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Closed Form Minimax Time-Delay Filters for Underdamped Systems

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Abstract— **This paper derives closed form solutions for the parameters of a time-delay filter designed to be robust to uncertainties in frequencies to be cancelled. It is shown that the slope of the magnitude plot of the two time-delay filter is zero at the nominal frequency indicating that it is a local maximum. This information is used for deriving the solution of the parameters of the time-delay filter in closed form. Three time-delay filters are also designed which force a zero of the filter to be located at the nominal frequency of the system. The applicability of the proposed technique for the control of multi-mode systems is also illustrated.**

I. INTRODUCTION

Vibration attenuation by shaping input to underdamped systems has been addressed by numerous researchers [1], [2], [3], [4] besides others. There has been an increased interest in the development of techniques to desensitize the controllers to uncertainties in the system model. Singer and Seering [2] proposed a technique to design a sequence of impulses with the objective of forcing the variation of the sensitivity of the residual energy of the system with respect to modeled damping or frequency to zero. The resulting controllers were called *Input Shapers*. Singh and Vadali [3] illustrated that robustness to modeling errors can be achieved by cascading multiple time-delay filters designed to cancel the nominal poles of the system. Singhose et al. [5], proposed a technique which they referred to as Multi-Hump Extra-Insensitive Input shaper where they determine the amplitudes of the impulses so as to maximize the uncertain domain where the residual vibration is below a specified threshold. Singh [6] proposed an optimization problem where the maximum magnitude of the residual energy in an uncertain domain is minimized. In this paper, the minimax problem is first addressed for a single mode system. In numerous applications, there exists one dominant mode which is the main contributor to the residual energy of the maneuvering structure. Thus, there is a motivation to derive the optimal time-delay filter which minimizes the maximum magnitude of the transfer function of the time-delay filter, in closed form. A simple technique for handling multiple modes is also proposed. The resulting filter is designed by addressing the problem as a series of single mode problems.

Section 2 focuses on the development of closed form solutions for the parameters of minimax time-delay filters. The development is then modified to permit differential weighting of the limiting and nominal frequencies. In Section 3, a simple approach is proposed to permit using the solution of the undamped systems to system with damping. The approach to design minimax filters for multimode systems is described in Section 4. The paper concludes with some remarks in the final section.

II. OPTIMAL MINIMAX FILTERS FOR UNDAMPED SYSTEMS

A. Two Time-Delay Filter

A time-delay filter to modify the reference input to a system to attenuate residual vibrations is shown in Figure 1.

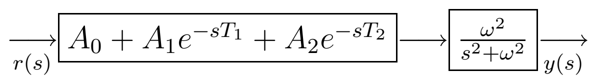

Fig. 1. Time-Delay Filter

The transfer function of the time-delay filter is given by the equation

$$A_0 + A_1 exp(-sT_1) + A_2 exp(-sT_2) \tag{1}$$

which is used to prefilter inputs to a system characterized by un-damped modes. The frequency of the mode to be cancelled is uncertain, but the region of uncertainty is known. Assume that the nominal frequency is ω_0 and the uncertain frequency ω lies in the range

$$\omega_l \leq \omega \leq 2\omega_0 - \omega_l \tag{2}$$

implying that the uncertainty is symmetric about the nominal frequency. To minimize, the maximum magnitude of the magnitude plot of the time-delay filter in the region of uncertainty, we require that the magnitude of the time-delay filter at the boundary be equal to that at the nominal frequency. Assuming further that

$$T_1 = \frac{\pi}{\omega_0}, \text{ and } T_2 = \frac{2\pi}{\omega_0}, \tag{3}$$

the magnitude of the transfer function of the time-delay filter can be shown to be

$$F(\omega) = A_0^2 + A_1^2 + A_2^2 + 2A_0A_1cos(\frac{\omega}{\omega_0}\pi) + 2A_0A_2cos(\frac{2\omega}{\omega_0}\pi) + 2A_1A_2cos(\frac{\omega}{\omega_0}\pi). \tag{4}$$

The location of the maximum of $F(\omega)$ can be determined from the equation

$$\frac{dF(\omega)}{d\omega} = -2A_0A_1\frac{\pi}{\omega_0}sin(\frac{\omega}{\omega_0}\pi) - 2A_0A_2\frac{2\pi}{\omega_0}sin(\frac{2\omega}{\omega_0}\pi) -$$

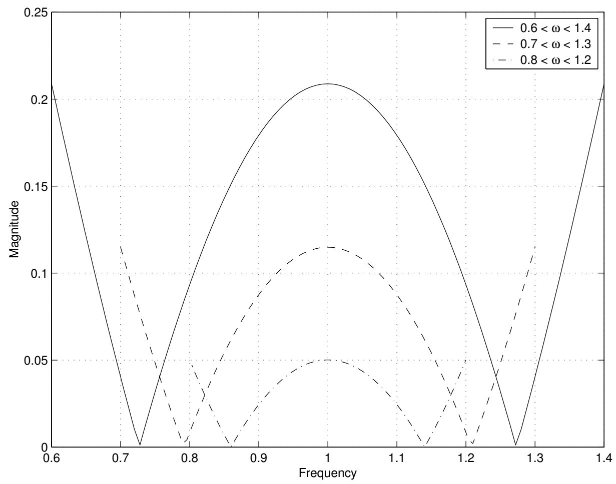

Fig. 2. Sensitivity Curve

$$2A_1A_2\frac{\pi}{\omega_0}sin(\frac{\omega}{\omega_0}\pi) = 0. \quad (5)$$

It can be seen that $\omega = \omega_0$ satisfies Equation 5. To determine the parameters of the minimax time-delay filter, the magnitude of the time-delay filter at the boundary and the nominal frequency are equated, resulting in the equation

$$(A_0 + A_1cos(\frac{\omega_l}{\omega_0}\pi) + A_2cos(\frac{2\omega_l}{\omega_0}\pi))^2 + (A_1sin(\frac{\omega_l}{\omega_0}\pi)+ A_2sin(\frac{2\omega_l}{\omega_0}\pi))^2 = (A_0 + A_1cos(\pi) + A_2cos(2\pi))^2. \quad (6)$$

For undamped systems, we can assume that A_2 is equal to A_0. We also require the constraint

$$A_0 + A_1 + A_2 = 1 \quad (7)$$

to be satisfied. Solving for A_0, we have

$$\frac{1}{A_0} = 2 + \frac{1}{2}\frac{1 - cos(\frac{2\omega_l}{\omega_0}\pi)}{1 + cos(\frac{\omega_l}{\omega_0}\pi)}. \quad (8)$$

Calculating the magnitude of the time-delay filter at $\omega = 2\omega_0 - \omega_l$ using the solution for A_0 given by Equation 8, we can show that it is equal to that at the nominal frequency ω_0. Thus, the magnitude of the time-delay filter at the two limits of the uncertain frequency range and the nominal frequency are the same.

Figure 2 illustrates the variation of the magnitude of the transfer function of the time-delay filter which is referred to as the sensitivity curve, for different uncertain regions. It is clear from the figure that as the uncertain region decreases, the maximum magnitude of the sensitivity curve becomes smaller.

Since, the magnitude of the sensitivity plot at the nominal frequency is

$$(A_0 + A_1cos(\pi) + A_2cos(2\pi)) = (A_0 - A_1 + A_0) = (-1 + 4A_0) = \frac{1 + cos(\frac{\pi\omega_l}{\omega_0})}{3 - cos(\frac{\pi\omega_l}{\omega_0})}, \quad (9)$$

TABLE I

ANALYTICAL AND NUMERICAL MINIMAX SOLUTIONS

Uncertain Range	Numerical Cost	Closed Form Cost
$0.6 < \omega < 1.4$	0.208818210	0.208818210
$0.7 < \omega < 1.3$	0.114893930	0.114893930
$0.8 < \omega < 1.2$	0.050139709	0.050139711

the uncertain region can be solved given a permissible maximum magnitude of the sensitivity curve. For a given magnitude M, the lower bound of the uncertain frequency range is

$$\omega_l = \frac{\omega_0}{\pi}cos^{-1}(\frac{3M-1}{M+1}). \quad (10)$$

or the width of the uncertainty is given by the equation

$$2(\omega_0 - \omega_l) = 2\omega_0(1 - \frac{1}{\pi}cos^{-1}(\frac{3M-1}{M+1})). \quad (11)$$

It is also clear from Equation 9 that A_0 lies in the range

$$\frac{1}{4} \leq A_0 \leq \frac{1}{2} \qquad \Rightarrow \qquad 0 \leq M \leq 1, \quad (12)$$

since the magnitude of the the sensitivity curve should not be greater than 1 at the nominal frequency.

The location of poles for the minimax filter can be solved for by substituting the closed form solutions for the parameters of the time-delay filter. The equations

$$A_0 + A_1cos(\omega T) + A_2cos(2\omega T) = 0, \quad (13)$$

$$A_1sin(\omega T) + A_2sin(2\omega T)) = 0, \quad (14)$$

where $T = \pi/\omega_0$, have to be solved to determine the locations of the zeros of the time-delay filter. Equation 14 can be rewritten as

$$(1 - 2A_0)sin(\omega T) + A_0sin(2\omega T) = sin(\omega T)(1 - 2A_0 + 2A_0cos(\omega T)) = 0. \quad (15)$$

which can be solved resulting in the equation

$$cos(\omega T) = \frac{2A_0 - 1}{2A_0}. \quad (16)$$

which satisfies Equation 13 as well. Thus the zeros of the time delay filter are located at

$$\omega = \pm\frac{1}{T}cos^{-1}(\frac{2A_0-1}{2A_0}) + \frac{2n\pi}{T} = \pm\frac{\omega_0}{\pi}cos^{-1}(\frac{2A_0-1}{2A_0}) + 2n\omega_0 \quad (17)$$

where n is an integer.

The proposed approach is compared to a numerical minimax optimization approach for three uncertain intervals and the results are tabulated in Table I. It can be seen that the difference between the numerical and the proposed approach is negligible.

B. Three Time-Delay Filter

It can be seen from the two time-delay filter that the magnitude of the time-delay filter transfer function at the nominal frequency is nonzero. To force the magnitude at the nominal frequency to zero, a three time-delay filter is proposed where the time-delays are assumed to be

$$T_1 = \frac{\pi}{\omega_0}, T_2 = \frac{2\pi}{\omega_0}, \text{ and } T_3 = \frac{3\pi}{\omega_0}. \quad (18)$$

Further, assuming that

$$A_0 = A_3, \text{ and } A_1 = A_2, \quad (19)$$

and with the requirement that

$$A_0 + A_1 + A_2 + A_3 = 1, \quad (20)$$

it can be shown that the magnitude of the transfer function of the time-delay filter

$$F(\omega) = 8cos^3(\frac{\omega\pi}{\omega_0})A_0^2 + (4A_0 - 8A_0^2)cos^2(\frac{\omega\pi}{\omega_0}) + (\frac{1}{2} - 8A_0^2)cos(\frac{\omega\pi}{\omega_0}) + \frac{1}{2} - 4A_0 + 8A_0^2 \quad (21)$$

is 0 at $\omega = \omega_0$, the nominal frequency. The location of the extremum of the sensitivity curve can be shown to be at

$$\omega = \omega_0 \text{ and } \omega = \frac{\omega_0}{\pi}cos^{-1}\left(\frac{4A_0 - 1}{4A_0}\right) + 2n\omega_0 \text{ n=1,2,3...} \quad (22)$$

which corresponds to the minimum at the nominal frequency. and

$$\omega = \frac{\omega_0}{\pi}cos^{-1}\left(-\frac{4A_0 + 1}{12A_0}\right) + 2n\omega_0 \text{ n=1,2,3...} \quad (23)$$

which corresponds to the maximum. The magnitude of the sensitivity curve at the maximum is

$$F\left(\omega = \frac{\omega_0}{\pi}cos^{-1}\left(-\frac{4A_0 + 1}{12A_0}\right)\right) = \frac{512A_0^3 - 192A_0^2 + 24A_0 - 1}{54A_0}. \quad (24)$$

Equating the magnitude of the transfer function of the time-delay filter at the lower limiting frequency to the maximum, we have

$$8cos^3(\frac{\omega_l\pi}{\omega_0})A_0^2 + (4A_0 - 8A_0^2)cos^2(\frac{\omega_l\pi}{\omega_0}) + (\frac{1}{2} - 8A_0^2)cos(\frac{\omega_l\pi}{\omega_0}) + \frac{1}{2} - 4A_0 + 8A_0^2 = \frac{512A_0^3 - 192A_0^2 + 24A_0 - 1}{54A_0}, \quad (25)$$

we can solve the cubic equation for A_0, resulting in the solutions

$$A_0 = -\frac{1}{3cos(\frac{\omega_l\pi}{\omega_0}) - 5}, A_0 = -\frac{1}{4(1 + 3cos(\frac{\omega_l\pi}{\omega_0}))} \text{ and } A_0 = -\frac{1}{4(1 + 3cos(\frac{\omega_l\pi}{\omega_0}))}. \quad (26)$$

The second and the third solutions which are identical force the boundary to be a maximum resulting in a suboptimal minimax filter. The first solution results in two maxima which lie within the uncertain interval resulting in the optimal minimax solution. Figure 3 illustrates the variation of the gains of the time-delay filter as a function of the normalized uncertain interval. It can be seen that the amplitudes are always positive as in the two time-delay case. Figure 4 illustrates the variation of the sensitivity curve for different uncertain regions. Compared to Figure 2, it can be seen that the maximum magnitude of the three time-delay filter is significantly smaller than the two time-delay filter. Figure 5 illustrates the variation of the uncertain region as a function of permissible maximum magnitude of the two and three time-delay filter. It is clear that as the maximum permissible magnitude of the time-delay filter M, is increased, the uncertain region monotonically increases. However, the rate of increase of the uncertain region of the three time-delay filter (dashed line) is large compared to the two time-delay filter in the vicinity of zero permissible magnitude. This implies that for small permissible magnitude, the uncertain

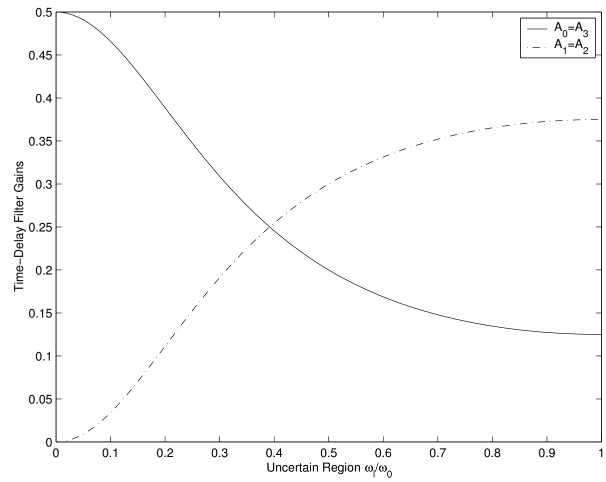

Fig. 3. Three Time-Delay Filter Parameters

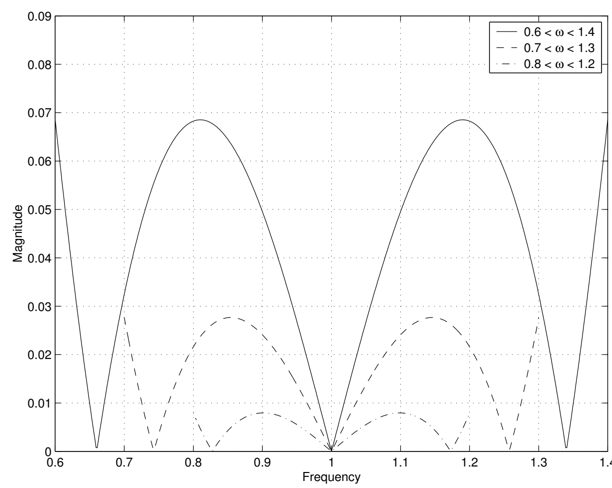

Fig. 4. Sensitivity Curve

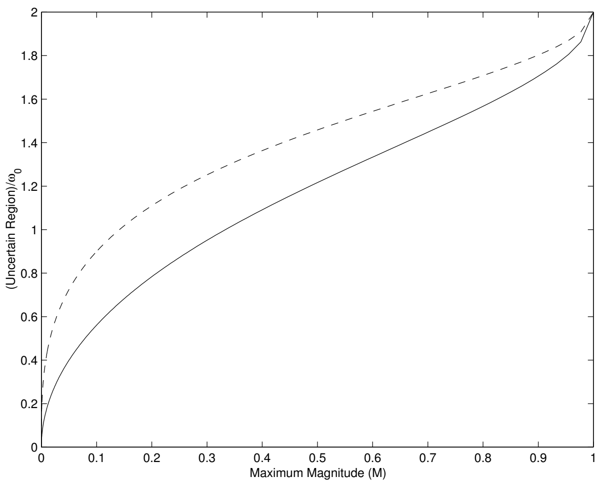

Fig. 5. Uncertain Range vs. Permissible Magnitude

region of the three time-delay filter is large.

III. Damped Systems

To design minimax filters for systems characterized by under-damped behaviour, a simple approach is proposed which uses a transformation to represent the damped system as a undamped system in the new space. For a under-damped system given by the transfer function

$$G(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}, \tag{27}$$

we define a transformation

$$s = p - \zeta\omega. \tag{28}$$

Substituting Equation 28 into Equation 27, we have

$$G(p) = \frac{\omega^2}{p^2 + \omega^2(1-\zeta^2)}, \tag{29}$$

which represents an undamped system with a natural frequency of $\omega\sqrt{1-\zeta^2}$. The closed form solution derived earlier for undamped systems can now be used for this transformed system. The time-delay filter can then be transformed back into the original space to arrive at the minimax filter for the damped system.

For instance, the transfer function of a time-delay filter designed to cancel the poles of an undamped system (Equation 29) is given by the equation

$$F(p) = 1 + exp(-p\frac{\pi}{\omega\sqrt{1-\zeta^2}}) \tag{30}$$

as shown by Singh and Vadali [3]. Transforming this filter into the original space, we have

$$F(s) = 1 + exp(-(s+\zeta\omega)\frac{\pi}{\omega\sqrt{1-\zeta^2}}) \tag{31}$$

which can be rewritten as

$$\begin{aligned} F(s) =& exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)\left(exp\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)\right. \\ &\left. + \, exp\left(-s\frac{\pi}{\omega\sqrt{1-\zeta^2}}\right)\right) \end{aligned} \tag{32}$$

The requirement that the final value of the output of the time-delay filter subject to a unity step input, be unity, requires scaling of the gains of the time-delay filter, resulting in the solution

$$F(s) = \frac{exp(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}) + exp(-s\frac{\pi}{\omega\sqrt{1-\zeta^2}})}{exp(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}) + 1} \tag{33}$$

which is identical to the time-delay filter designed to cancel the damped poles as shown by Singh and Vadali [3].

Therefore, the closed form solution for the gains of the two time-delay minimax filter for a damped system with uncertainty in frequency is given by the equations

$$A_0 = \frac{2(1+cos(\frac{\omega_l}{\omega_0}\pi))}{5+4cos(\frac{\omega_l}{\omega_0}\pi) - cos(\frac{2\omega_l}{\omega_0}\pi)} exp(\frac{2\zeta\pi}{\sqrt{1-\zeta^2}}) \tag{34}$$

$$A_1 = \left(1 - 2\frac{2(1+cos(\frac{\omega_l}{\omega_0}\pi))}{5+4cos(\frac{\omega_l}{\omega_0}\pi) - cos(\frac{2\omega_l}{\omega_0}\pi)}\right) exp(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}) \tag{35}$$

and

$$A_2 = \frac{2(1+cos(\frac{\omega_l}{\omega_0}\pi))}{5+4cos(\frac{\omega_l}{\omega_0}\pi) - cos(\frac{2\omega_l}{\omega_0}\pi)} \tag{36}$$

and the delay times are

$$T_1 = \frac{\pi}{\omega\sqrt{1-\zeta^2}} \text{ and } T_2 = \frac{2\pi}{\omega\sqrt{1-\zeta^2}}. \tag{37}$$

and the parameters of the three time-delay minimax filter for a damped system are given by the equations

$$A_0 = \frac{1}{5 - 3cos(\frac{\omega_l\pi}{\omega_0})} exp(\frac{3\zeta\pi}{\sqrt{1-\zeta^2}}) \tag{38}$$

$$A_1 = \frac{1}{2}\left(1 - 2\frac{1}{5-3cos(\frac{\omega_l\pi}{\omega_0})}\right) exp(\frac{2\zeta\pi}{\sqrt{1-\zeta^2}}) \tag{39}$$

$$A_2 = \frac{1}{2}\left(1 - 2\frac{1}{5-3cos(\frac{\omega_l\pi}{\omega_0})}\right) exp(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}) \tag{40}$$

$$A_3 = \frac{1}{5 - 3cos(\frac{\omega_l\pi}{\omega_0})} \tag{41}$$

$$T_1 = \frac{\pi}{\omega\sqrt{1-\zeta^2}}, T_2 = \frac{2\pi}{\omega\sqrt{1-\zeta^2}} \text{ and } T_3 = \frac{3\pi}{\omega\sqrt{1-\zeta^2}}. \tag{42}$$

The gains A_0, A_1, A_2 and A_3 have to be normalized which is achieved by dividing each of the gains by $\sum_i A_i$

Equations 34-37 and 38-42 are the exact minimax solutions for an uncertain regions which lies along the vertical line in the complex plane passing through the nominal

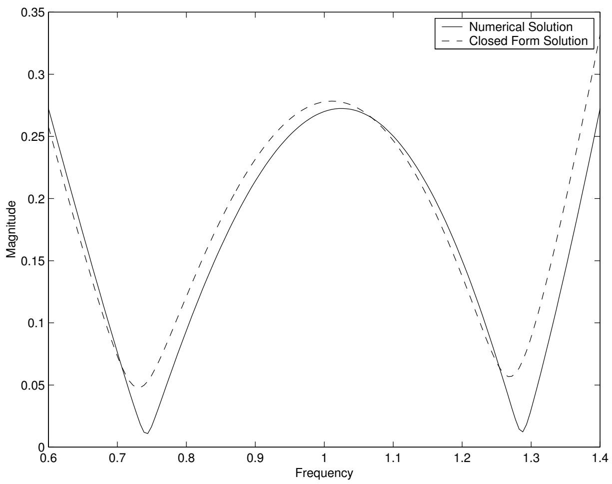

Fig. 6. Filter Comparison

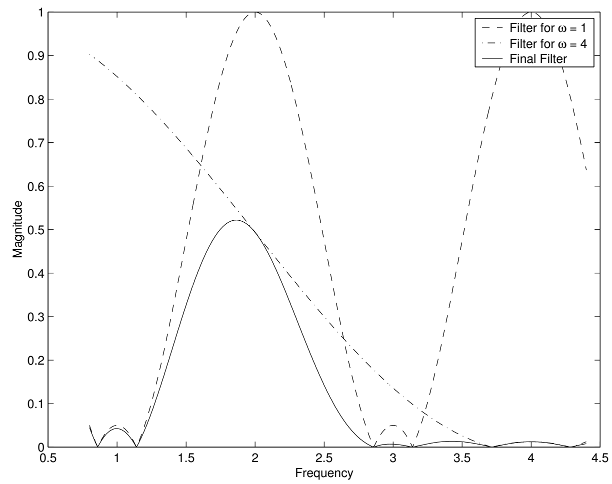

Fig. 7. Robust Minimax Control: Multi-Mode System

damped poles. This implies that both the damping ratio and the natural frequency are varying. The optimal solution when uncertainty exists in the estimated natural frequency implies that the uncertain region is along a straight line passing through the damped nominal pole and the origin. Figure 6 illustrates the difference between the numerical and the closed form solution for a system with a nominal damping ratio of 0.1 and uncertainty in the natural frequency . The difference between the numerical and closed form increases with increasing uncertainty and it is clear that the difference is not large and occurs at the limit of the uncertain region.

IV. Multi-Mode Systems

The proposed approach can be used to design robust filters for systems whose transfer function includes multiple modes. Time-delay filters are designed for each uncertain frequency and they are subsequently convolved to arrive at a time-delay filter which is robust to all uncertain frequencies.

To illustrate the proposed technique, assume that the two frequencies to be attenuated lie in the range

$$0.8 \leq \omega_1 \leq 1.2, \text{ and } 3.6 \leq \omega_2 \leq 4.4, \tag{43}$$

and the nominal frequencies are selected to be at the midpoint of the uncertain regions. The transfer functions of the minimax time-delay filters for each of the frequencies are

$$F_1(s) = 0.2625 + 0.4749exp(-\pi s) + 0.2625exp(-2\pi s) \tag{44}$$

and

$$F_2(s) = 0.2531 + 0.4938exp(-\frac{\pi}{4}s) + 0.2531exp(-\frac{2\pi}{4}s). \tag{45}$$

Figure 7 illustrates the magnitude plot of the minimax filters. The dash line and the dash-dot lines are the magnitude plots of the filters $F_1(s)$ and $F_2(s)$ respectively. The solid line is the magnitude plot of the final filter which is

$$F(s) = F_1(s)F_2(s). \tag{46}$$

It can be seen that the filter $F(s)$ is robust to uncertainties in both the frequencies.

V. Performance Analysis

The proposed technique minimizes the maximum magnitude of the transfer function of a time-delay filter given knowledge of the uncertain domain of the under-damped mode. From a practical viewpoint, the residual energy of the maneuvering structure is the metric of interest to control designers. It is therefore of interest to determine how the proposed technique for the design of prefilters compares to the minimax filter designed to minimize the maximum residual energy over the uncertain domain, a technique developed by Singh [6]. The residual energy defined by

$$F = \frac{1}{2}\dot{y}^T M \dot{y} + \frac{1}{2}y^T K y \tag{47}$$

where M and K are the mass and stiffness matrices and $\dot{y}$ and y are the velocity and position vectors of the system in consideration, is used to compare the performance of the two filters.

The two time-delay minimax filter which minimizes the maximum magnitude of the residual energy is

$$G(s) = 0.2439 + 0.4554e^{-3.0527s} + 0.3007e^{-2(3.0527)s} \tag{48}$$

and the minimax filter using the closed form solution proposed in this work is

$$G(s) = 0.2787 + 0.4426e^{-\pi s} + 0.2787e^{-2\pi s} \tag{49}$$

where the uncertain frequency ω lies in the range

$$0.7 \leq \omega \leq 1.3. \tag{50}$$

It can be seen that the time-delay of the filter which minimizes the residual energy is smaller compared to that of the closed form solution.

Figure 8 illustrates the variation of the square root of the residual energy over the uncertain frequency for a two time-delay filter. It can be seen that the minimax filter designed

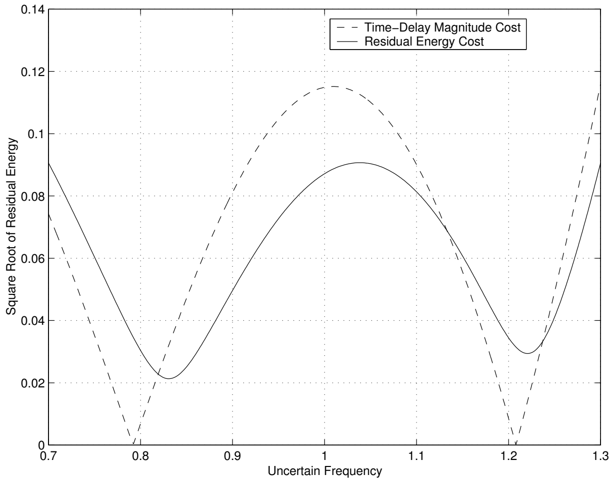

Fig. 8. Filter Performance: 2 Time-Delay Filter

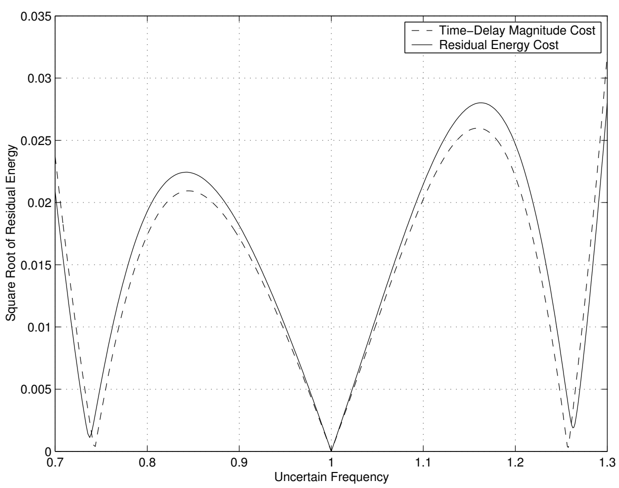

Fig. 9. Filter Performance: 3 Time-Delay Filter

using the residual energy as the cost, performs better than the proposed filter.

The next filter which constrains the residual energy to be zero at the nominal frequency results in the filters

$$G(s) = 0.15 + 0.35e^{-3.1415s} + 0.35e^{-2(3.1415)s} + 0.15e^{-3(3.1415)s} \tag{51}$$

for the residual energy cost and

$$G(s) = 0.1479 + 0.3521e^{-\pi s} + 0.3521e^{-2\pi s} + 0.1479e^{-3\pi s} \tag{52}$$

for the cost proposed in this work. Figure 9 illustrates that the difference between the two filters is not very significant with the proposed prefilter providing a performance which is better than the filter designed to minimize the maximum residual energy, over a large segment of the uncertain range. The benefit of the proposed technique is that the closed form solution permits its use in real-time filter design which would be of interest for the design of adaptive filters.

VI. Conclusions

The contribution of this paper is the development of a closed form solution to the parameters of a time-delay filter which minimizes the maximum magnitude of the transfer function of the time-delay filter. A simple technique to design minimax time-delay filters for underdamped systems is proposed. The minimax two time-delay filter results in non-zero magnitude at the nominal frequency. This magnitude can be reduced by penalizing the magnitude at the nominal frequency compared to the limiting frequencies in the uncertain domain. Closed form solutions for the three time-delay filter are also derived which force the magnitude at the nominal frequency to zero.

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