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Information Structures to Secure Control of Globally Rigid Formations

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***Abstract*— Sensor and network topologies of rigid formations with distance information between mobile autonomous agents are considered. An approach based on rigidity for creating such topologies were suggested in our previous work. Here, we first illustrate some potential scenarios on formations that require unambiguity in the knowledge of distances between every pair of agents in a formation. Then, we show how a stronger type of rigidity, namely global rigidity, plays a role in creating such unambiguous formations. We draw out and summarize some relevant results from the related mathematical theory of global rigidity; and present some new results on globally rigid formations.**

I. Introduction

In previous papers ([5], [6], [8]), we suggested an approach based on rigidity for maintaining formations of mobile autonomous agents with sensor and network topologies that use distance, direction, bearing and angle information between agents. In this paper, we investigate globally rigid formations.

By a *formation*, we mean a group of mobile autonomous agents moving in real 2- or 3-dimensional space. A formation is called *rigid* if the distance between each pair of agents does not change over time under ideal conditions. A formation is called *globally rigid*, if the distance between each pair of agents is unambiguous. Sensing and communication links are used for maintaining distances between agents fixed. Distances between *all* agent pairs can be held fixed by directly measuring distances between only *some* agents and keeping them at desired values [5]. It is also true that it is not necessary to have sensing and communication links between every pair of agents to create a globally rigid formation, which we will explore in this paper.

First, we present some reasons why it is desirable to have globally rigid formations. Departure of an agent from a rigid formation creates a need for generating new links between remaining agents to maintain a formation while preserving the links between remaining agents. In Eren et al. [4], an approach was given to determine which pairs of agents these new links should be created between. With these new links, the remaining agents will be able to maintain *a* formation. However, if the remaining agents desire to maintain exactly the same relative distances between themselves as before the agent departure, then there is a need for additional information about distances between agents where new links are created. In other words, by knowing relative distances between *some* pairs of agents, there is a need to compute the relative distances between *all* pairs of agents (or all new pairs to be added).

As another example, consider a formation splitting into two when it encounters an obstacle so that one half can proceed to one side and the other half to the other, instead of making a maneuver in which the whole formation proceeds to the same side. When the formation splits, the links between agents in different sub-formations are broken while the links between agents in the same sub-formations are preserved. As this takes place, there is a need to create new links in each sub-formation to maintain rigid formation. A method of generating such links is given in a companion paper by Eren et al. [4]. If, furthermore, there is a need to preserve the relative distances between agents in each sub-formation that are exactly the same as before, then we encounter the same need to know distances between all pairs of agents.

There are two steps needed in overcoming these problems. The first is finding a globally rigid formation and the second is, given such a globally rigid formation, finding relative distances between every pair of agents. The first step is characterizing globally rigid formations [7], which we will explore in this paper. The second step is related to the Euclidean distance matrix completion problem [9]. Because of space constraints, we present only globally rigid formations with distance information between agents, and we leave out globally rigid formations with direction, bearing and angle information. We refer the reader to Eren [3] for a detailed treatment of such formations.

The paper is organized as follows: We start with an overview of point formations and rigidity in §II. Globally rigid formations are investigated in §III.

II. Point Formations and Rigidity

By a d-dimensional *point formation* at $p \triangleq \text{column } \{p_1, p_2, \ldots, p_n\}$, written $\mathbb{F}_p$, is meant a set

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of n points $\{p_1, p_2, \ldots, p_n\}$ in $\mathbb{R}^d$ together with a set $\mathcal{L}$ of k *maintenance links*, labelled (i, j), where i and j are distinct integers in $\{1, 2, \ldots, n\}$; the *length* of link (i, j) is the Euclidean distance between points p_i and p_j. For our purposes, a point formation $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$ provides a natural high-level model for a set of n agents moving in real 2- or 3- dimensional space. In this context, the points p_i represent the positions of agents in $\mathbb{R}^d$ $\{d = 2 \text{ or } 3\}$ and the links in $\mathcal{L}$ label those specific agent pairs whose inter-agent distances are to be maintained over time. In practice actual agent positions cannot be expected to move exactly in formation because of sensing errors, vehicle modelling errors, etc. The ideal benchmark formation against which the performance of an actual agent formation is to be measured is called a *reference formation.*

Each point formation $\mathbb{F}_p$ uniquely determines a graph $\mathbb{G} \triangleq (\mathcal{V}, \mathcal{L})$ with vertex set $\mathcal{V} \triangleq \{1, 2, \ldots, n\}$ and edge set $\mathcal{L}$, as well as a distance function $\delta : \mathcal{L} \to \mathbb{R}$ whose value at $(i, j) \in \mathcal{L}$ is the distance between p_i and p_j. Let us note that the distance function of $\mathbb{F}_p$ is the same as the distance function of any point formation $\mathbb{F}_q$ with the same graph as $\mathbb{F}_p$ provided q is *congruent* to p in the sense that there is a distance preserving map $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $T(q_i) = p_i, i \in \{1, 2, \ldots, n\}$. In the sequel we will say that two point formations $\mathbb{F}_p$ and $\mathbb{F}_q$ are *congruent* if they have the same graph and if q and p are congruent.

By a *trajectory* of $\mathbb{F}_p$, we mean a continuously parameterized, one-parameter family of points $\{q(t) : t \geq 0\}$ in $\mathbb{R}^{nd}$, which contains p. We can define a rigid point formation as follows: A formation is said to undergo *rigid motion* along a trajectory $q([0, \infty)) \triangleq \{\text{column } \{q_1(t), q_2(t), \ldots, q_n(t)\} : t \geq 0\}$ if the Euclidean distance between each pair of points $q_i(t)$ and $q_j(t)$ remains constant all along the trajectory. Let us note that $\mathbb{F}_p$ undergoes rigid motion along a trajectory $q([0, \infty))$ just in case each pair of points $q(t_1), q(t_2) \in q([0, \infty))$ are congruent. The set of points $\mathcal{M}_p$ in $\mathbb{R}^{nd}$ which are congruent to p is known to be a smooth manifold. It is clear that any trajectory along which $\mathbb{F}_p$ undergoes rigid motion must lie completely within $\mathcal{M}_p$; conversely any trajectory of $\mathbb{F}_p$ that lies within $\mathcal{M}_p$ is one along which $\mathbb{F}_p$ undergoes rigid motion. A point formation $\mathbb{F}_p$ is said to be *rigid* if rigid motion is the only kind of motion it can undergo along any trajectory on which the lengths of all links in $\mathcal{L}$ remain constant. Thus, if $\mathbb{F}_p$ is rigid, it is possible to "keep formation" by making sure that the lengths of the formation's maintained links do not change as the formation moves.

Whether a given point formation is rigid or not can be studied by examining what happens to the given point formation $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$ with m maintenance links, along a trajectory $q([0, \infty)) \triangleq \{\{q_1(t), q_2(t), \ldots, q_n(t)\} : t \geq 0\}$ on which the Euclidean distances $d_{ij} \triangleq ||p_i - p_j||$ between pairs of points (p_i, p_j) for which (i, j) is a link are constant. Along such a trajectory

$$(q_i - q_j) \cdot (q_i - q_j) = d_{ij}^2, \quad (i, j) \in \mathcal{L}, \quad t \geq 0 \qquad (1)$$

Assuming a smooth (piecewise analytic) trajectory, we can differentiate to get

$$(q_i - q_j) \cdot (\dot{q}_i - \dot{q}_j) = 0, \quad (i, j) \in \mathcal{L}, \quad t \geq 0 \qquad (2)$$

The m equations can be collected into a single matrix equation

$$R(q)\dot{q} = 0 \qquad (3)$$

where $\dot{q} = \text{column } \{\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\}$ and $R(q)$ is a specially structured $m \times dn$ matrix called the *rigidity matrix* [13].

Because any trajectory of $\mathbb{F}_p$ which lies within $\mathcal{M}_p$, is one along which $\mathbb{F}_p$ undergoes rigid motion, (2) automatically holds along any trajectory which lies within $\mathcal{M}_p$. It follows that the tangent space to $\mathcal{M}_p$ at q, written $\mathcal{T}_q$, must be contained in the kernel of $R(q)$. Since p must be on any such trajectory, it must be true that $\mathcal{T}_q \subset$ kernel $R(q)$. If $\dot{q}$ satisfies (3), then it lies in the tangent space. If the affine span of the points $p_1, p_2, \ldots, p_n$ is $\mathbb{R}^n$, then $\mathcal{M}_p$ is $n(n+1)/2$ dimensional since it arises from the $n(n-1)/2$-dimensional manifold of orthogonal transformations of $\mathbb{R}^n$ and the n-dimensional manifold of translations of $\mathbb{R}^n$. Thus $\mathcal{M}_p$ is 6-dimensional for $\mathbb{F}_p$ in $\mathbb{R}^3$, and 3-dimensional for $\mathbb{F}_p$ in $\mathbb{R}^2$. We have $\text{rank } R(q) = nd -$ dimension kernel $R(q) \leq nd - n(n+1)/2$. We have the following theorem [11]:

Theorem 1. *Assume $\mathbb{F}_p$ is a formation with at least d points in d-space $\{d = 2, \text{ or } 3\}$ where $\text{rank } R(p) = \max\{\text{rank } R(x) : x \in \mathbb{R}^d\}$. $\mathbb{F}_p$ is rigid in $\mathbb{R}^d$ if and only if*

$$\text{rank } R(p) = \begin{cases} 2n - 3 & \text{if } d = 2, \\ 3n - 6 & \text{if } d = 3. \end{cases}$$

This theorem leads us to the notion of the "generic" rigidity behavior of the graph. When the maximum rank for $R(x)$ over all x is less than this upper bound, the formation $\mathbb{F}_p$ may still be rigid for some particular p. However, this is unstable. For almost all changes in the positions p (or in the lengths of maintenance links which are realizable), the formation will no longer be rigid. We are interested in "generic rigidity", a property that will hold for all small changes in p.

A point formation $\mathbb{F}_p$ is *generically rigid* if it is rigid for almost all choices of p in $\mathbb{R}^{dn}$. It is possible to characterize generic rigidity in terms of the "generic rank" of R where by R's *generic* or maximal rank we mean the largest value of $\text{rank}\{R(q)\}$ as q ranges over all values in $\mathbb{R}^{nd}$. The following theorem is due to Roth [11].

Theorem 2. *A formation $\mathbb{F}_p$ with at least d points in d-space $\{d = 2, \text{ or } 3\}$ is generically rigid if and only if*

$$\text{generic rank } \{R\} = \begin{cases} 2n - 3 & \text{if } d = 2, \\ 3n - 6 & \text{if } d = 3. \end{cases}$$

To understand this type of rigidity, it is useful to observe that the set of points p that satisfy the condition $\text{rank}\, R(p) = \max\{\text{rank}\, R(x) : x \in \mathbb{R}^d\}$ is a dense open subset of $\mathbb{R}^{nd}$ [11]. Generic rigidity is a property of only the set of maintenance links, or the underlying graph. It does not even claim that $\mathbb{F}_p$ itself is rigid but only that almost all nearby points q give rigid formations $\mathbb{F}_q$. The concept of generic rigidity does not depend on the precise distances between the points of $\mathbb{F}_p$ but examines how well the rigidity of formations can be judged by knowing the vertices and their incidences, in other words, by knowing the underlying graph. A point formation $\mathbb{F}_p$ is *strongly generically rigid* if it is generically rigid and if $\text{rank}\, R(p) = \text{generic rank}\, \{R\}$. Hence, a strongly generically rigid point formation is rigid and it remains rigid under small perturbations. For this reason, it is a desirable specialization of the concept of a "rigid formation" for our purposes. We have the following theorem for a strongly generically rigid point formation and a generically rigid graph [13]:

Theorem 3. *For a formation $\mathbb{F}_p$ in d-space with at least d points, the following are equivalent:*

1) *the formation's underlying graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is generically rigid in d-dimensional space ($d = 2, 3$);*
2) *for some p,*

$$rank\ \{R(p)\} = \begin{cases} 2n-3 & \text{if } d = 2, \\ 3n-6 & \text{if } d = 3. \end{cases}$$

3) *for almost all p, the formation $\mathbb{F}_p$ is strongly generically rigid.*

For 2-dimensional space, we have a complete combinatorial characterization of generically rigid graphs, which was first proved by Laman in 1970. In the theorem below, $|\,.\,|$ is used to denote the cardinal number of a set.

Theorem 4 (Laman). *A graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ (where $\mathcal{L} \neq \emptyset$ or $n > 1$) is generically rigid in 2-dimensional space if and only if there is a subset $\mathcal{L}' \subseteq \mathcal{L}$ satisfying the following two conditions: (1) $|\mathcal{L}'| = 2|\mathcal{V}| - 3$, (2) For all $\mathcal{L}'' \subseteq \mathcal{L}', \mathcal{L}'' \neq \emptyset, |\mathcal{L}''| \leq 2|\mathcal{V}(\mathcal{L}'')| - 3$, where $|\mathcal{V}(\mathcal{L}'')|$ is the number of vertices that are end-vertices of the edges in $\mathcal{L}''$.*

There is no comparable complete result for 3-dimensional space, though there are useful partial results [13], [14]. Although we lack a characterization in 3-dimensional space, there are sequential techniques to generate rigid classes of graphs both in 2- and 3-dimensional space [1], [13]. We explain these techniques in the sequel.

A *dependence* on the maintenance link set of a point formation $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$ is an assignment $\lambda : \mathcal{L} \to \mathbb{R}$, with $\lambda(i,j) = \lambda_{i,j} = \lambda_{j,i}$ (and not all zero), such that for each vertex i: $\sum_{j|(i,j)\in\mathcal{L}} \lambda_{ij}(p_i - p_j) = 0$. This equation gives a row dependence of the rigidity matrix. A point formation is *independent* if its maintenance link set is such that the rows of the rigidity matrix are all independent. A point formation is *minimally rigid* if removing any link makes it non-rigid. There are $2n-3$ and $3n-6$ maintenance links in minimally rigid formations in 2- and 3-dimensional space respectively. A graph is called (generically)[1] minimally rigid in d-space if it is rigid and has exactly $dn - \binom{d+1}{2}$ edges. Minimally rigid point formations are also maximally independent point formations, corresponding to bases in vector spaces as minimally spanning sets and maximally independent sets.

If a point formation is rigid but not minimally rigid, we say that there is *redundancy* in the link set $\mathcal{L}$. Let us suppose that a link (i,j) is removed from a rigid point formation. If the formation remains rigid then (i,j) is called a *redundant link* in the initial formation (*redundant edge* in the underlying graph). If adding a link (i,j) does not increase the rank of the rigidity matrix, then we call (i,j) an *implicit link* (*implicit edge* in the underlying graph).

In 2-dimensional space, by Theorem 4, a set of edges $\mathcal{L}$ is *independent* or an *independent edge set*, if $|\mathcal{L}| \leq 2|\mathcal{V}(\mathcal{L})| - 3$ and for every $\mathcal{L}' \subseteq \mathcal{L}$, $|\mathcal{L}'| \leq 2|\mathcal{V}(\mathcal{L}')| - 3$. If the set of edges $\mathcal{L}$ of a graph in the plane $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is independent, then $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is called an *independent graph*.

Sequential Techniques: First, we introduce some additional terminology. If (i,j) is an edge, then we say that i and j are *adjacent* or that j is a *neighbor* of i and i is a neighbor of j. [2] The vertices i and j are *incident* with the edge (i,j). Two edges are *adjacent* if they have exactly one common end-vertex. The *degree* or *valency* of a vertex i is the number of neighbors of i. If a vertex has k neighbors, it is called a *vertex of degree* k or a k*-valent vertex*.

One operation is the *vertex addition*: given a minimally rigid graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$, we add a new vertex i with d edges between i and d other vertices in $\mathcal{V}$. The other is the *edge splitting*: given a minimally rigid graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$, we remove an edge (j,k) in $\mathcal{L}$ and then we add a new vertex i with $d+1$ edges by inserting two edges (i,j), (i,k) and $d-1$ edges between i and $d-1$ vertices (other than j,k) in $\mathcal{V}$. Now we are ready to present the following theorems:

Theorem 5 (vertex addition [12]). *Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph with a vertex i of degree d in d-space; let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$ denote the subgraph obtained by deleting i and the edges incident with it. Then G is generically minimally rigid if and only if $\mathbb{G}^*$ is generically minimally rigid.*

Theorem 6 (edge splitting [12]). *Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph with a vertex i of degree $d+1$ in d-space; let $\mathcal{V}_i$ be the set of vertices incident to i; and let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$ be the subgraph obtained by deleting i and its $d+1$ incident edges. Then $\mathbb{G}$ is generically minimally rigid if and only if there is a pair j,k of vertices of $\mathcal{V}_i$ such that the edge (j,k) is not in $\mathcal{L}^*$ and the graph $\mathbb{G}' = (\mathcal{V}^*, \mathcal{L}^* \bigcup (j,k))$ is*

[1] In the sequel, we use the term rigid graph instead of generically rigid graph unless there is a danger of confusion.

[2] The neighbor relation is symmetric throughout the paper, i.e., if agent i senses or communicates with agent j, so does agent j with agent i.

generically minimally rigid.

Henneberg sequences are a systematic way of generating minimally rigid graphs based on the vertex addition and edge splitting operations. In d-space, we are given a sequence of graphs: $\mathbb{G}_d, \mathbb{G}_{d+1}, \ldots, \mathbb{G}_{|\mathcal{V}|}$ such that: 1) $\mathbb{G}_d$ is the complete graph on d vertices; 2) $\mathbb{G}_{i+1}$ comes from $\mathbb{G}_i$ by adding a new vertex either by (i) the vertex addition or (ii) the edge splitting operation.

Note that $\mathbb{G}_i$ and $\mathbb{G}_{i+1}$ correspond to $\mathbb{G}^*$ and $\mathbb{G}$ in the statements of Theorem 5 and Theorem 6. All graphs in the sequence are minimally rigid in d-space. In 2-dimensional space, the two operations of vertex addition and edge splitting are sufficient to generate all minimally rigid graphs starting from a single edge. In 3-dimensional space, they generate a proper subclass of minimally rigid graphs starting from a triangle. The reason behind this difference between planar and spatial cases is the following observation.

A minimally rigid graph in 2-dimensional space may have all vertices of degree larger than 2; $|\mathcal{L}| = 2|\mathcal{V}| - 3$ or equivalently $2|\mathcal{L}| = 4|\mathcal{V}| - 6$ guarantees that some vertices have degree 3 or less. If $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is a graph with $|\mathcal{V}|$ vertices which is minimally rigid, the existence of at least one vertex of degree 2 or 3 means that by Theorem 5 or Theorem 6 there exists a minimally rigid graph $\mathbb{G}^*$ with $|\mathcal{V}| - 1$ vertices, and one can go from $\mathbb{G}$ to $\mathbb{G}^*$ by the vertex addition or edge splitting operations. Then one uses the same procedure on $\mathbb{G}^*$. Hence, all minimally rigid graphs can be generated by the vertex addition and edge splitting operations alone. It is also true that starting with a single edge only minimally rigid graphs are generated with these operations in 2-dimensional space.

On the other hand, a minimally rigid graph in 3-dimensional space may have all vertices of degree larger than 4; $|\mathcal{L}| = 3|\mathcal{V}| - 6$ or equivalently $2|\mathcal{L}| = 6|\mathcal{V}| - 12$ guarantees only that some vertices have degree 5 or less. A quick check with the vertex addition and edge splitting operations in 3-dimensional space tells us that we can generate vertices of degree 3 and 4 with these operations, but not of degree 5. We need other types of operations to generate minimally rigid graphs in 3-dimensional space with all vertices having a degree of 5 or higher, and to remove a vertex of degree 5 from a minimally rigid graph in 3-dimensional space. The following theorem is about removing a 5-valent vertex in a minimally rigid graph.

Theorem 7 (Removing a 5-valent vertex [12]). *Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a minimally rigid graph with a 5-valent vertex a and edges (a, b_i), $1 \leq i \leq 5$. Let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$ be a graph obtained by removing vertex a and the edges (a, b_i), $1 \leq i \leq 5$ from $\mathbb{G}$. Then one of the following is true: 1) for some choice of two non-adjacent edges with vertices drawn from $b_1, b_2, \ldots, b_5$, the graph obtained by inserting these edges is minimally rigid in 3-dimensional space; 2) for two choices of adjacent pairs of edges with vertices drawn from $b_1, b_2, \ldots, b_5$ (not all adjacent with a single vertex), the two graphs obtained from $\mathbb{G}^*$ by inserting these pairs are both minimally rigid in 3-dimensional space.*

Adding a 5-valent vertex to a minimally rigid graph in 3-dimensional space to guarantee preservation of minimal rigidity is a long-standing problem. So far there are only partial results [12]. There are two sequential operations for adding 5-valent vertices to minimally rigid graphs. They are conjectured to preserve minimal rigidity in 3-dimensional space. The first operation is replacing an X (i.e., two edges that do not share any vertices) by a vertex of degree five by connecting the vertex to the end vertices of these two edges plus an additional vertex. The second operation is the double V replacement. This operation takes two graphs $\mathbb{G}_1$ and $\mathbb{G}_2$ that are minimally rigid in 3-dimensional space to a graph $\mathbb{G}$ that is generically rigid in 3-dimensional space.

Conjecture 8 (Adding a 5-valent vertex by replacing a single X [12]). *If $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is a minimally rigid graph in 3-dimensional space containing edges $(a, b), (c, d)$, then the graph obtained by deleting these two edges and adding a 5-valent vertex i attached to the vertices a, b, c, d and another vertex $e \in \mathcal{V}$ is minimally rigid in 3-dimensional space.*

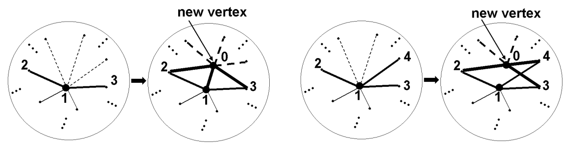
Conjecture 9 (Adding a 5-valent vertex by replacing 2 V's [12]). *If $\mathbb{G}_1 = \mathbb{G} \cup \{(a, b), (b, c)\}$ and $\mathbb{G}_2 = \mathbb{G} \cup \{(\tilde{a}, \tilde{b}), (\tilde{b}, \tilde{c})\}$ are minimally rigid graphs in 3-dimensional space with $b \neq \tilde{b}$, then the graph $\mathbb{G}^*$ obtained from $\mathbb{G}$ by adding a 5-valent vertex i, attached to vertices including $a, b, c, \tilde{a}, \tilde{b}, \tilde{c}$, is also a minimally rigid graph in 3-dimensional space.*

Every minimally rigid graph can be generated by this *extended Henneberg sequence*, which includes the two operations in the conjectures with the vertex addition and the simpler edge splitting operations. What is unproven is that only minimally rigid graphs in 3-dimensional space are generated in this way. The lack of a simpler technique for adding 5-valent vertices is connected to the failure of any simple spatial analogues of Laman's Theorem.

There are two partial results which give sufficient conditions for the replacement of two edges by a 5-valent vertex and three edges by a 6-valent vertex. These operations are called *vertex splitting*. If the graph $\mathbb{G}'$ is a vertex split of a generically rigid graph $\mathbb{G}$ on d edges in d-space or a vertex split on $d-1$ edges, then $\mathbb{G}'$ is generically rigid in d-space [13]. Vertex 3-splits on two edges and three edges are depicted in Figures 1a and 1b. By vertex d-split, we mean a split in d-space.

III. Globally Rigid Formations

A non-rigid point formation has infinitely many "realizations" for the given values of the constraints or dimensions. Assigning coordinates to the vertices of a graph is called *graph realization*. More precisely, given a graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$, $\bigcup\{(p_i, p_j) : (i, j) \in \mathcal{L}\} \cup \bigcup\{p_i\} \subset \mathbb{R}^d$ is said to be a realization of $\mathbb{G}$ in d-dimensional space where (p_i, p_j) is the straight line segment with endpoints p_i and



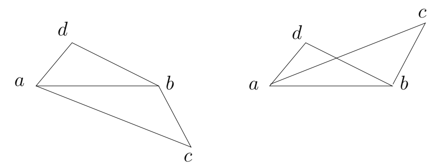
(a) (b)

Fig. 1. (a) Vertex 3-split on two edges. Edges which are shifted to the new vertex 0 are shown with dashed edges, and new edges after split are shown with wider edges. (b) Vertex 3-split on three edges. Edges which are shifted to the new vertex 0 are shown with dashed edges, and new edges after split are shown with wider edges.

p_j. Translations, rotations and reflections are not considered to be different realizations. It turns out that even a rigid formation may have several distinct realizations in this sense.

Each point formation $\mathbb{F}_p$ uniquely determines a graph $\mathbb{G} \triangleq \{\mathcal{V}, \mathcal{L}\}$ with vertex set $\mathcal{V} \triangleq \{1, 2, \ldots, n\}$ and edge set $\mathcal{L}$, as well as a distance function $\delta : \mathcal{L} \rightarrow \mathbb{R}$ whose value at $(i, j) \in \mathcal{L}$ is the distance between p_i and p_j. Recall that the distance function of $\mathbb{F}_p$ is the same as the distance function of any point formation $\mathbb{F}_q$ with the same graph as $\mathbb{F}_p$ provided q is *congruent* to p in the sense that there is a distance-preserving map $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $T(q_i) = p_i, i \in \{1, 2, \ldots, n\}$. Furthermore, recall that two point formations $\mathbb{F}_p$ and $\mathbb{F}_q$ are *congruent* if they have the same graph and if q and p are congruent. It is clear that $\mathbb{F}_p$ is uniquely determined by its graph and distance function *at most* up to a congruence transformation. A formation which is *exactly* determined up to congruence by its graph and distance function is called "globally rigid." More precisely, a d-dimensional point formation $\mathbb{F}_p$ is said to be *globally rigid* if each d-dimensional point formation $\mathbb{F}_q$ with the same graph and distance function as $\mathbb{F}_p$ is congruent to $\mathbb{F}_p$.

As we have already noted, we need formations whose point formations are uniquely determined up to congruence by their graphs and distance functions. Unfortunately rigidity is not a strong enough property of a formation to ensure that this is so. In other words it is possible to construct two rigid formations $\mathbb{F}_p$ and $\mathbb{F}_q$ which both have the same graph and distance function, but are not congruent. The subtlety here stems from the fact that rigidity of $\mathbb{F}_p$ stipulates that only those formations encountered on trajectories containing $\mathbb{F}_p$ be congruent to $\mathbb{F}_p$. Unfortunately there are formations with the same graph and distance function as $\mathbb{F}_p$ which cannot be reached from $\mathbb{F}_p$ on any trajectory; such formations are typically not congruent to $\mathbb{F}_p$. From a different perspective, a rigid formation is a formation which is impossible to deform *continuously* while holding fixed the lengths of all of its links. There are examples of rigid formations which can indeed be deformed, but not continuously; such formations are rigid but not globally rigid. In the end, the key feature which distinguishes globally rigid formations from all others including those which are merely



(a) (b)

Fig. 2. Two rigid formations with the same graph and distance function.

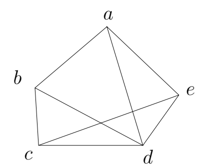


Fig. 3. A globally rigid formation.

rigid is that the former cannot be deformed by any means whatever, continuous or not, whereas the latter always can.

An example of a rigid formation which can be deformed discontinuously is shown in Figure 2(a). Observe that a discontinuous deformation can be obtained by reflecting the triangle formed by points a, b, c about the line determined by points a and b. The resulting rigid formation is shown in Figure 2(b). Adding a link from point c to d in Figure 2(a) would make the formation globally rigid. An example of a globally rigid formation whose graph is not complete is shown in Figure 3.

Let us agree to say that a formation $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$ of n points in $\mathbb{R}^d$ is *generically globally rigid* if for each q in some open neighborhood of p in $\mathbb{R}^{dn}$, formation $\mathbb{F}_q = (\{q_1, q_2, \ldots, q_n\}, \mathcal{L})$ is globally rigid. There is a graph-theoretic characterization of generic global rigidity for 2-dimensional formations analogous to the characterization of generic rigidity provided by Laman's theorem. To explain the result we need a few more concepts.

A connected graph $\mathbb{G}$ is k-*connected* if it is possible to obtain from it a new graph with at least two distinct connected components by removing at least one set of k vertices from $\mathbb{G}$ along with all of those edges of $\mathbb{G}$ which are incident on the k vertices being removed. A graph $\mathbb{G}$ which is generically rigid in $\mathbb{R}^d$ is *redundantly rigid* in $\mathbb{R}^d$ if removal of any single edge results in a graph which is also generically rigid in $\mathbb{R}^d$. Finally, a connected simple graph $\mathbb{G} = \{\mathcal{V}, \mathcal{L}\}$ with n vertices is *generically globally rigid* in $\mathbb{R}^2$ if there is an open dense set of points $p \in \mathbb{R}^{dn}$ at which $\mathbb{F}_p$ is a globally rigid formation with link set $\mathcal{L}$. The following recent result settles the generic global rigidity question for $d = 2$ in graph theoretic terms [2], [10].

Theorem 10. *A connected simple graph* $\mathbb{G}$ *with* $n \geq 4$ *vertices is generically globally rigid in* $\mathbb{R}^2$ *if and only if it is* 3*-connected and redundantly rigid in* $\mathbb{R}^2$.

The proof is built on the sequential construction from the base case of $\mathbb{K}_4$, the complete graph with four vertices, by

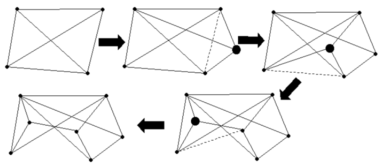


Fig. 4. A sequence for generating a globally rigid formation. The sequence starts with K_4, and a new vertex (shown as a larger circle) is adjoined at each step by edge splitting operation. Edges about to be split are shown as dashed lines.

a sequence of edge splits, in a manner which extends the Henneberg sequences as shown in Figure 4. This suggests that a sequential approach might give some results for global rigidity for other classes of formations and for higher dimensions.

Much like the situation with generic rigidity, the generalization of Theorem 10 to higher dimensions does not yet exist [2]. There is no general algebraic test for generic global rigidity of a graph. There is even a question, in 3-dimensional space, what one means by generic global rigidity: (a) a graph may be globally rigid for points forming an open dense subset of $\mathbb{R}^3$; (b) a graph may be globally rigid for points forming an open non-empty subset of $\mathbb{R}^3$. It appears that (b) does not imply (a) in 3-dimensional space (though it does in 2-dimensional space). We mean (a) in our discussions on generic global rigidity. In 2-dimensional space, it might be possible to turn Theorem 10 into some algebraic condition involving repeated uses of the rigidity matrix.

The known bad example in 3-dimensional space is $\mathbb{K}_{5,5}$, a complete bipartite graph [2]. (A graph $\mathbb{G}$ is called *bipartite* if its vertex set can be partitioned into two parts $\mathcal{V}_1$ and $\mathcal{V}_2$ such that every edge has one end in $\mathcal{V}_1$ and one in $\mathcal{V}_2$.) This is also the counterexample to 3-dimensional version of Theorem 10, since it is 5-connected and redundantly rigid. It fails to be globally rigid in some open neighborhood of points lying on a quadric surface. Therefore, it cannot be generically globally rigid. It is not known, for sure, that it is globally rigid at some other open neighborhood of points. It is suspected that it is, but this is unproven. So it is suspected that global rigidity is not a generic property, but the last piece of that proof/counterexample is still missing. We know that there are classes of graphs for which global rigidity is a generic property. For example, graphs generated from $\mathbb{K}_5$ by a sequence of edge splits (as explained in the previous section) and edge additions are generically globally rigid. By *edge addition* we mean inserting edges into a graph.

At the moment we do not have a conjecture for which graphs are generically globally rigid in 3-dimensional space. However, we have a partial result and a conjecture for subclasses of graphs. The result uses the same techniques described for 2-dimensional space in Connelly [2].

Theorem 11. *A graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ with at least five vertices, is generically globally rigid in 3-dimensional space if there is an ordering of vertices $1, 2, \ldots, |\mathcal{V}|$ and a sequence of graphs $\mathbb{G}_5, \ldots, \mathbb{G}_{|\mathcal{V}|}$ such that: 1) $\mathbb{G}_5$ is K_5; 2) for $5 \le i \le |\mathcal{V}|$, $\mathbb{G}_{i+1}$ is generated by (i) adding a 4-valent vertex (ii) edge splitting; 3) $\mathbb{G}_{|\mathcal{V}|}$ is $\mathbb{G}$.*

In 2-dimensional space, because of duality [1], vertex splitting creating two vertices of at least degree 3 is also known to preserve generic global rigidity.

Conjecture 12. *If, in the previous result we add a third step: (iii) vertex splitting of either type (vertex split on d edges or on $d-1$ edges in d dimensional space as explained in §II) such that each of the new vertices is at least 4-valent; then the resulting graph is also generically globally rigid in 3-dimensional space.*

While these steps will not generate all globally rigid formations, they will generate classes of formations for any number of vertices. Adding a sequence of 4-valent vertices to a set of n points which are in general position (no four coplanar) will generate a globally rigid formation with $4n - 10$ edges, and using edge splitting alone will generate a globally rigid formation with $3n - 5$ edges. The number of edges in these graphs is less than the number of edges in the complete graph, which has $n(n+1)/2$ edges.

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