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Delay-dependent robust output feedback stabilization of uncertain state-delayed systems with saturating actuators

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Abstract. The robust output feedback stabilization problem for state-delayed systems with time-varying delays and saturating actuators is addressed here. The systems considered are continuous-time, with parametric uncertainties entering all the matrices in the system representation. A saturating control law is designed and a region of initial conditions is specified within which local asymptotic stability of the closed-loop system is ensured. The designed controller is dependent on the time-delay and its rate of change. The controller is constructed in terms of the solution to a set of matrix inequalities.

1. Introduction

In recent years, much work has been devoted to the analysis and synthesis of controllers for state-delayed systems with or without parametric uncertainties; see [1] for earlier work. To the authors best knowledge, the issue of output feedback stabilization of uncertain state-delayed systems with saturating control was only addressed in [2].

In this context, the contribution of this paper can be summarized as follows:

- A delay-dependent robust output feedback stabilization problem is addressed for state-delayed systems in full generality, including actuators constraints and norm-bounded parametric uncertainties entering all the matrices in the system representation.
- In attempt to make the approach the least conservative, unlike in [2], the most efficient descriptor model transformation and Lyapunov-Krasovskii functional are employed, as in [3].
- The least conservative model for the actuator saturation; see [4], as delivered by the differential inclusions approach is employed here.
- The value of the time-delay as well as its rate of change are taken into account in the design method presented and further permit to reduce the conservativeness of the approach.
- The method developed here is applied to an example system which was previously used in [5], and shows that the new design yields less conservative results, in that stabilization is ensured for a larger set of initial conditions.

2. Problem statement

Consider the following uncertain time-delay system:

$$\dot{x}(t) = A(t)x(t) + A_h(t)x(t-h(t)) + B(t)u_{sat}(t),$$
$$y(t) = C(t)x(t) + D(t)u_{sat}(t), \quad u_{sat}(t) = sat(u(t)),$$
$$sat(u(t)) = \left[sat(u_1(t)) \quad sat(u_2(t)) \quad \dots \quad sat(u_m(t)) \right]$$
$$x(t_0+\psi) = \phi(\psi),\ \forall \psi \in [-h_{\max}, 0],\ (t_0, \phi) \in R^+ \times C^w_{h_{\max}, n} \quad (1)$$

where $x(t) \in R^n$ is the state vector, $y(t) \in R^r$ is the output vector, $u(t) \in R^m$ is the control input vector to the actuator (generated from the designed controller), $u_{sat}(t) \in R^m$ is the control input vector to the plant. The time-delay h is a function of time and is assumed to be continuously differentiable, with its amplitude and rate of change bounded as follows:

$$0 \le h(t) \le h_{\max}, \quad 0 \le \dot{h}(t) \le \beta < 1, \qquad \text{for all} \quad t \ge 0 \quad (2)$$

where $h_{\max}$ and α are given positive constants. The uncertain matrices $A(t)$, $A_h(t)$, $B(t)$, $C(t)$ and $D(t)$ are:

$$A(t) = A + \Delta A(t),\ A_h(t) = A_h + \Delta A_h(t),\ B(t) = B + \Delta B(t),$$
$$C(t) = C + \Delta C(t),\ D(t) = D + \Delta D(t) \quad (3)$$

where the matrices A, A_h, B, C and D are real, and are assumed to be known exactly. The matrices $\Delta A(t)$, $\Delta A_h(t)$, $\Delta B(t)$, $\Delta C(t)$ and $\Delta D(t)$ are real-valued, represent the norm-bounded parameter uncertainties, and are assumed to be of the following form:

$$\begin{bmatrix} \Delta A(t) & \Delta B(t) \\ \Delta C(t) & \Delta D(t) \end{bmatrix} = \begin{bmatrix} H_A \\ H_C \end{bmatrix} F(t) \begin{bmatrix} E_A & E_B \end{bmatrix},$$
$$\Delta A_h(t) = H_h F_h(t) E_h \quad (4)$$

where $F(t) \in R^{i \times j}$ and $F_h(t) \in R^{i_h \times j_h}$ are real, uncertain and time-varying matrices with Lebesgue measurable entries which, additionally, meet the following requirements: $F(t)F^T(t) \le I$ and $F_h(t)F_h^T(t) \le I$. The matrices H_A, H_C, H_h, E_A, E_B and E_h are known and real, and characterize the way in which the uncertain parameters in $F(t)$ and $F_h(t)$ enter the nominal matrices A, A_h, B, C and D.

Finally, the following is assumed to hold:

Assumption 1: $(A + A_h, B)$ is stabilizable and (C, A) is detectable.

Assumption 2: The input vector is subject to amplitude constraints, i.e. $u \in U_0 \subset R^m$, with

$$U_0 \triangleq \left\{ u \in R^m \ ;\ -\bar{u}_i \le u_i \le \bar{u}_i\ ,\ \bar{u}_i > 0,\ i = 1 \dots m \right\}. \quad (5)$$

The results to be presented are concerned with providing sufficient conditions for the design of an observer-based dynamic output feedback law for system (1). This law is assumed to take the following form:

$$u(t) = K\hat{x}(t) \tag{6}$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_{sat}(t) + L\left(y(t) - C\hat{x}(t) - Du_{sat}(t)\right) \tag{7}$$

where $\hat{x}(t) \in R^n$ is the observer state vector, $K \in R^{m\times n}$ and $L \in R^{n\times r}$ are the constant controller and observer gains, respectively. Using (1), (4) and (7), permits to write,

$$\begin{aligned}
\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} &= \left\{ \begin{bmatrix} A & 0 \\ LC & A-LC \end{bmatrix} + \begin{bmatrix} H_A \\ LH_C \end{bmatrix} F(t) \begin{bmatrix} E_A & 0 \end{bmatrix} \right\} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \\
&+ \left\{ \begin{bmatrix} A_h & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} H_h \\ 0 \end{bmatrix} F_h(t) \begin{bmatrix} E_h & 0 \end{bmatrix} \right\} \begin{bmatrix} x(t-h(t)) \\ \hat{x}(t-h(t)) \end{bmatrix} \\
&+ \left\{ \begin{bmatrix} B \\ B \end{bmatrix} + \begin{bmatrix} H_A \\ LH_C \end{bmatrix} F(t) E_B \right\} u_{sat}(t)
\end{aligned} \tag{8}$$

Finally, introducing the following definitions,

$$\begin{aligned}
&\xi(t) \triangleq \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix},\ \hat{A} \triangleq \begin{bmatrix} A & 0 \\ LC & A-LC \end{bmatrix},\ \hat{A}_h \triangleq \begin{bmatrix} A_h & 0 \\ 0 & 0 \end{bmatrix}, \\
&\Delta\hat{A}(t) \triangleq \hat{H}F(t)\hat{E} \triangleq \begin{bmatrix} H_A \\ LH_C \end{bmatrix} F(t) \begin{bmatrix} E_A & 0 \end{bmatrix},\ \hat{B} \triangleq \begin{bmatrix} B \\ B \end{bmatrix}, \\
&\Delta\hat{A}_h(t) \triangleq \hat{H}_h F_h(t) \hat{E}_h \triangleq \begin{bmatrix} H_h \\ 0 \end{bmatrix} F_h(t) \begin{bmatrix} E_h & 0 \end{bmatrix}, \\
&\Delta\hat{B}(t) \triangleq \hat{H}F(t)E_B \triangleq \begin{bmatrix} H_A \\ LH_C \end{bmatrix} F(t) E_B,\ \hat{A}(t) \triangleq \hat{A} + \Delta\hat{A}(t), \\
&\hat{A}_h(t) \triangleq \hat{A}_h + \Delta\hat{A}_h(t),\ \hat{B}(t) \triangleq \hat{B} + \Delta\hat{B}(t),\ \hat{K} = \begin{bmatrix} 0 & K \end{bmatrix}
\end{aligned} \tag{9}$$

permits to re-write equation (8) in yet more aggregated compact form,

$$\dot{\xi}(t) = \hat{A}(t)\xi(t) + \hat{A}_h(t)\xi(t-h(t)) + \hat{B}(t)u_{sat}(t),$$

with $u_{sat}(t) = sat\left(\hat{K}\xi(t)\right)$. (10)

The robust output feedback stabilization problem with saturating actuators:

Find a matrix $K \in R^{m\times n}$, a matrix $L \in R^{n\times r}$ and a set of initial conditions $S_0 \subset R^{2n}$ such that the closed-loop system (10) is asymptotically stable.

3. Main result

The following theorem provides sufficient conditions for robust output feedback stabilization of the uncertain time-delay system (10) with control saturation.

Theorem 1. *Consider the system* (10)*. Suppose that there exist,* $n\times n$*-matrices:* P_{i11}, P_{i12}, P_{i21}, P_{i22}*;* $i = 1,...,3$, W_{i11}, W_{i12}, W_{i21}, W_{i22}*;* $i = 1,...,4$, R_{i11}, R_{i12}, R_{i21}, R_{i22}*;* $i = 1,...,3$, $S = S^T > 0$*, an* $m\times n$ *matrix* K *and an* $n\times r$ *matrix* L*, and a vector* $\underline{\alpha} \in R^m$*, and a positive scalar* γ*, which together with some suitable positive scalars* ε_i*;* $i = 1,...,3$*, satisfy the following matrix conditions:*

$$\begin{bmatrix} P_{111} & P_{112} \\ P_{121} & P_{122} \end{bmatrix} > 0,\ P_{111} = P_{111}^T,\ P_{112} = P_{121}^T,\ P_{122} = P_{122}^T \tag{11}$$

$$\begin{bmatrix} R_{311} & R_{312} \\ R_{321} & R_{322} \end{bmatrix} > 0,\ R_{311} = R_{311}^T,\ R_{312} = R_{321}^T,\ R_{322} = R_{322}^T \tag{12}$$

$$\begin{bmatrix} R_{111} & R_{112} & R_{211} & R_{212} \\ R_{121} & R_{122} & R_{221} & R_{222} \\ R_{211}^T & R_{221}^T & R_{311} & R_{312} \\ R_{212}^T & R_{222}^T & R_{321} & R_{322} \end{bmatrix} > 0,\ R_{111} = R_{111}^T,\ R_{112} = R_{121}^T,\ R_{122} = R_{122}^T \tag{13}$$

$$\begin{bmatrix} \Omega_{1,j} & \Omega_2 \\ \Omega_2^T & \Omega_3 \end{bmatrix} < 0, \qquad \forall j = 1,...,2^m \tag{14}$$

$$\begin{bmatrix} P_{111} & P_{112} & 0 \\ P_{121} & P_{122} & \underline{\alpha}_i K_{(i)}^T \\ 0 & \underline{\alpha}_i K_{(i)} & \gamma\bar{u}_i^2 \end{bmatrix} \geq 0, \qquad \forall i = 1,...,m \tag{15}$$

$$\underline{\alpha}_i \in (0,1], \qquad \forall i = 1,...,m \tag{16}$$

where $K_{(i)}$ *is the* $i-th$ *row of* K*, and where,*

$$\Omega_{1,j} = \begin{bmatrix}
\Psi_{111} & \Psi_{112,j} & \Psi_{211} & \Psi_{212} & h_{\max}(W_{111}^T + P_{111}^T) & h_{\max}(W_{121}^T + P_{121}^T) & h_{\max}(W_{311}^T + P_{211}^T) & h_{\max}(W_{321}^T + P_{221}^T) & -W_{311}^T A_h - \varepsilon_3 E_h^T E_h \\
 & \Psi_{122,j} & \Psi_{221,j} & \Psi_{222,j} & h_{\max}(W_{112}^T + P_{112}^T) & h_{\max}(W_{122}^T + P_{122}^T) & h_{\max}(W_{312}^T + P_{212}^T) & h_{\max}(W_{322}^T + P_{222}^T) & -W_{312}^T A_h \\
 & & \Psi_{311} & \Psi_{312} & h_{\max}W_{211}^T & h_{\max}W_{221}^T & h_{\max}(W_{411}^T + P_{311}^T) & h_{\max}(W_{421}^T + P_{321}^T) & -W_{411}^T A_h \\
 & & & \Psi_{322} & h_{\max}W_{212}^T & h_{\max}W_{222}^T & h_{\max}(W_{412}^T + P_{312}^T) & h_{\max}(W_{422}^T + P_{322}^T) & -W_{412}^T A_h \\
 & & & & -h_{\max}R_{111} & -h_{\max}R_{112} & -h_{\max}R_{211} & -h_{\max}R_{212} & 0 \\
 & & & & & -h_{\max}R_{122} & -h_{\max}R_{221} & -h_{\max}R_{222} & 0 \\
 & & & & & & -h_{\max}R_{311} & -h_{\max}R_{312} & 0 \\
 & & & & & & & -h_{\max}R_{322} & 0 \\
 & & & & & & & & -(1-\beta)S + \varepsilon_3 E_h^T E_h
\end{bmatrix} \tag{17}$$

$$\Omega_2 = \begin{bmatrix} P_{211}^T H_A + P_{221}^T LH_C & P_{211}^T H_h & W_{311}^T H_h \\ P_{212}^T H_A + P_{222}^T LH_C & P_{212}^T H_h & W_{312}^T H_h \\ P_{311}^T H_A + P_{321}^T LH_C & P_{311}^T H_h & W_{411}^T H_h \\ P_{312}^T H_A + P_{322}^T LH_C & P_{312}^T H_h & W_{412}^T H_h \end{bmatrix} \tag{18}$$

$$\Omega_3 = diag\{-\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I\} \tag{19}$$

$$\begin{aligned}
\Psi_{111} &= (A + A_h)^T P_{211} + P_{211}^T (A + A_h) + (LC)^T P_{221} + P_{221}^T LC \\
&\quad + W_{311}^T A_h + A_h^T W_{311} + S + \varepsilon_1 E_A^T E_A + (\varepsilon_2 + \varepsilon_3) E_h^T E_h
\end{aligned}$$

$$\begin{aligned}
\Psi_{112,j} &= (A + A_h)^T P_{212} + (LC)^T P_{222} + P_{211}^T B\Gamma_j(\underline{\alpha})K \\
&\quad + P_{221}^T \left(A - LC + B\Gamma_j(\underline{\alpha})K\right) + A_h^T W_{312} + \varepsilon_1 E_A^T E_B \Gamma_j(\underline{\alpha})K
\end{aligned}$$

$$\begin{aligned}
\Psi_{122,j} &= \left(B\Gamma_j(\underline{\alpha})K\right)^T P_{212} + P_{212}^T B\Gamma_j(\underline{\alpha})K \\
&\quad + \left(A - LC + B\Gamma_j(\underline{\alpha})K\right)^T P_{222} + P_{222}^T \left(A - LC + B\Gamma_j(\underline{\alpha})K\right)
\end{aligned}$$

$$+\varepsilon_1\left(E_B\Gamma_j(\underline{\alpha})K\right)^T E_B\Gamma_j(\underline{\alpha})K$$

$$\Psi_{211}=P_{111}^T-P_{211}^T+\left(A+A_h\right)^T P_{311}+\left(LC\right)^T P_{321}+A_h^T W_{411},$$

$$\Psi_{212}=P_{121}^T-P_{221}^T+\left(A+A_h\right)^T P_{312}+\left(LC\right)^T P_{322}+A_h^T W_{412}$$

$$\Psi_{221,j}=P_{112}^T-P_{212}^T+\left(B\Gamma_j(\underline{\alpha})K\right)^T P_{311}+\left(A-LC+B\Gamma_j(\underline{\alpha})K\right)^T P_{321}$$

$$\Psi_{222,j}=P_{122}^T-P_{222}^T+\left(B\Gamma_j(\underline{\alpha})K\right)^T P_{312}+\left(A-LC+B\Gamma_j(\underline{\alpha})K\right)^T P_{322}$$

$$\Psi_{311}=-P_{311}-P_{311}^T+h_{\max}\hat{A}_{h11,\max}^T R_{311}\hat{A}_{h11,\max},$$

$$\Psi_{312}=-P_{312}-P_{321}^T, \qquad \Psi_{322}=-P_{322}-P_{322}^T$$

Under these conditions, system (10) *is locally asymptotically stable for any initial condition* $\phi(\sigma)$ *in the ball* $\Phi(\sigma)=\left\{\phi\in C_{h_{\max},n}^{w};\ \|\phi\|_c^2\le\sigma\right\}$, *with* $\sigma=\dfrac{1}{\gamma\pi_2}$ (20)

where, $$\pi_2=\lambda_{\max}\left(\begin{bmatrix}P_{111} & P_{112}\\ P_{121} & P_{122}\end{bmatrix}\right)+\frac{h_{\max}}{(1-\beta)}\lambda_{\max}(S)+\frac{3}{2}h_{\max}^2\lambda_{\max}\left(\hat{A}_{h11,\max}^T R_{311}\hat{A}_{h11,\max}\right) \quad (21)$$

4. Numerical example.

Consider system (1) with the following state space matrices:

$$A=\begin{bmatrix}0.1 & -0.1\\ 0.1 & -3\end{bmatrix},\ B=\begin{bmatrix}5 & 0\\ 0 & 1\end{bmatrix},\ C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ H_A=\frac{1}{20}\begin{bmatrix}1\\ 1\end{bmatrix},$$

$$E_A=\begin{bmatrix}1 & 1\end{bmatrix},\quad E_B=\begin{bmatrix}1 & 1\end{bmatrix},\quad \overline{u}_1=5,\quad \overline{u}_1=2 \quad (22)$$

with all the other matrices being equal to zero. It is exactly the same example that was used previously in [5], where the authors using relaxation techniques achieved an initial state set (assuming zero initial conditions for $\hat{x}$),

$$D_0=\left\{x:\ x^T Zx\le 1,\ Z=10^{-6}\begin{bmatrix}290.5 & -7.966\\ -7.966 & 0.500\end{bmatrix}\right\}.$$

The corresponding volume of this set was computed as $\log\left(Vol\left(D_0\right)\right)=\log\left(\sqrt{\det(Z)}\right)=11.61$. In an attempt to achieve a larger set of initial conditions guaranteeing asymptotic stability of the closed-loop system, the following relaxation schemes were used in the present paper:

LMIR 1: Given K, L, $\underline{\alpha}_1$ and $\underline{\alpha}_2$, solve for all the P matrices and γ, the problem $Min\left(w\text{Trace}(P_1)+\gamma\right)$, such that the LMIs of Theorem 1 are satisfied, where w is a column vector whose four entries are weights multiplying each diagonal entry of matrix P_1 to form its trace.

LMIR 2: Given all the P matrices, K, $\underline{\alpha}_1$ and $\underline{\alpha}_2$, solve for L and γ the problem $Min(\gamma)$, such that the LMIs of Theorem 1 are satisfied.

Using the above relaxation schemes, the achieved set of initial conditions (assuming zero initial conditions for $\hat{x}$), is $S_c=\left\{x:\ x^T\gamma P_{111}x\le 1,\ \gamma P_{111}=10^{-6}\begin{bmatrix}183.3 & -4.396\\ -4.396 & 0.355\end{bmatrix}\right\}$, and the corresponding computed volume is $\log\left(Vol\left(S_c\right)\right)=\log\left(\sqrt{\det\left(\gamma P_{111}\right)}\right)=11.90$.

Figure 1 shows the sets of initial conditions achieved in [5] and in the present paper, where it is seen that there is a substantial increase in the size of the set of initial conditions guaranteeing asymptotic stability.

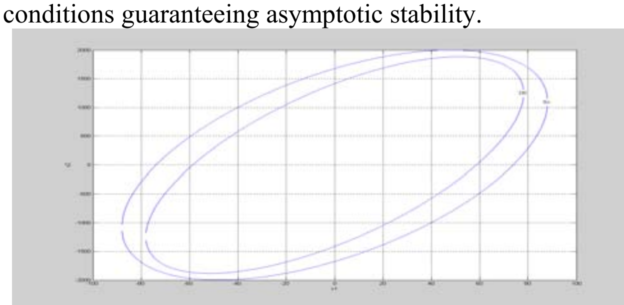

figure 1: Set S_c of initial conditions achieved in this paper as compared to set D_0 of [5]

5. Conclusion

The problem formulation employed here is believed to be the first and the most general considered so far with reference to the delay-dependent robust output feedback stabilization of state-delayed systems with saturating actuators. A major innovation of the approach adopted here is that the stabilizing control design is made dependent on both the value of the time-delay as well as on its rate of change. As it was demonstrated by way of an example, the result of Theorem 1 ensures stability for larger sets of initial conditions than previous results obtained in the literature.

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