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Convex synthesis of controllers for consensus

Gustavo Ayres de Castro and Fernando Paganini
Electrical Engineering Department, UCLA

Abstract— **We develop convex conditions that are necessary and sufficient for the existence of a controller that yields a closed loop that achieves consensus. The conditions generate controllers with no particular communication structure, but with optimal $\mathcal{H}_2$ performance on the non-consensus part of the closed loop. We further explore the conditions to impose topology on the interconnection structure generated by the controllers. This is achieved by restricting a certain Lyapunov matrix to be block diagonal, in order to produce convex synthesis results.**

I. INTRODUCTION

Recently there has been a large interest in the coordination of groups of mobile agents, some examples are [4], [5], [7], [9]. One very important problem in that setting is that of agreement, or consensus, between the agents. A system composed of subsystems (or agents) achieves consensus with respect to a certain state variable of interest if for any set of initial conditions, the value of that variable for each agent converges to the same value for all agents.

Research on this problem started in the field of computer science, but only recently acceptable proofs of convergence for consensus protocols have been made. Moreover, the connection between physical consensus phenomena present in schooling fish, flocking birds, herds, have motivated investigation of such properties, along with potential applications in the design of controllers for systems composed of agents such as formation flight [4], [5], [6], platoons of vehicles [12], large segmented telescopes [8]. For example, in [7] and some references therein a motivating example is the heading of agents moving in the plane with constant velocity. The authors prove that a "nearest-neighbor" type of rule guarantees that the system achieves a consensus in the headings, as long as the graph that defines the interconnections remains connected most of the time. In much of that and other work in this area, the focus is on relating graph-theoretical tools to consensus of the interconnected system. One such example is in [4], where a decentralized control is used to yield a system that stabilizes a formation, therefore achieving consensus. Also the work in [9] shows that for the case where each agent has scalar dynamics, a protocol defined by the Laplacian of the interconnection graph yields a system that achieves consensus even under the assumption of bounded control, or in the presence of time delay in the links.

Most of the work in this area (e.g, [7], [9]) involves imposing a priori a specific control law and subsequently analyzing its dynamics. For example, in [9] it is shown that consensus can be achieved if the controller interconnection graph has certain properties. Since they are essentially analyses, these methods provide no means for a direct search for a controller while optimizing a performance criterion.

The objective of this paper is to characterize *all* consensus achieving controllers, and then to provide synthesis methods to impose topological information structure on the resulting controllers. In this way we allow, in principle, for a broad search for possible "protocols" that achieve consensus. Another departure worth mentioning; in [7], [9] the focus is on systems composed of one dimensional dynamical agents, while here we develop an approach that allows to tackle systems with higher-order dynamics modelling each agent.

The paper is organized as follows. In Section II we review some results needed for our discussion. Most importantly, we present the concept from [9], which says that consensus is equivalent to stability with respect to an invariant manifold. Then in Section III we develop necessary and sufficient Linear Matrix Inequality (LMI) conditions for consensus analysis in a system. In their most direct form, these conditions do not yield convex synthesis, but after a suitable transformation we are able in Section IV to obtain LMI conditions for state-feedback synthesis achieving consensus. The same transformation can be used to incorporate an $\mathcal{H}_2$ performance condition on the synthesis as a convex constraint. Both LMI conditions are necessary and sufficient, therefore totally characterizing the class of all controllers that achieve consensus. Even though the conditions are convex, they provide no means to directly impose a certain communication topology. That issue is tackled in Section V, in which we use decentralized Lyapunov functions [1], [2] to develop convex sufficient conditions for synthesis of structured controllers for consensus, also with an $\mathcal{H}_2$ norm objective. An illustrative example is presented in Section VI, and some concluding remarks are made in Section VII.

II. PRELIMINARIES AND BACKGROUND

We consider a system Σ composed of N interconnected agents, where each agent is assumed to have the following state space description:

$$\dot{x}_i = A_{ii}x_i + B_i^{(1)}w_i + B_i^{(2)}u_i + \sum_{j\neq i} A_{ij}x_j. \qquad (1)$$

We assume that all agents have arbitrary state space descriptions, but to save on notation, we restrict those descriptions to have the same dimensions, namely $A_{ij} \in \mathbb{R}^{n\times n}$, $B_i^{(1)} \in$

Box 951594, Los Angeles, CA 90095-1594; Email: gcastro@ee.ucla.edu, paganini@ee.ucla.edu. Research supported by CNPq–Brazil and NSF under Grant ECS-9875056

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$\mathbb{R}^{n\times m}$, $B_i^{(2)} \in \mathbb{R}^{n\times r}$, for all i, j. It is straightforward to generalize the results in this paper for subsystems of different dimensions, i.e, a totally heterogeneous network.

The system (1) may have a preexisting communication structure. A nonzero matrix A_{ij} in (1), with $i \neq j$, represents information being sent from subsystem j to subsystem i.

Our model assumes the control action u_i only influences the state x_i. From a practical point of view, it is a reasonable assumption, since for most networked systems the control actions taken at one location do not directly influence other locations. Nevertheless, the local control law could be in general a function of all the states $x_1, x_2, \dots, x_N$ in the system Σ, in other words, the control action need not be decentralized, i.e, $u_i = f(x_i)$.

Notice that we can stack up the local signals of the subsystems G_i, $i = 1, \dots, N$, to form the global version of such signals. For example, the global state is

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix},$$

where $x(t) \in \mathbb{R}^{Nn\times Nn}$ at each time t. Denote by $\text{diag}(B_1^{(1)}, \cdots, B_N^{(1)})$ the block diagonal matrix obtained in the obvious way. Then the system Σ describes the evolution of the global state $x(t)$ of the interconnection of all N subsystems, and its state space description is induced by the collection of systems:

$$\dot{x} = Ax + B_1 w + B_2 u, \tag{2}$$

with

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{bmatrix},$$

where $B_1 = \text{diag}(B_1^{(1)}, \cdots, B_N^{(1)})$ and $B_2 = \text{diag}(B_1^{(2)}, \cdots, B_N^{(1)})$. For the rest of this paper we will focus on the global state space description (2). For motivation, the reader should keep in mind the "interconnection-based" view of the system (2) given by (1).

A. LMI control

We briefly review some results in linear matrix inequalities (LMI) control, see e.g. [3]. We start with stability analysis and stabilization. A matrix A is stable if and only if there exists a symmetric positive definite matrix $\mathbf{X} > 0$ such that the LMI

$$A\mathbf{X} + \mathbf{X}A^* < 0$$

is feasible, where A^* stands for the transpose of A. For state feedback stabilization of a linear system such as (2) with $B_1 = 0$, consider the feedback $u = \mathbf{F}x$ and cast the analysis condition for the closed loop matrix $A_{cl} = A + B_2\mathbf{F}$:

$$(A + B_2\mathbf{F})\mathbf{X} + \mathbf{X}(A + B_2\mathbf{F})^* < 0.$$

The condition above is not an LMI in $\mathbf{F}$ and $\mathbf{X}$ but the transformation $\mathbf{Y} = \mathbf{F}\mathbf{X}$ turns that inequality in an LMI:

$$A\mathbf{X} + B_2\mathbf{Y} + \mathbf{X}A^* + \mathbf{Y}^* B_2^* < 0.$$

If the LMI is feasible, the stabilizing feedback matrix can be reconstructed from the feasible solution $\mathbf{X}$ and $\mathbf{Y}$ by computing $\mathbf{F} = \mathbf{Y}\mathbf{X}^{-1}$.

The same idea applies to state feedback $\mathcal{H}_2$ control. Define the performance variable z as:

$$z = C_1 x + D_{12} u.$$

We remark the $\mathcal{H}_2$ norm of the transfer from w to z is defined by

$$||T_{zw}||_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}(T_{zw}(j\omega)^* T_{zw}(j\omega))d\omega.$$

Then the following proposition [3] gives LMI conditions for synthesis of a state feedback $u = \mathbf{F}x$ controller that achieves a certain $\mathcal{H}_2$ norm.

Proposition 1 (State Feedback $\mathcal{H}_2$ Control): Given $\gamma > 0$, there exists a static state feedback law $u = \mathbf{F}x$ that internally stabilizes the system (2) and satisfies $\|T_{zw}\|_2^2 < \gamma$ if and only if there exist matrices $\mathbf{X} > 0$, $\mathbf{Z}$ and $\mathbf{Y}$ such that the following inequalities are satisfied

$$A\mathbf{X} + B_2\mathbf{Y} + \mathbf{X}A^* + \mathbf{Y}^* B_2^* + B_1 B_1^* < 0 \tag{3}$$

$$\begin{bmatrix} \mathbf{X} & (\bullet)^* \\ (C_1\mathbf{X} + D_{12}\mathbf{Y}) & \mathbf{Z} \end{bmatrix} > 0 \tag{4}$$

$$\text{trace}(\mathbf{Z}) < \gamma. \tag{5}$$

In this case, a suitable feedback is $\mathbf{F} = \mathbf{Y}\mathbf{X}^{-1}$.

We remark the LMI conditions above can be used to find the $\mathcal{H}_2$ optimal state feedback by minimizing γ.

B. Consensus

Now we define the main concept in the paper, that of consensus or agreement for dynamical systems. Consider an autonomous version of the system Σ with local state space description for each subsystem is given by:

$$\dot{x_i} = A_{ii}x_i + \sum_{j\neq i} A_{ij}x_j, \tag{6}$$

and the global autonomous system given by:

$$\dot{x} = Ax. \tag{7}$$

Definition 1 (consensus to $\mathcal{S}$): Let S be an orthonormal matrix in $\mathbb{R}^{Nn\times p}$, for some p. The system (7) achieves consensus to the subspace $\mathcal{S} = \text{span}\{S\}$ if $\mathcal{S}$ is a minimal set such that for any initial condition, the state $x(t)$ converges to a point in $\mathcal{S}$.

The definition, extracted from concepts in [9] (and also in [7]), implies that every point in $\mathcal{S}$ is a (marginally) stable equilibrium point of the equation (7), i.e, $AS = 0$, where S is simply an orthonormal basis for $\mathcal{S}$, the nullspace of A. Since the state of the system has to converge to a fixed point (a function of initial conditions only) it follows that the system must be marginally stable with no poles in $j\omega$ for $\omega \neq 0$. In other words, the geometric and algebraic multiplicity of the zero eigenvalues must be the same.

Example 1: Consider a system where each agent's dynamics is given by

$$\dot{x}_i(t) = u_i(t)$$

Namely, each system is a scalar integrator. For example, x could model the heading particles moving in the plane with constant velocity [7]. In this case, the subspace in which we are interested in achieving consensus is that where all the headings coincide, namely, the span of the vector

$$S = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

If a controller $u(t) = f(x)$ is designed in a way that the system achieves consensus with respect to S, then the headings of all subsystems will converge to the same value.

III. Consensus analysis

We now seek convex conditions for analysis of consensus for autonomous systems such as (7) in the sense of Definition 1. So assume that the autonomous system (7) achieves consensus to $\mathcal{S}$. We begin by exploring the essential properties of the matrix A. Define the orthonormal complement of the matrix S as $S_\perp$, i.e, $S_\perp^* S_\perp = I$ and $S_\perp^* S = 0$. Then, any $x \in \mathbb{R}^{nN}$ can be written as:

$$x = \begin{bmatrix} S_\perp & S \end{bmatrix} \begin{bmatrix} \zeta \\ \eta \end{bmatrix}, \quad (8)$$

for unique $\zeta \in \mathbb{R}^{Nn-p}$ and $\eta \in \mathbb{R}^p$. Substituting the relation (8) in the state space equations (7), we obtain:

$$\begin{bmatrix} S_\perp & S \end{bmatrix} \begin{bmatrix} \dot{\zeta} \\ \dot{\eta} \end{bmatrix} = A \begin{bmatrix} S_\perp & S \end{bmatrix} \begin{bmatrix} \zeta \\ \eta \end{bmatrix} = AS_\perp \zeta, \quad (9)$$

where the last equality holds from $AS = 0$. Now, since $\begin{bmatrix} S_\perp & S \end{bmatrix}$ is an invertible (unitary) matrix, its inverse is given by:

$$\begin{bmatrix} S_\perp & S \end{bmatrix}^{-1} = \begin{bmatrix} S_\perp^* \\ S^* \end{bmatrix},$$

and we can write (9) as:

$$\begin{aligned} \begin{bmatrix} \dot{\zeta} \\ \dot{\eta} \end{bmatrix} &= \begin{bmatrix} S_\perp^* \\ S^* \end{bmatrix} AS_\perp \zeta = \\ &= \begin{bmatrix} S_\perp^* AS_\perp & 0 \\ S^* AS_\perp & 0 \end{bmatrix} \begin{bmatrix} \zeta \\ \eta \end{bmatrix}. \end{aligned} \quad (10)$$

It follows that the eigenvalues of the system above are the union of the eigenvalues of $S_\perp^* AS_\perp$ and the matrix $0 \in \mathbb{R}^{p\times p}$. Thus consensus is equivalent to stability of $S_\perp^* AS_\perp$ and $AS = 0$. The next Lemma formalizes this statement and provides an immediate LMI formulation for consensus analysis.

Lemma 2: The autonomous system (7) achieves consensus to $\mathcal{S}$ in the sense of Definition 1 if and only if

i. $AS = 0$, and
ii. there exists a matrix $\mathbf{P} > 0$ such that

$$S_\perp^* AS_\perp \mathbf{P} + \mathbf{P} S_\perp^* A^* S_\perp \;<\; 0. \quad (11)$$

Proof: The solution of (10) is given by $\zeta(t) = e^{S_\perp^* AS_\perp t}\zeta(0)$, $\eta(t) = \eta(0) + \int_0^t e^{S_\perp^* AS_\perp \tau}\zeta(0) d\tau$, that is, ζ is totally decoupled from η. To show necessity, assume the Lyapunov inequality (11) is satisfied, then it is clear that $S_\perp^* AS_\perp$ is stable and $\zeta(t) \to 0$ as $t \to \infty$, and $\eta(t) \to \eta(0) + \int_0^\infty e^{S_\perp^* AS_\perp \tau}\zeta(0) d\tau$. The integral converges by stability of $S_\perp^* AS_\perp$, and therefore the state $x(t) = S_\perp \zeta(t) + S\eta(t)$ converges to a point in S as $t \to \infty$. Conversely, assume the system achieves consensus and there exists no solution $\mathbf{P} > 0$ to the Lyapunov LMI (ii). Then it follows that $S_\perp^* AS_\perp$ has some eigenvalues with real$(\lambda) \geq 0$. That means it we can always find a $\zeta(0) = \zeta_0$ such that $\lim_{t\to\infty} \zeta(t) \neq 0$. Since $x(t) = S_\perp \zeta(t) + S\eta(t)$, is is clear that $x(t)$ will not converge to S, contradicting Definition 1. ■

The conditions in Lemma 2 are by itself already convex conditions equivalent to consensus; nevertheless, the matrix A is "boxed in" between the basis matrix $S_\perp$. Such a condition is not very attractive when we are interested in the problem from a synthesis perspective since it poses a difficulty for convexifying the synthesis problem. The next theorem gives us a necessary and sufficient condition for consensus without the presence of $S_\perp$ between A and the Lyapunov function.

Theorem 3 (LMI for Consensus Analysis): Given a matrix $S \in \mathbb{R}^{Nn\times p}$, the autonomous system (7) achieves consensus to $\mathcal{S}$ if and only if

i. $AS = 0$, and
ii. there exists $\mathbf{X} > 0$ such that

$$S_\perp^*(A\mathbf{X} + \mathbf{X}A^*)S_\perp < 0 \quad (12)$$

where $\mathbf{X}$ satisfies:

$$\mathbf{X} = S_\perp S_\perp^* \mathbf{X} S_\perp S_\perp^* + SS^* \mathbf{X} SS^*. \quad (13)$$

Proof: We first tackle sufficiency. Assume $AS = 0$ and there exist $\mathbf{X} > 0$ such that the conditions (i) and (ii) are feasible. Then, note that $\mathbf{X}S_\perp = S_\perp S_\perp^* \mathbf{X} S_\perp$, and substitute that expression in (12) to obtain:

$$\begin{aligned} 0 > S_\perp^* A\mathbf{X}S_\perp + S_\perp^* \mathbf{X}A^* S_\perp &= \\ S_\perp^* AS_\perp \mathbf{P} + \mathbf{P} S_\perp^* A^* S_\perp, & \end{aligned}$$

where we have defined $\mathbf{P} := S_\perp^* \mathbf{X} S_\perp$. Invoking Lemma 2 we conclude the system achieves consensus.

We now prove necessity. Assume the system (7) achieves consensus. Then by Lemma 2 the conditions (11) are satisfied, in particular there exists a solution $\mathbf{P} > 0$ to the

LMI (11) in Lemma 2. Now, let $\mathbf{M} > 0$ be an arbitrary matrix and define

$$\mathbf{X} = \left[\begin{array}{cc} S_\perp & S \end{array} \right] \left[\begin{array}{cc} \mathbf{P} & 0 \\ 0 & \mathbf{M} \end{array} \right] \left[\begin{array}{c} S_\perp^* \\ S^* \end{array} \right]. \quad (14)$$

It is clear that $\mathbf{X} > 0$ and (13) is satisfied. Moreover, we again have

$$\mathbf{X}S_\perp = S_\perp \mathbf{P}, \quad (15)$$

and since by assumption the LMI (11) in Lemma 2 is feasible, we can substitute the relation (15) in that LMI, to arrive at (12):

$$\begin{aligned} 0 &> S_\perp^* A S_\perp \mathbf{P} + \mathbf{P} S_\perp^* A^* S_\perp = \\ &= S_\perp^* A \mathbf{X} S_\perp + S_\perp^* \mathbf{X} A^* S_\perp = \\ &= S_\perp^* \left(A\mathbf{X} + \mathbf{X}A^* \right) S_\perp. \end{aligned} \quad (16)$$

■

IV. SYNTHESIS FOR CONSENSUS

Assume we are interested in finding a control law $u = \mathbf{F}x$, such that the closed loop system

$$\dot{x} = (A + B_2 \mathbf{F})\, x \quad (17)$$

achieves consensus. Namely, we want the closed loop system to satisfy the conditions in Theorem 3, i.e:

i. $(A + B_2\mathbf{F})S = 0$;
ii. there exists $\mathbf{X} > 0$ such that

$$S_\perp^* \left((A + B_2\mathbf{F})\mathbf{X} + \mathbf{X}(A + B_2\mathbf{F})^* \right) S_\perp < 0,$$

where $\mathbf{X} = S_\perp S_\perp^* \mathbf{X} S_\perp S_\perp^* + SS^*\mathbf{X}SS^*$.

The conditions are not convex in the variables $\mathbf{F}$, $\mathbf{X}$, etc. Note that the standard change of variables for state feedback can be used to convexify the inequality (ii), namely, if we define $\mathbf{Y} = \mathbf{F}\mathbf{X}$, then that inequality becomes:

$$\begin{aligned} S_\perp^* \left((A + B_2\mathbf{F})\mathbf{X} + \mathbf{X}(A + B_2\mathbf{F})^* \right) S_\perp &= \\ S_\perp^* \left(A\mathbf{X} + B_2\mathbf{Y} + \mathbf{X}A^* + \mathbf{Y}^* B_2^* \right) S_\perp &< 0, \end{aligned}$$

which is now an LMI in $\mathbf{X}$ and $\mathbf{Y}$. Now the condition (i) under the change of variables $\mathbf{Y} = \mathbf{F}\mathbf{X}$ becomes

$$(A + B_2\mathbf{Y}\mathbf{X}^{-1})S = 0, \quad (18)$$

and is still non convex. Note that Theorem 3 guarantees that $\mathbf{X} = S_\perp S_\perp^* \mathbf{X} S_\perp S_\perp^* + SS^*\mathbf{X}SS^*$, and moreover since the matrix $\left[\begin{array}{cc} S_\perp & S \end{array} \right]$ is unitary, we can conclude:

$$\begin{aligned} \mathbf{X}^{-1} &= \left(\left[\begin{array}{cc} S_\perp & S \end{array} \right] \left[\begin{array}{cc} S_\perp^*\mathbf{X}S_\perp & 0 \\ 0 & S^*\mathbf{X}S \end{array} \right] \left[\begin{array}{c} S_\perp^* \\ S^* \end{array} \right] \right)^{-1} = \\ &= \left[\begin{array}{cc} S_\perp & S \end{array} \right] \left[\begin{array}{cc} (S_\perp^*\mathbf{X}S_\perp)^{-1} & 0 \\ 0 & (S^*\mathbf{X}S)^{-1} \end{array} \right] \left[\begin{array}{c} S_\perp^* \\ S^* \end{array} \right] = \\ &= S_\perp (S_\perp^*\mathbf{X}S_\perp)^{-1} S_\perp^* + S(S^*\mathbf{X}S)^{-1}S^*. \end{aligned} \quad (19)$$

Now making use of this expression in (18) we obtain

$$\begin{aligned} &(A + B_2\mathbf{Y}\mathbf{X}^{-1})S = \\ &= AS + B_2\mathbf{Y}\left(S_\perp (S_\perp^*\mathbf{X}S_\perp)^{-1} S_\perp^* + S(S^*\mathbf{X}S)^{-1}S^* \right) S = \\ &= AS + B_2\mathbf{Y}S(S^*\mathbf{X}S)^{-1}, \end{aligned} \quad (20)$$

where we have made use of the orthonormal identities $S_\perp^* S = 0$ and $S^*S = I$. In view of (20), we right multiply $(A + B_2\mathbf{Y}\mathbf{X}^{-1})S = 0$ by $S^*\mathbf{X}S$, and noting that $SS^*\mathbf{X}S = \mathbf{X}S$, we can obtain the equivalent condition:

$$A\mathbf{X}S + B_2\mathbf{Y}S = 0,$$

which is a linear condition in the variables $\mathbf{X}$ and $\mathbf{Y}$. We have just proved the following theorem.

Theorem 4 (Convex synthesis for consensus): Assume $B_1 = 0$. Then, there exists a state feedback $u = \mathbf{F}x$ such that the system (2) achieves consensus to $\mathcal{S}$ if and only if there exist matrices $\mathbf{X} > 0$ and $\mathbf{Y}$ such that

i. $A\mathbf{X}S + B_2\mathbf{Y}S = 0$
ii. $S_\perp^* \left(A\mathbf{X} + \mathbf{X}A^* + B_2\mathbf{Y} + \mathbf{Y}^*B_2^* \right) S_\perp < 0$.
iii $\mathbf{X} = S_\perp S_\perp^* \mathbf{X} S_\perp S_\perp^* + SS^*\mathbf{X}SS^*$

The control law can be reconstructed by $\mathbf{F} = \mathbf{Y}\mathbf{X}^{-1}$.

We now tackle the design of a state feedback controller that minimizes the $\mathcal{H}_2$ norm of the transfer from w to z in (1) and such that the closed loop achieves consensus.

Before we look at the general case, let us assume for now that the open loop matrix satisfies $AS = 0$, and that B_2 is full column rank. We consider the performance variable

$$z_\zeta = C_1 S_\perp \zeta(t) + D_{12} u(t). \quad (21)$$

If we are looking for a state feedback controller $u = \mathbf{F}x$, since $AS = 0$, the equality constraint (i) in Theorem 4 is simply given by $\mathbf{F}S = 0$. That is equivalent to the relation $\mathbf{F} = \tilde{\mathbf{F}} S_\perp^*$, for some $\tilde{\mathbf{F}}$. Then the performance variable is:

$$\begin{aligned} z_\zeta &= C_1 S_\perp \zeta(t) + D_{12} u(t) = \\ &= C_1 S_\perp \zeta(t) + D_{12} \tilde{\mathbf{F}} S_\perp^* x(t) = \\ &= C_1 S_\perp \zeta(t) + D_{12} \tilde{\mathbf{F}} \zeta(t), \end{aligned}$$

and since $\mathbf{F}S_\perp = \tilde{\mathbf{F}}$, we can write the performance variable as:

$$z_\zeta = C_1 S_\perp \zeta(t) + D_{12} \mathbf{F} S_\perp \zeta(t).$$

Intuitively, if $AS = 0$ the steady state control output will be zero. In case $AS = 0$ is not satisfied, then the more suitable definition for performance would be (22), which excludes the constant control terms that will be present in steady state. The reasoning is that since the system is marginally stable we can only hope to minimize the transfer with respect to the stable part of the state x, namely $S_\perp \zeta$.

That is, for the consensus problem in the general case, when $AS \neq 0$, the meaningful definition for performance variable is:

$$z_\zeta = C_1 S_\perp \zeta(t) + D_{12} \mathbf{F} S_\perp \zeta(t). \quad (22)$$

Such a definition of performance allows us to prove the following theorem.

Theorem 5 ($\mathcal{H}_2$ State feedback for consensus): Given $\gamma > 0$, there exists a static state feedback law $u = \mathbf{F}x$ that yields consensus to $\mathcal{S}$ in the system (2) and satisfies $\|T_{zw}\|_2^2 < \gamma$ if and only if there exist matrices $\mathbf{X} > 0$, $\mathbf{Z}$, and $\mathbf{Y}$ such that the following conditions are satisfied

$$
\begin{aligned}
A\mathbf{X}S + B_2\mathbf{Y}S &= 0 \\
S_\perp^* \left(A\mathbf{X} + \mathbf{X}A^* + B_2\mathbf{Y} + \mathbf{Y}^* B_2^* + B_1 B_1^*\right) S_\perp &< 0 \quad (23)\\
\begin{bmatrix} S_\perp^* \mathbf{X} S_\perp & (\bullet)^* \\ (C_1\mathbf{X}S_\perp + D_{12}\mathbf{Y}S_\perp) & \mathbf{Z} \end{bmatrix} &> 0 \quad (24)\\
\mathrm{trace}(\mathbf{Z}) &< \gamma. \quad (25)
\end{aligned}
$$

$$
\mathbf{X} = S_\perp S_\perp^* \mathbf{X} S_\perp S_\perp^* + SS^*\mathbf{X}SS^* \qquad (26)
$$

In this case, a suitable feedback is $\mathbf{F} = \mathbf{Y}\mathbf{X}^{-1}$.

Remark 1: **Imposing left nullspace conditions** Note that consensus with respect to a certain $\mathcal{S}$ guarantees that the nullspace of $(A+B\mathbf{F})$ has dimension p. Therefore, there exists a matrix $T \in \mathbb{R}^{Nn\times p}$ such that $T^*(A+B\mathbf{F}) = 0$. In some cases it may be interesting to include this as a condition in the synthesis problem. Under the linearizing transformation $\mathbf{Y} = \mathbf{F}X$, this condition becomes $T^*(A + B\mathbf{Y}\mathbf{X}^{-1}) = 0$, which is easily turned linear by right multiplying by $\mathbf{X}$, yielding the condition:

$$
T^* A\mathbf{X} + T^* B\mathbf{Y} = 0, \qquad (27)
$$

By adding condition (27) to those of Theorems 4 or 5, we obtain necessary and sufficient conditions for consensus synthesis with a specific left nullspace. It can be shown [9] that the left nullspace of $(A + BF)$ plays a role in the value to which the state converges. In the particular case of integrators modelling headings, where $S^* = [1\ 1 \cdots 1]$, forcing $T = S$ yields a system that converges to the average of the initial conditions on the headings.

V. Synthesis for a given network topology

From a practical point of view, the synthesis result in Theorem 4 lacks an important feature, it does not include in the design a restriction in the communication between units. In other words, in general a solution obtained from Theorem 4 would have a *full block* state feedback matrix $\mathbf{F}$, i.e, each unit G_i would receive information from every other unit in the system. That is a very strong condition and typically undesirable. It would be more interesting to be able to impose a certain communication structure in the system, by designing the structure of the feedback matrix $\mathbf{F}$. We address the solution to this problem with an approach similar to that of [1], [2].

For example, assume we are dealing with a system composed of 4 subsystems, and we wish to design a feedback law with the particular structure:

$$
\mathbf{F} = \begin{bmatrix} \mathbf{F}_{11} & 0 & \mathbf{F}_{13} & 0 \\ \mathbf{F}_{21} & \mathbf{F}_{22} & 0 & 0 \\ 0 & 0 & \mathbf{F}_{33} & \mathbf{F}_{34} \\ \mathbf{F}_{11} & 0 & 0 & \mathbf{F}_{44} \end{bmatrix}, \qquad (28)
$$

where the partition is made in the obvious way, with $\mathbf{F}_{ij} \in \mathbb{R}^{m\times n}$. Now, assume we wish to find a feasible solution for the conditions in Theorem 4 such that $\mathbf{F}$ has the form (28).

Imposing structure in $\mathbf{F}$ would mean imposing the formula for the controller $\mathbf{F} = \mathbf{Y}\mathbf{X}^{-1}$ to yield a structured matrix. That is, some terms in the multiplication $\mathbf{Y}\mathbf{X}^{-1}$ would have to vanish. That cancellation feature is not convex. Efficient solutions to such problems typically rely on some form of relaxation of the problem, or consider a special structure, e.g, [1], [2], [11], [13].

Now assume we impose that $\mathbf{X} \in \mathbb{R}^{4n\times 4n}$ be a block diagonal matrix $diag(X_1, X_2, X_3, X_4)$, with each block $\mathbf{X}_i \in \mathbb{R}^{n\times n}$. Then the resulting $\mathbf{F}$ will have the desired structure if and only if $\mathbf{Y}$ has the desired structure for $\mathbf{F}$. Specifically, in our 4 system case example we obtain:

$$
\begin{aligned}
\mathbf{F} &= \mathbf{Y}\mathbf{X}^{-1} = \\
&= \begin{bmatrix} \mathbf{Y}_{11} & 0 & \mathbf{Y}_{13} & 0 \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} & 0 & 0 \\ 0 & 0 & \mathbf{Y}_{33} & \mathbf{Y}_{34} \\ \mathbf{Y}_{11} & 0 & 0 & \mathbf{Y}_{44} \end{bmatrix} \begin{bmatrix} \mathbf{X}_1^{-1} & 0 & 0 & 0 \\ 0 & \mathbf{X}_2^{-1} & 0 & 0 \\ 0 & 0 & \mathbf{X}_3^{-1} & 0 \\ 0 & 0 & 0 & \mathbf{X}_4^{-1} \end{bmatrix} = \\
&=: \begin{bmatrix} \mathbf{F}_{11} & 0 & \mathbf{F}_{13} & 0 \\ \mathbf{F}_{21} & \mathbf{F}_{22} & 0 & 0 \\ 0 & 0 & \mathbf{F}_{33} & \mathbf{F}_{34} \\ \mathbf{F}_{11} & 0 & 0 & \mathbf{F}_{44} \end{bmatrix}
\end{aligned}
$$

This example illustrates the idea behind our approach. For the general case, let $\mathbf{F} = \mathbf{F}_\chi$ denote that a matrix $\mathbf{F}$ belongs to the set of prescribed structures $\chi = \{\mathbf{F} : \mathbf{F}_{ij} = 0 \in \mathbb{R}^{m\times n},\ \text{for}(i,j) \in \mathcal{I}\}$ (hard zeros in some blocks indexed by some set $\mathcal{I}$). Then, if in the synthesis conditions of Theorems 4 and 5 we restrict the Lyapunov-like matrix $\mathbf{X}$ to be block diagonal, ie, $\mathbf{X} = \mathrm{diag}(\mathbf{X}_1, \cdots, \mathbf{X}_N)$, $\mathbf{F}$ will have the desired structure $\mathbf{F} = \mathbf{F}_\chi$ if and only if it $\mathbf{Y} = \mathbf{Y}_\chi$. The next two corollaries result immediately from applying this idea to Theorems 4 and 5. The conservativeness behind such a restriction on the Lyapunov matrix has been studied in the context of spatially invariant systems in [2], where it is shown to be equivalent to an IQC in the local state space.

Corollary 6 (Structured synthesis for consensus): Assume $B_2 = 0$, and let $\chi = \{\mathbf{F} : \mathbf{F}_{ij} = 0 \in \mathbb{R}^{m\times n},\ \text{for}(i,j) \in \mathcal{I}\}$. Then, there exists a structured state feedback $u = \mathbf{F}_\chi x$, such that the system (2) achieves consensus to $\mathcal{S}$ if there exist matrices $\mathbf{Y} = \mathbf{Y}_\chi$, and $\mathbf{X}_i > 0$, $i = 1, \ldots, N$, such that

i. $A\mathbf{X}S + B_2\mathbf{Y}_\chi S = 0$
ii. $S_\perp^* \left(A\mathbf{X} + \mathbf{X}A + B_2\mathbf{Y}_\chi + \mathbf{Y}_\chi^* B_2^*\right) S_\perp < 0$.
iii. $\mathrm{diag}(\mathbf{X}_1, \cdots, \mathbf{X}_N = S_\perp S_\perp^* \mathrm{diag}(\mathbf{X}_1, \cdots, \mathbf{X}_N) S_\perp S_\perp^* + SS^* \mathrm{diag}(\mathbf{X}_1, \cdots, \mathbf{X}_N) SS^*$.

The control law can be reconstructed by $\mathbf{F}_\chi = \mathbf{Y}_\chi \mathrm{diag}(\mathbf{X}_1^{-1}, \cdots, \mathbf{X}_N^{-1})$.

Proof: The proof is immediate by substituting the restricted $\mathbf{Y} = \mathbf{Y}_\chi$, and $\mathbf{X} = \mathrm{diag}(\mathbf{X}_1, \cdots, \mathbf{X}_N)$, with $\chi = \{\mathbf{Y} : \mathbf{Y}_{ij} = 0 \in \mathbb{R}^{m\times n},\ \text{for}(i,j) \in \mathcal{I}\}$ in the original conditions in Theorem 4. ■

Similarly, we can prove the following corollary to the state feedback $\mathcal{H}_2$ control Theorem 5.

Corollary 7 (Structured $\mathcal{H}_2$ control for consensus): Let $\chi = \{\mathbf{F} : \mathbf{F}_{ij} = 0 \in \mathbb{R}^{m\times n},\ \text{for}(i,j) \in \mathcal{I}\}$ be a desired structure. Given $\gamma > 0$, there exists a static state feedback law $u = \mathbf{F}_\chi x$ that internally stabilizes the system (2) and satisfies $\|T_{zw}\|_2^2 < \gamma$ if there exist matrices $\mathbf{Z}$, $\mathbf{Y} = \mathbf{Y}_\chi$, and $\mathbf{X}_i > 0$, $i = 1, \ldots, N$, such that the following conditions are satisfied

$$\begin{array}{rcl} A\mathbf{X}S + B_2\mathbf{Y}_\chi S & = & 0 \\ S_\perp^* \left(A\mathbf{X} + \mathbf{X}A + B_2\mathbf{Y}_\chi + \mathbf{Y}_\chi^* B_2^* + B_1 B_1^*\right) S_\perp & < & 0 \\ \left[\begin{array}{cc} S_\perp^* \mathbf{X} S_\perp & (\bullet)^* \\ (C_1 \mathbf{X} S_\perp + D_{12}\mathbf{Y}_\chi S_\perp) & \mathbf{Z} \end{array}\right] & > & 0 \\ \mathrm{trace}(\mathbf{Z}) & < & \gamma, \end{array}$$

$$\begin{array}{rcl} \mathrm{diag}(\mathbf{X}_1, \cdots, \mathbf{X}_N) & = & S_\perp S_\perp^* \mathrm{diag}(\mathbf{X}_1, \cdots, \mathbf{X}_N) S_\perp S_\perp^* + \\ & + & SS^* \mathrm{diag}(\mathbf{X}_1, \cdots, \mathbf{X}_N) SS^*. \end{array}$$

In this case, a suitable feedback is $\mathbf{F}_\chi = \mathbf{Y}_\chi \mathrm{diag}(\mathbf{X}_1^{-1}, \cdots, \mathbf{X}_N^{-1})$.

Remark 2: **Synthesis for networks of integrators** Assume each local system is an integrator $\dot{x}_i = u_i$, where $x_i \in \mathbb{R}$, i.e, $n = 1$, $A = 0$, $B_2 = I$ and $B_1 = 0$ in (2). In [9] it is shown that their solution always accepts a Lyapunov function $\mathbf{P} = I$. Now since $A = 0$, condition (29) becomes $\mathbf{Y}S = 0$ and does not depend on $\mathbf{M}$. In such a case we can set $\mathbf{M} = 1$ without loss of generality and obtain $\mathbf{X} = I$, therefore block diagonal. So any solutions in [9] are also feasible points for the LMIs of Corollary 6.

VI. Example

We consider a network of 20 integrators, namely each unit is described by the state space equation:

$$\frac{d}{dt} x_k(t) = u_k(t) + w_k(t),$$

where u_k is the local control input and w_k is an input noise. We wish to find a control law to minimize the $\mathcal{H}_2$ norm of the transfer T_{zw} where:

$$z = C_1 S_\perp \zeta(t) + D_{12} u(t).$$

Moreover, that control should achieve consensus to the subspace where all the headings coincide:

$$S = (\sqrt{20})^{-1} \left[\begin{array}{c} 1 \\ \vdots \\ 1 \end{array}\right].$$

We take the following values:

$$C_1 \;=\; \left[\begin{array}{c} S_\perp^* \\ 0 \end{array}\right] \qquad D_{12} = \left[\begin{array}{c} 0 \\ I \end{array}\right].$$

The Laplacian associated with the information structure considered is given by:

```
      -3  0  1  0  0  0  0  0  0  0  1  0  0  1  0  0  0  0  0  0
       0 -2  0  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  1  0
       1  0 -1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
       0  0  0 -4  0  0  0  1  0  1  0  0  0  0  1  0  0  1  0  0
       0  0  0  0 -4  0  1  0  0  1  0  0  0  1  0  1  0  0  0  0
       0  0  0  0  0 -1  0  0  0  0  1  0  0  0  0  0  0  0  0  0
       0  0  0  0  1  0 -1  0  0  0  0  0  0  0  0  0  0  0  0  0
       0  0  0  1  0  0  0 -2  0  0  0  1  0  0  0  0  0  0  0  0
       0  0  0  0  0  0  0  0 -1  0  0  0  0  0  0  0  0  0  1  0
       0  0  0  1  1  0  0  0  0 -2  0  0  0  0  0  0  0  0  0  0
L= -   1  0  0  0  0  1  0  0  0  0 -2  0  0  0  0  0  0  0  0  0
       0  1  0  0  0  0  0  1  0  0  0 -3  1  0  0  0  0  0  0  0
       0  0  0  0  0  0  0  0  0  0  0  1 -3  0  0  1  1  0  0  0
       1  0  0  0  1  0  0  0  0  0  0  0  0 -3  0  0  1  0  0  0
       0  0  0  1  0  0  0  0  0  0  0  0  0  0 -3  0  0  0  1  1
       0  0  0  0  1  0  0  0  0  0  0  0  1  0  0 -3  0  0  1  0
       0  0  0  0  0  0  0  0  0  0  0  0  1  1  0  0 -2  0  0  0
       0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  0  0 -1  0  0
       0  1  0  0  0  0  0  0  1  0  0  0  0  0  1  1  0  0 -4  0
       0  0  0  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  0 -1
```

An application of the $\mathcal{H}_2$ synthesis method of Theorem 5 results in a controller feedback matrix with the structure above and with an $\mathcal{H}_2$ performance of 5.3744.

In order to compare the solution with the best possible $\mathcal{H}_2$ norm generated by this Laplacian, we search over $\alpha > 0$ for the smallest $\mathcal{H}_2$ norm generated by $-\alpha L$. After a bisection search over α, the optimal $\mathcal{H}_2$ norm with this Laplacian is found to be 5.6034.

VII. Future work and conclusions

We have derived necessary and sufficient LMIs that characterize all the state feedback controllers that solve the problem of consensus. The consequences are twofold; first, this shows that consensus is a convex problem up to a problem of structured control design. Second, using those LMIs, we have showed that an assumption on the matrix $\mathbf{X}$ yields convex conditions for structured synthesis for consensus. Our results apply to systems composed by a heterogenous collection of agents (or subsystems), in other words, the agents may have arbitrary dynamics.

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