m can be computed with:

$$m = 2\times(n-2)+1 = 7.$$

Step 2: Specify $\xi_{2i}, (i = 1,2,3)$ by random, and must be satisfied with:

$$\sum_{i=1}^{n-2} \xi_{2i} = Z_0^{goal} - z_0^{start}, \ (\xi_{2i} \neq 0)$$

Step 3: Calculate the matrix A and vector B:

$$\mathbf{A} = \begin{bmatrix} \sum_{i=1}^{3}\xi_{2i} & \sum_{i=2}^{3}\xi_{2i} & \sum_{i=3}^{3}\xi_{2i} \\ \dfrac{\left(\sum_{i=1}^{3}\xi_{2i}\right)^2}{2} & \dfrac{\left(\sum_{i=2}^{3}\xi_{2i}\right)^2}{2} & \dfrac{\left(\sum_{i=3}^{3}\xi_{2i}\right)^2}{2} \\ \dfrac{\left(\sum_{i=1}^{3}\xi_{2i}\right)^3}{3!} & \dfrac{\left(\sum_{i=2}^{3}\xi_{2i}\right)^3}{3!} & \dfrac{\left(\sum_{i=3}^{3}\xi_{2i}\right)^3}{3!} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} Z_3^{goal} - Z_3^{start} - (Z_2^{start} \quad 0 \quad 0)\cdot \mathbf{A}(:,1) \\ Z_4^{goal} - Z_4^{start} - (Z_3^{start} \quad Z_2^{start} \quad 0)\cdot \mathbf{A}(:,1) \\ Z_5^{goal} - Z_5^{start} - (Z_4^{start} \quad Z_3^{start} \quad Z_2^{start})\cdot \mathbf{A}(:,1) \end{bmatrix}$$

where $\mathbf{A}(:,1)$ denotes the Column 1 of the matrix **A**.

Step 4: Compute $\xi_{2i-1}, (i = 1,2,3)$ with:

$$\boldsymbol{\xi} = \mathbf{A}^{-1} \cdot \mathbf{B}$$

where $\boldsymbol{\xi} = [\xi_1 \quad \xi_3 \quad \xi_5]^T$

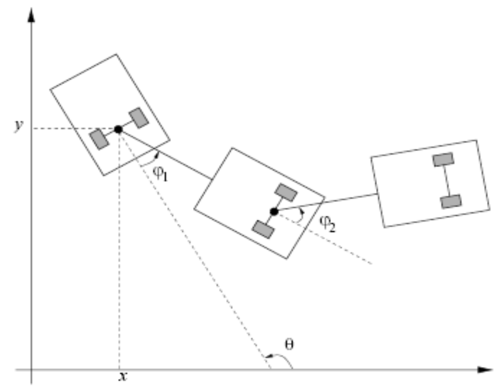

Fig.1 A mobile robot with two trailers

Step 5: Calculate the switch time T_i:

$$\begin{cases} T^0 = 0 \\ T^i = T^{i-1} + |\xi_i|, (i = 1,2,\cdots,7) \end{cases}$$

and v_1, v_2:

$$v_1 = \begin{cases} 0, (t \in [T^{2i} \quad T^{2i+1}), (i = 0,1,2,3)) \\ \operatorname{sgn}(\xi_{2i}), (t \in [T^{2i-1} \quad T^{2i}), (i = 1,2,3)) \end{cases}$$

$$v_2 = \begin{cases} 0, (t \in [T^{2i-1} \quad T^{2i}), (i = 1,2,3)) \\ \operatorname{sgn}(\xi_{2i+1}), (t \in [T^{2i} \quad T^{2i+1}), (i = 0,1,2,3)) \end{cases}$$

Here, a numerical result is shown below.

ξ =[-1.333 –1 –3.125 –2 –0.417 –2 –0.125]

T = [0 1.333 2.333 5.458 7.458 7.875 9.875 10]

v_1 = [0 –1 0 –1 0 –1 0]

v_2 = [–1 0 –1 0 –1 0 –1]

And the computer simulations are carried out using MATLAB/SIMULINK and are presented to show the effectiveness of the results. Fig.2 and Fig.3 show, respectively, the control input: v_1, v_2 and the trajectories of configuration variables in chained form systems (4).

IV. CONCLUSION

A novel method based on bang-bang controls is proposed in this paper. This method can be used in solving the motion planning problem for nonholonomic systems which can be transformed into chained form. It is proved that can steer an n-dimensional chained form system between an arbitrary pair of initial and final configuration. And a numerical example and simulation results show the effectiveness of proposed method. Future research directions include how to deduce the number of switching, how to get the time optimal switching and how to extend this method to multi-chained form systems.

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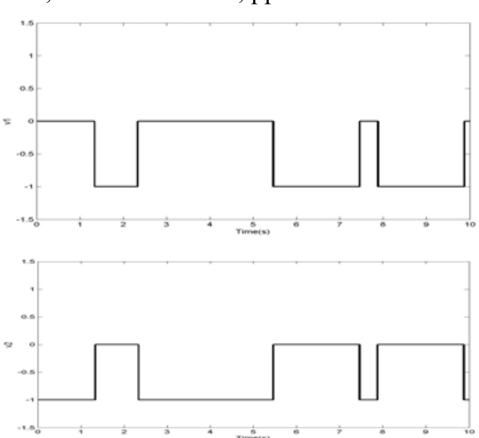

Fig.2 Control Input: v_1, v_2

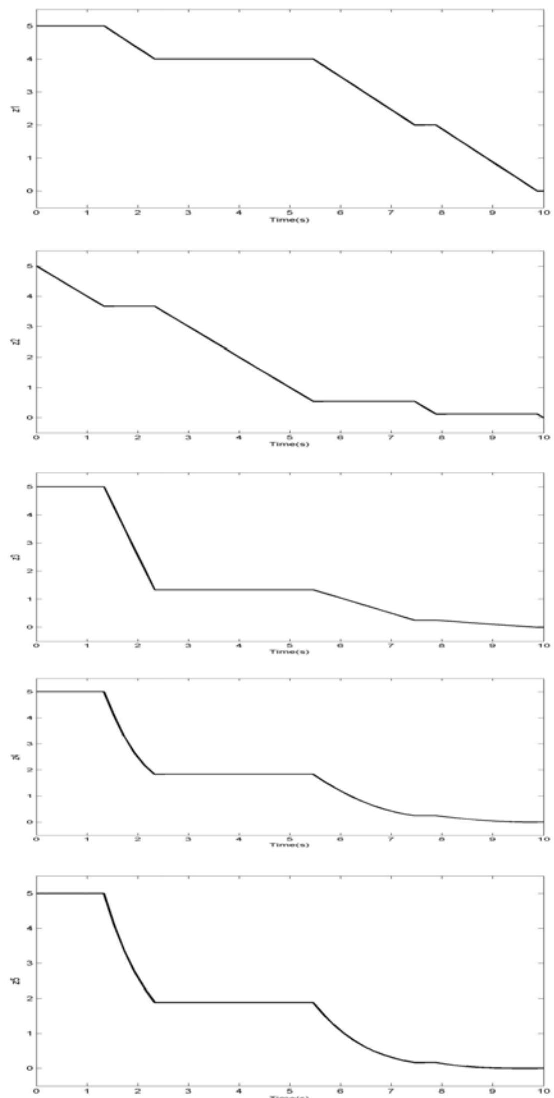

Fig.3 Trajectory of configuration variables $[z_1\ z_2\ z_3\ z_4\ z_5]$ in chained form system (6)