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Simulation Studies on the Boundary Stabilization and Disturbance Rejection for Fractional Diffusion-Wave Equation

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***Abstract*— A class of evolution systems described by the one-dimensional fractional diffusion-wave equation subject to a boundary controller at the boundary is considered. Both boundary stabilization and disturbance rejection are considered. This paper, for the first, has confirmed, via hybrid symbolic and numerical simulation studies, that the existing two schemes for boundary stabilization and disturbance rejection for (integer order) wave/beam equations are still valid for fractional order diffusion-wave equations. The problem definition, the hybrid symbolic and numerical simulation techniques, outlines of the methods for boundary stabilization and disturbance rejection are presented together with extensive simulation results. Different dynamic behaviors are revealed for different fractional orders which create new future research opportunities.**

***Index Terms*— Fractional order calculus, fractional diffusion equation, fractional wave equation, boundary control, disturbance rejection.**

I. Introduction

Fractional diffusion and wave equations are obtained from the classical diffusion and wave equations by replacing the first and second order time derivative term by a fractional derivative $(0, 1)$ and $(1, 2)$, respectively. There has been a growing interest in investigating the solutions and properties of these equations because many of the universal phenomenons can be modeled accurately using the fractional diffusion and wave equations [1]. Research has been focused on the analytical solution to the fractional diffusion and wave equations. In [2], the solution to the fractional diffusion equation was given in closed form in terms of Fox functions. In [3], the fractional diffusion and wave equations were reformulated as integrodifferential equations. Analytical solutions for the Green's functions of the latter were then found. In [4], the fractional wave equation was shown to govern the propagation of stress waves in viscoelastic solids. The transition from a pure diffusion process to a pure wave process was also shown when the time derivative increases from 1 to 2. In [5], a general solution was given for a fractional diffusion-wave equation defined in a bounded space domain.

In this paper, we study the boundary control of a string governed by the fractional wave equation using the simulation method proposed in [6]. To the best of our knowledge, research on the boundary control of the fractional wave equation is still relatively new. In [7], the fractional derivative control was applied to the boundary control of the wave equation, which is similar, but still very different from the topic in this paper. The major contribution of this paper is that, for the first, we have confirmed, via hybrid symbolic and numerical simulation studies, that the existing two schemes for boundary stabilization and disturbance rejection for (integer order) wave/beam equations are still valid for fractional order diffusion-wave equations. The problem definition, the hybrid symbolic and numerical simulation techniques, outlines of the methods for boundary stabilization and disturbance rejection are presented together with extensive simulation results. Different dynamic behaviors are revealed for different fractional orders which create new future research opportunities.

The paper is organized as follows. In Sec. II, we give the mathematical description of the boundary control of fractional wave equation. In Sec. III, we briefly summarize the simulation method used. In Sec. IV, the effectiveness of a static boundary velocity controller, a controller mostly used in the boundary control of wave equation [8] [9] is studied. In Sec. V, we study the disturbance rejection performance of a dynamic controller proposed in [10] for the boundary control of wave equation. Finally, Sec. VI concludes this paper.

II. Problem definition

We consider a string, which might be made from special materials, governed by the fractional wave equation, fixed at one end, and stabilized by a boundary control at the other end. The system can be represented by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2}, \quad 1 < \alpha < 2, \quad x \in [0, 1], \quad t \geq 0 \quad (1)$$

$$u(0, t) = 0, \quad (2)$$

$$u_x(1, t) = f(t), \quad (3)$$

$$u(x, 0) = u_0(x), \quad (4)$$

$$u_t(x, 0) = v_0(x), \quad (5)$$

where $u(x, t)$ is the displacement of the string at $x \in [0, 1]$ and $t \geq 0$, $f(t)$ is the boundary control force at the free end of the string, $u_0(x)$ and $v_0(x)$ are the initial conditions of displacement and velocity, respectively.

The control objective is to stabilize $u(x, t)$, given the initial conditions (4) and (5).

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We adopt the following definition for the fractional derivative of order α of function $f(t)$ [4][11],

$$\frac{d^\alpha}{dt^\alpha} f(t) \doteq \begin{cases} f^{(n)}(t) & \text{if } \alpha = n \in N, \\ \frac{t^{n-\alpha-1}}{\Gamma(n-\alpha)} * f^{(n)}(t) & \text{if } n-1 < \alpha < n, \end{cases} \quad (6)$$

where the $*$ denotes the time convolution between two causal functions.

Based on the definition of (6), the Laplace transform of the fractional derivative is

$$\mathcal{L}\left\{\frac{d^\alpha}{dt^\alpha}\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} f^k(0^+) s^{\alpha-1-k} \quad (7)$$

III. A Hybrid Symbolic-Numerical Simulation Method

In [6], a method combining symbolic algebra and numerical methods was developed to simulate some typical boundary control problems, including the boundary control of beam equation studied in [12]. Here we summarize the steps.

1) Take the Laplace transform of (1), (2), (3), (4) and (5) with respect to t, including the controller $f(t)$. Thus, the original PDE of $u(x,t)$ with initial and boundary conditions is transformed into an ODE of $U(x,s)$ with boundary conditions. The resulting ODE is usually hard to solve manually, due to the complicity of the fractional wave equation, the dynamic controller $f(t)$, and the initial conditions.
2) Call the Matlab Symbolic Math Toolbox function `dsolve()` to symbolically solve the ODE(s) and the boundary or initial condition(s). Although `dsolve()` is able to determine the arbitrary constants in the solution using the boundary or initial condition(s), we find that its capability is very weak. Here, we feed only the ODE of $U(x,s)$ to `dsolve()` rather than provide both the ODE of $U(x,s)$ and the boundary conditions. The expression of $U(x,s)$ with two arbitrary constants $C1$ and $C2$, which are to be determined later, can be obtained.
3) Using Matlab Symbolic Math Toolbox function `diff()`, differentiate $U(x,s)$ with respect to x to get the derivatives of $U(x,s)$. Substituting $U(x,s$ and its derivatives into the Laplace transform of (2) and (3), we can get two equations with two unknowns $C1$ and $C2$.
4) Passing the two equations obtained in the last step to the Matlab Symbolic Math Toolbox function `solve()` to determine the constants $C1$ and $C2$. Now, we have obtained the explicit expression of $U(x,s)$.
5) Due to the complicity of $U(x,s)$, its analytical inverse Laplace transform is usually unavailable. We apply the numerical inverse Laplace transform to $U(x,s)$ to obtain the numerical solution of (1), (2), (3), (4), and (5).

IV. Boundary Stabilization

In this section, we study the performance and properties of the following boundary controller:

$$f(t) = k_d u_t(1,t) \quad (8)$$

where k_d is the controller gain and the suffix d means it is a derivative gain.

Although the control law (8) has been widely used in the boundary control of wave equation and beam equation [9] [13][14], its effectiveness when applied to the boundary control of fractional wave equation is still unknown.

The initial conditions are chosen as

$$u(x,0) = -0.5\sin(0.5\pi x), \quad (9)$$

$$u_t(x,0) = 0. \quad (10)$$

First, we choose $k_d = 1$ and study the response for $\alpha = 1.25, 1.50, 1.75, 2.00$.

The tip end movement over time is shown in Fig. 1. The evolution of the whole string displacement is shown in Fig. 2 through Fig. 5.

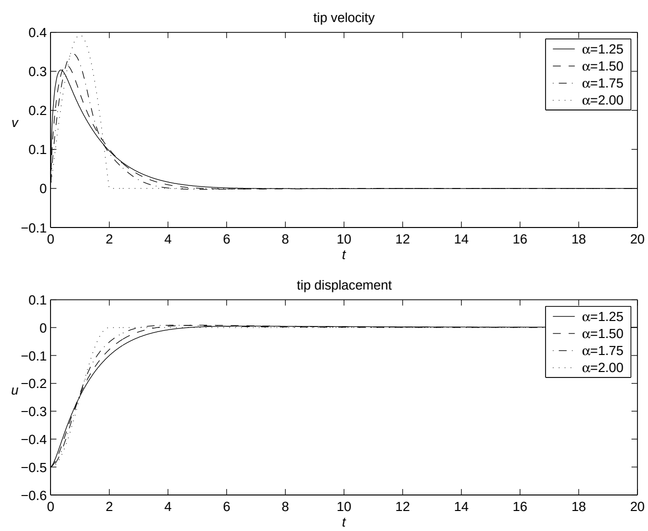

Fig. 1. Tip end movement over time for different α

We can see that controller (8) still stabilizes the fractional wave equation. The fixed controller gain $k_d = 1$ provides the shortest convergence time on the system with $\alpha = 2$. The response for $\alpha = 2$ (wave equation) becomes zero for $t > 2$, an already well-known result [15].

Next, for a fixed $\alpha = 1.5$, we study the response for different controller gains $k_d = 0.25, 0.50, 1.00, 2.00$, among which $k_d = 1.00$ is already studied.

The tip end movement over time is shown in Fig. 6. The evolution of the whole string displacement is shown in Fig. 7 through Fig. 9.

We can see that when k_d increases from 0.25 to 2.00, the system changes from underdamped to overdamped, because k_d is a derivative gain. Once again, the simulation results show that controller (8) still works for the boundary control of the fractional wave equation.

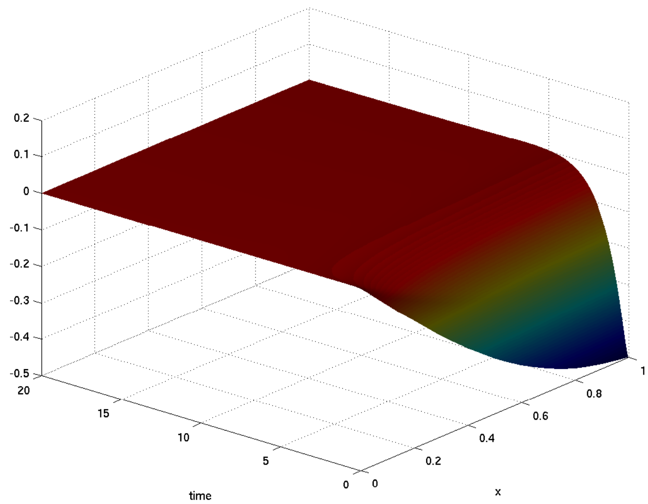

Fig. 2. Evolution of the whole string for $\alpha = 1.25$

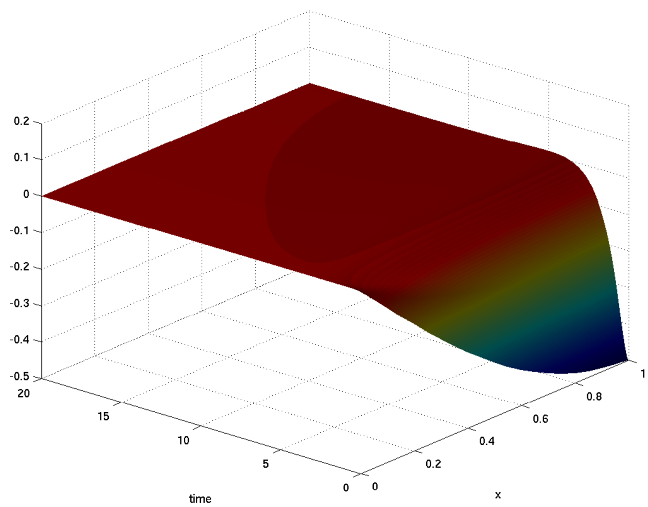

Fig. 3. Evolution of the whole string for $\alpha = 1.50$

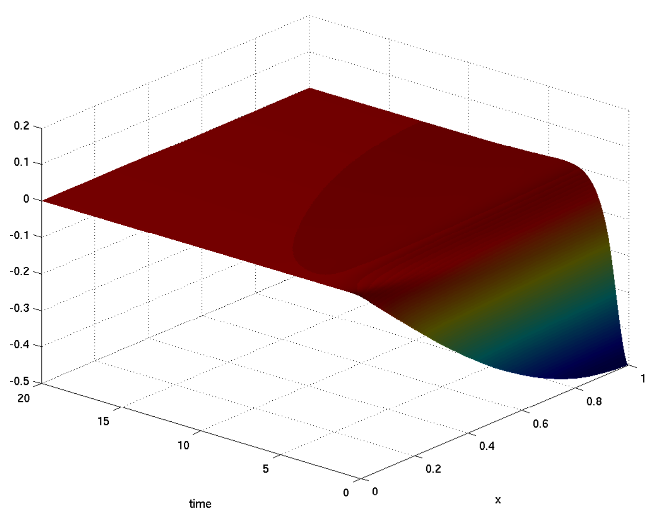

Fig. 4. Evolution of the whole string for $\alpha = 1.75$

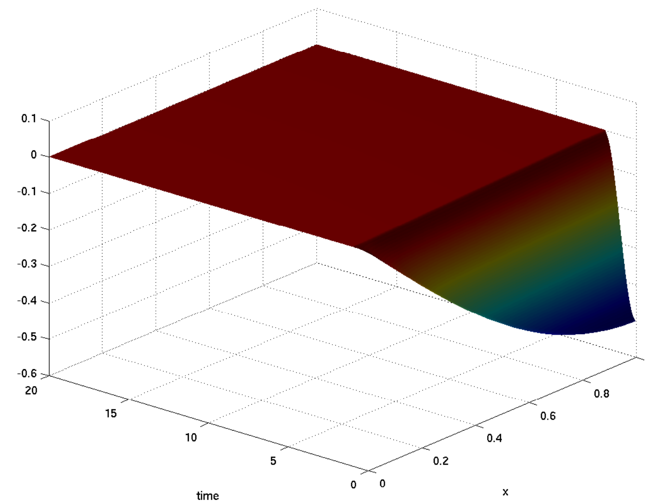

Fig. 5. Evolution of the whole string for $\alpha = 2.00$

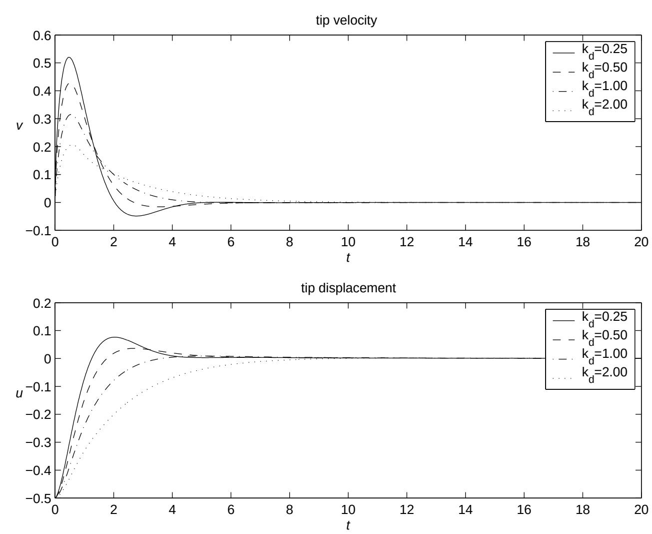

Fig. 6. Tip end movement over time for different k_d.

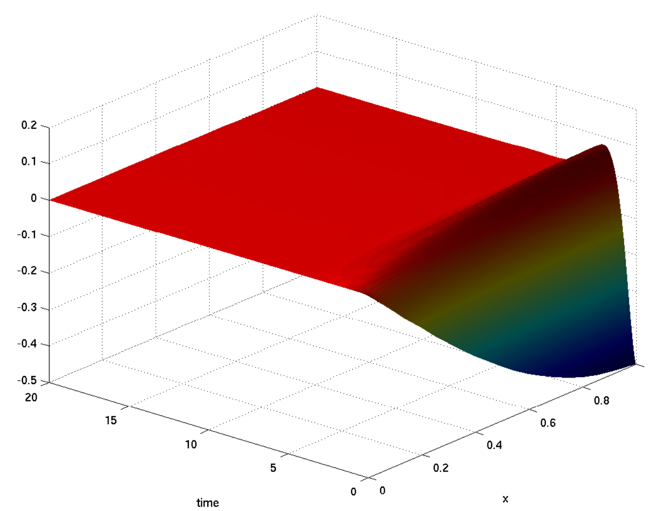

Fig. 7. Evolution of the whole string for $k_d = 0.25$

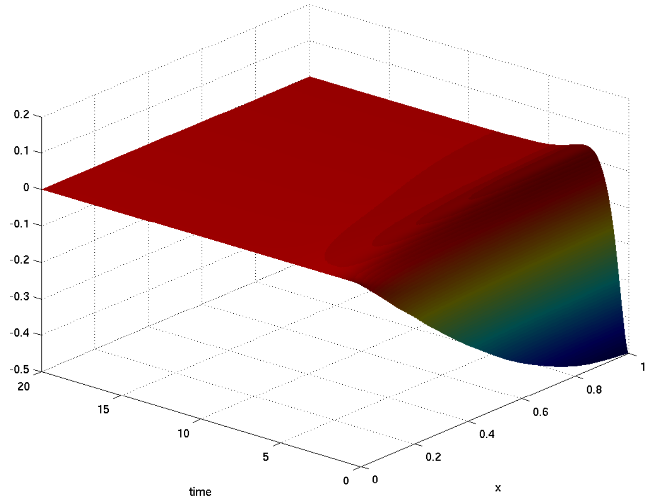

Fig. 8. Evolution of the whole string for $k_d = 0.50$

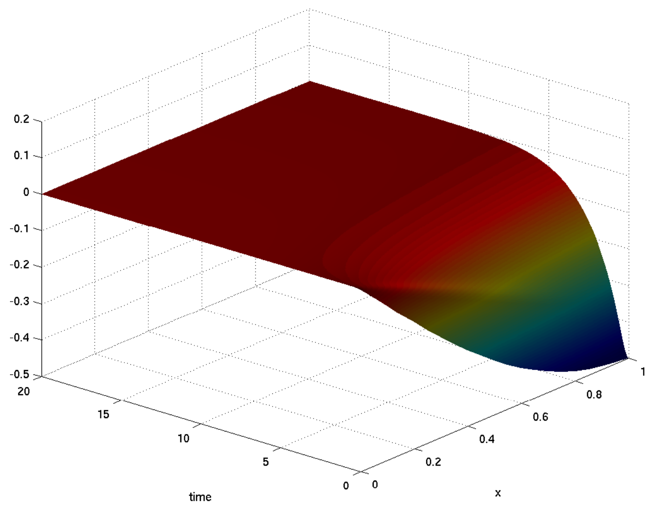

Fig. 9. Evolution of the whole string for $k_d = 2.00$

V. DISTURBANCE REJECTION

In this section, we assume that a disturbance force $n(t)$ is added at the same point where the boundary control signal enters. We further assume that $n(t)$ is a sinusoidal disturbance signal with unknown amplitude and phase but with a known frequency ω. Together with the boundary control signal [10] $(k_d + \frac{ks}{s^2+\omega^2})\hat{u}_t(1,s)$, the overall applied boundary force is given by

$$\hat{f}(s) = (k_d + \frac{ks}{s^2+\omega^2})\hat{u}_t(1,s) + \hat{n}(s), \quad (11)$$

where $\hat{f}(s)$ is the Laplace transform of the combination of boundary control force and disturbance force $n(t)$; $\hat{n}(s)$ is the Laplace transform of $n(t)$; $\hat{u}_t(1,s)$ is the Laplace transform of the velocity of the free end; k_d and k are the control gains.

The above boundary controller (11) was proposed in [10] to reject the noise for the boundary control of wave equation. The effectiveness of (11) when applied to the boundary control of the fractional wave equation is still unknown. Here, again, we use simulation results to verify the feasibility. In our simulation, the disturbance $n(t)$ is chosen as

$$n(t) = \sin(10t). \quad (12)$$

We first study the response for $k_d = 1$ and $k = 0$, i.e., the dynamic part of the controller is turned off, for $\alpha = 1.25, 1.50, 1.75, 2.00$.

The tip end movement over time is shown in Fig. 10. The evolution of the whole string displacement is shown in Fig. 11 through Fig. 14.

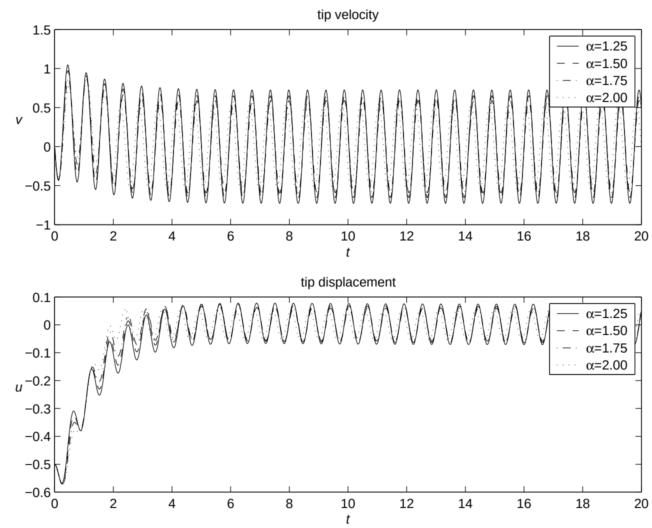

Fig. 10. Tip end movement over time for different α

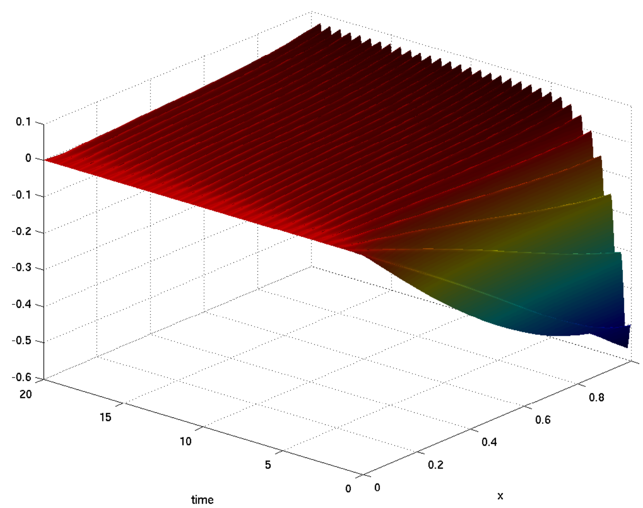

Fig. 11. Evolution of the whole string for $\alpha = 1.25$

We can see that although the performance is severely degraded with the presence of the noise for all α, the smaller α is, the less the response is affected by the noise. This is because the lower order derivatives help reduce the level of noise [7].

Next we will study the response for $k_d = 1$ and $k = 10$. The tip end movement over time is shown in Fig. 15.

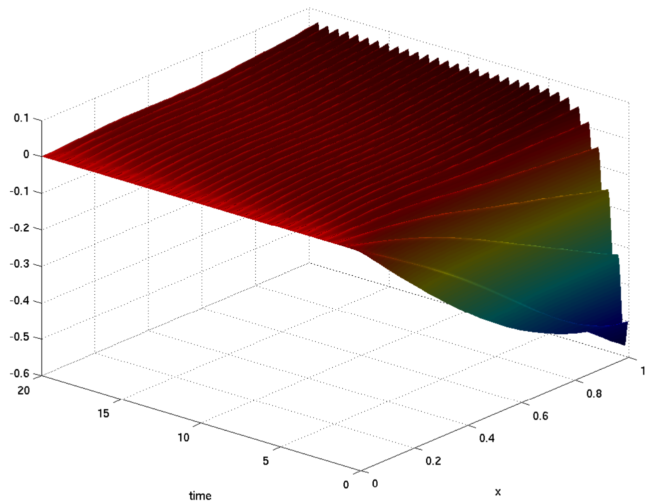

Fig. 12. Evolution of the whole string for $\alpha = 1.50$

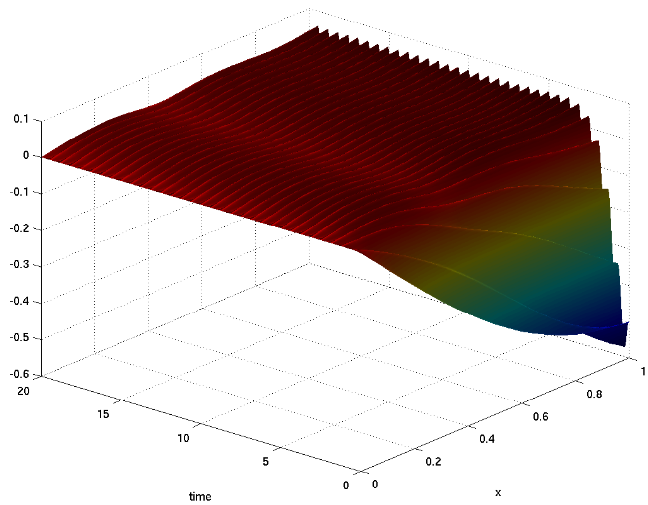

Fig. 13. Evolution of the whole string for $\alpha = 1.75$

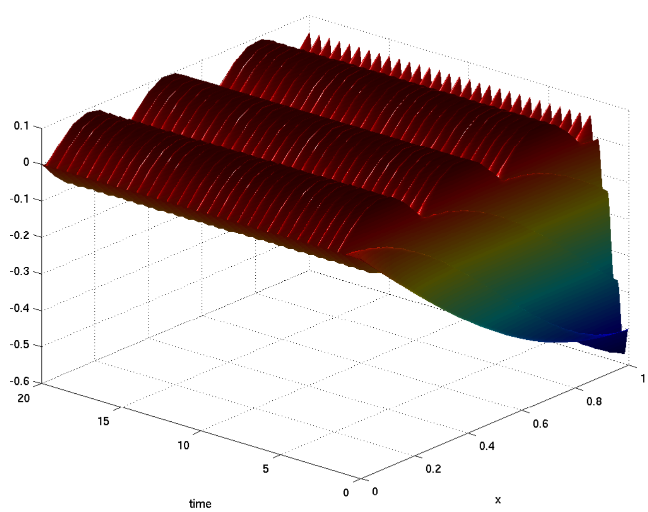

Fig. 14. Evolution of the whole string for $\alpha = 2.00$

The evolution of the whole string displacement is shown in Fig. 16 through Fig. 19.

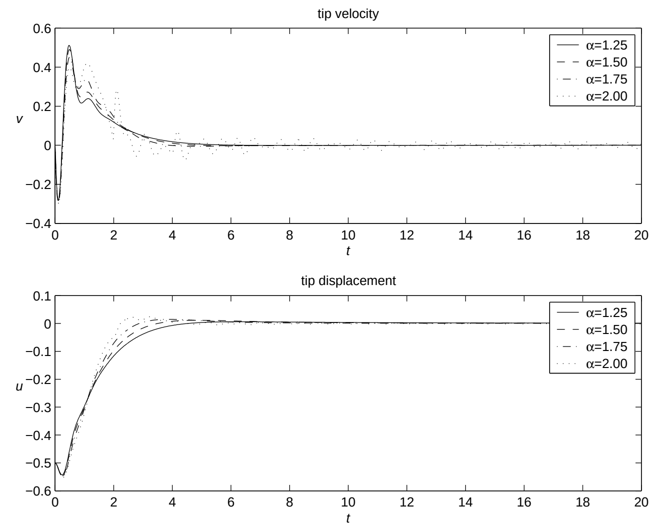

Fig. 15. Tip end movement over time for different α

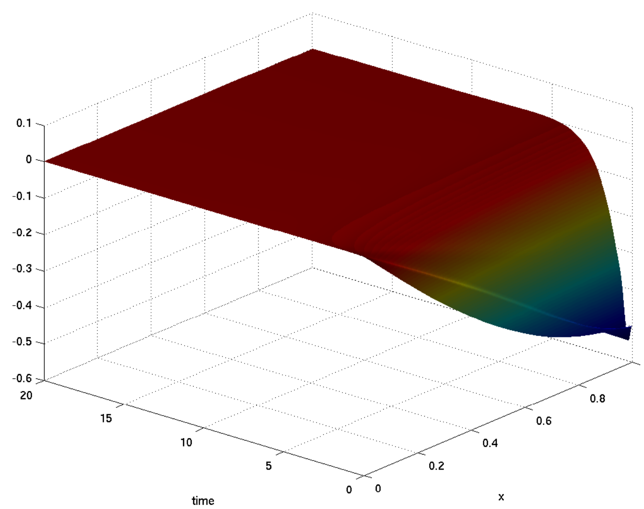

Fig. 16. Evolution of the whole string for $\alpha = 1.25$

Simulation results show that controller (11) still applies to the boundary control of the fractional wave equation. Since the lower order derivatives reduce the affect of the noise, better performance than in the wave equation case can be obtained.

VI. Concluding Remarks

We have studied two controllers, already applied to the boundary control of wave equation, on the boundary control of fractional wave equation. Simulation results show that the studied controllers are applicable for the boundary control of the fractional wave equation. For noise rejection, since the time derivative of fractional wave equation is lower than that of the wave equation, the performance is even better. Boundary control of fractional diffusion and wave equations

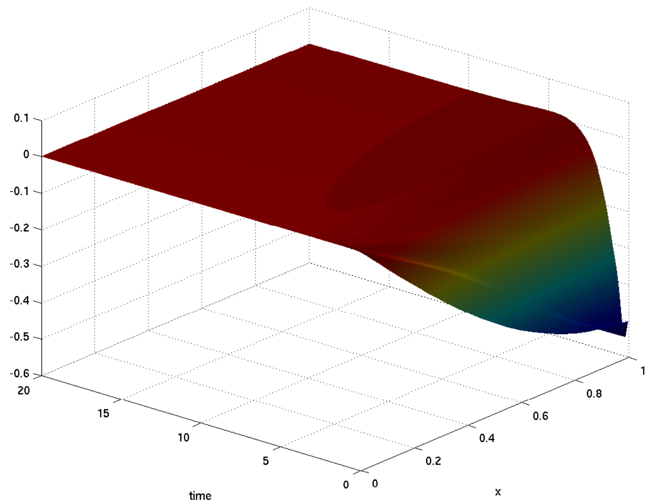

Fig. 17. Evolution of the whole string for $\alpha = 1.50$

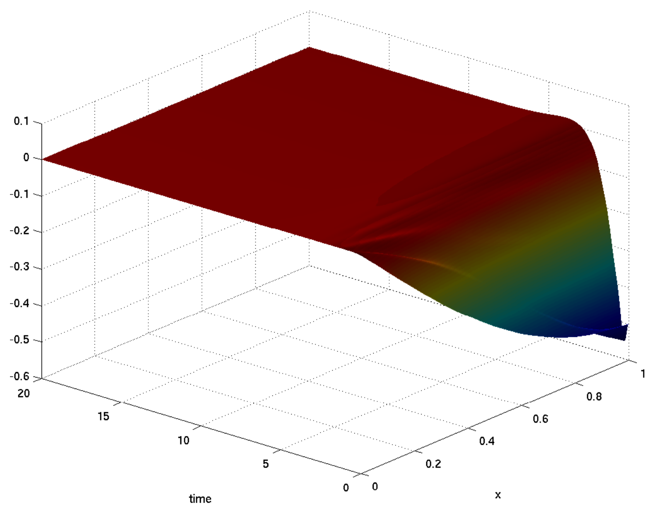

Fig. 18. Evolution of the whole string for $\alpha = 1.75$

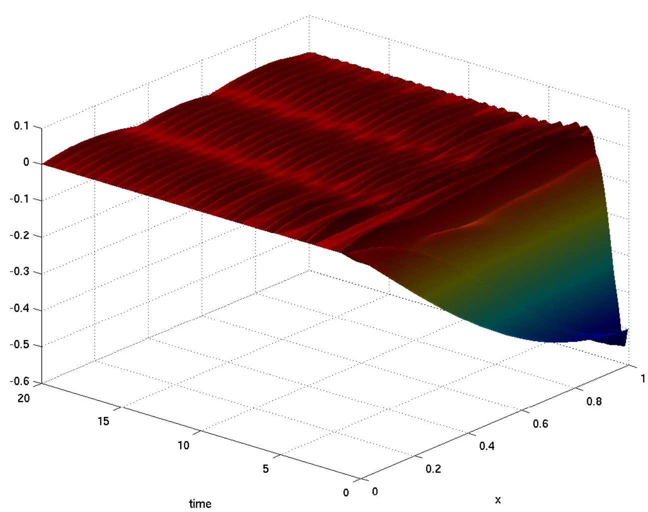

Fig. 19. Evolution of the whole string for $\alpha = 2.00$

is a new research topic. Based on the experience from the boundary control of the diffusion and wave equations, controllers better suited for the fractional diffusion and wave equations are expected to be explored in the future.

VII. REFERENCES

[1] R. R. Nigmatullin, "Realization of the generalized transfer equation in a medium with fractal geometry," *Phys. Stat. Sol. (b)*, vol. 133, pp. 425–430, 1986.

[2] W. Wyss, "The fractional diffusion equation," *J. Math. Phys.*, vol. 27, no. 11, 1986.

[3] W. R. Schneider, "Fractional diffusion and wave equation," *J. Math. Phys.*, vol. 30, no. 1, pp. 134–144, 1989.

[4] F. Mainardi and P. Paradisi, "A model of diffusive waves in viscoelasticity based on fractional calculus," in *Proceedings of the 36th IEEE Conference on Decision and Control, Hyatt Regency San Diego, California*, 1997.

[5] O. P. Agrawal, "Solution for a fractional dissusion-wave equation defined in a boundary domain," *Nonlinear Dynamics*, vol. 29, pp. 145–155, 2002.

[6] J. Liang, Y. Chen, and B.-Z. Guo, "A hybrid symbolic-numeric simulation method for some typical boundary control problems," Department of electrical and computer engineering, Utah State University, Tech. Rep., 2003.

[7] B. Mbodje and G. Montseny, "Boundary fractional derivative control of the wave equation," *IEEE Trans. Automat. Contr.*, vol. 40, no. 2, pp. 378–382, 1995.

[8] Ö. Morgül, "An exponential stability result for the wave equation," *Automatica*, vol. 38, pp. 731–735, 2002.

[9] G. Chen, "Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain," *J. Math. Pure. Appl.*, vol. 58, pp. 249–273, 1979.

[10] Ö. Morgül, "Stabilization and disturbance rejection for the wave equation," *IEEE Trans. Automat. Contr.*, vol. 43, no. 1, pp. 89–95, 1998.

[11] F. Mainardi, "Fractional relaxation-oscillation and fractional diffusion-wave phenomena," *Chaos, solitons, and Fractals*, vol. 7, no. 9, pp. 1461–1477, 1996.

[12] J. Liang, Y. Chen, and B. Guo, "A new boundary control method for beam equation with delayed boundary measurement using modified smith predictors," in *Proceedings of the 41th IEEE Conference on Decision and Control*, 2003.

[13] F. Conrad and Ö. Morgül, "On the stability of a flexible beam with a tip mass," *SIAM journal of control and optimization*, vol. 36, no. 6, pp. 1962–1986, 1998.

[14] G. Chen, M. C. Delfour, A. M. Krall, and G. Payre, "Modelling, stabilization and control of serially connected beams," *SIAM J. Contr. Optimiz.*, vol. 25, pp. 526–546, 1987.

[15] G. Chen and J. Zhou, "The wave propagation method for the analysis of boundary stabilization in vibrating structures," *SIAM J. Appl. Math.*, vol. 50, no. 5, pp. 1254–1283, 1990.