Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Robust H_2 Control for Two-Degree of Freedom Control Systems

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***Abstract*— In this paper we consider a robust H_2 control problem for two-degree of freedom (2DOF) control systems with structured uncertainties. We show that it can be reduced to a scaled H_∞ synthesis problem with one additional Linear Matrix Inequality (LMI) condition. Particularly it can be reduced to an LMI optimization problem if the uncertainty is unstructured. As a result, we can obtain the optimal robust H_2 controller for 2DOF control systems.**

I. INTRODUCTION

Many design methods for achieving both robust stability and nominal or robust performance have been proposed so far, such as robust H_2 control [5] as well as mixed H_2/H_∞ control, μ synthesis and so on. These methods are usually effective for general control system design problems from the theoretical point of view, while they are not so effective for those control system design problems with practical specifications such as robustness of output trajectories. Robustness of output trajectories is one of the most important performances that should be achieved in control system designs. Along this line we have already proposed a new design method for robust servo systems based on LMI [4], where we used a special type of two-degree of freedom (2DOF) control systems. In this paper we aim at generalizing this result to general 2DOF control systems, and propose a novel robust H_2 control design method for them as a preliminary study. To be more specific, we consider a robust H_2 control problem for 2DOF control systems with structured uncertainties, and show that it can be reduced to a scaled H_∞ synthesis problem with one additional LMI condition. Particularly it can be reduced to an LMI optimization problem if the uncertainty is unstructured. As a result, we can obtain the optimal robust H_2 controller for 2DOF control systems.

II. PROBLEM FORMULATION

A. Preliminaries

We consider the following system:

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_w & B \\ C_z & D_{zw} & D_u \\ C & D_w & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix},\ w = \Delta(t)z \qquad (1)$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the control input, $y \in \mathbf{R}^l$ is the output, $z \in \mathbf{R}^p$ and $w \in \mathbf{R}^p$ are the exogenous signals to describe the uncertainty. The coefficient matrices except $\Delta(t)$ are also known constant matrices of appropriate sizes, respectively. The uncertainty $\Delta(t) \in \mathbf{\Delta}$ is a norm-bounded time-varying structured uncertainty where $\mathbf{\Delta}$ is defined by

$$\mathbf{\Delta} := \{\Delta \in \mathbf{R}^{p\times p} : \Delta = \mathrm{diag}(\Delta_1, \cdots, \Delta_j), \|\Delta\| \le 1\} \quad (2)$$

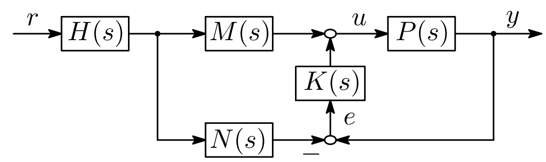

Fig. 1. Configuration of the two-degree of freedom control system

B. Two-Degree of Freedom Control Systems

Let us consider the system with $\Delta(t) = 0$ in (1). It is known that a general 2DOF system has the configuration of Fig. 1 [6], in which, $N(s)$ and $M(s)$ are right coprime factors of the nominal system $P(s) := C(sI - A)^{-1}B$ of (1), and they are represented as follows:

$$P(s) = N(s)M^{-1}(s) \qquad (3)$$
$$M(s) := F(sI - A_f)^{-1}B + I,\ N(s) := C(sI - A_f)^{-1}B$$

where F is a constant matrix such that $A_f := A + BF$ is stable. Then in Fig. 1, $H(s) := C_h(sI - A_h)^{-1}B_h + D_h$ is any stable transfer function and $K(s) := C_k(sI - A_k)^{-1}B_k + D_k$ is any feedback controller that stabilize the closed loop system. Finally we obtain the following state space realization of the 2DOF control system with the uncertainty $\Delta(t)$ in the framework of Fig. 1.

$$\begin{bmatrix} \dot{x}_q \\ z \\ e \end{bmatrix} = \begin{bmatrix} A_q & B_{q1} & B_{q2} \\ C_{q1} & D_{q11} & D_{q12} \\ C_{q2} & D_{q21} & 0 \end{bmatrix} \begin{bmatrix} x_q \\ w \\ r \end{bmatrix},\ w = \Delta(t)z \quad (4)$$

where

$$\left[\begin{array}{c|c} A_q & B_{q1} \\ \hline C_{q1} & D_{q11} \end{array}\right] := \left[\begin{array}{cc|c} \bar{A} & 0 & \bar{B}_1 \\ 0 & 0 & 0 \\ \hline \bar{C}_1 & 0 & D_{zw} \end{array}\right] + \left[\begin{array}{cc} \bar{B}_2 & 0 \\ 0 & I \\ \hline D_u & 0 \end{array}\right] \begin{bmatrix} D_k & C_k \\ B_k & A_k \end{bmatrix} \left[\begin{array}{cc|c} \bar{C}_2 & 0 & D_w \\ 0 & I & 0 \end{array}\right] \qquad (5)$$

$B_{q2} := \begin{bmatrix} B_r \\ 0 \end{bmatrix},\ C_{q2} := \begin{bmatrix} C_r & 0 \end{bmatrix},\ D_{q21} := D_w,\ D_{q12} := D_u D_h$

$\bar{A} := \begin{bmatrix} A_f & BC_h & 0 \\ 0 & A_h & 0 \\ 0 & 0 & A \end{bmatrix} \in \mathbf{R}^{\bar{n}\times\bar{n}},\ \bar{B}_1 := \begin{bmatrix} 0 \\ 0 \\ B_w \end{bmatrix},\ \bar{B}_2 := \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix}$

$\bar{C}_1 := \begin{bmatrix} C_z + D_u F & D_u C_h & C_z \end{bmatrix},\ \bar{C}_2 := \begin{bmatrix} 0 & 0 & C \end{bmatrix}$

$C_r := \begin{bmatrix} 0 & 0 & C \end{bmatrix},\ B_r := \begin{bmatrix} BD_h \\ B_h \\ 0 \end{bmatrix}$

Note that $A_q,\ B_{q1},\ C_{q1}$ and D_{q11} have the affine relation with respect to $\mathcal{K} := \begin{bmatrix} D_k & C_k \\ B_k & A_k \end{bmatrix}$ in (5).

C. Robust H_2 Performance Measure

Definition 1: [2] The robust H_2 performance measure is defined by

$$\Gamma := \sup_{r_0,\Delta}\left\{\int_0^\infty e^T(t)e(t)dt : \|r_0\|\le 1,\ \Delta(t)\in\mathbf{\Delta}\right\}$$

where $e(t)$ is the error signal in the presence of the uncertainty $\Delta(t)$ when the system (4) is excited by an impulsive input $r(t) := r_0\delta(t)$ with the zero initial state $x_q(0) = 0$.

Let the transfer function from w to z be $G_{zw}(s)$ and the set $\mathcal{S}$ of scaling matrix be defined by

$$\mathcal{S} := \{S > 0 : S\Delta(t) = \Delta(t)S,\ {}^\forall\Delta(t)\in\mathbf{\Delta}\}$$

Then we can obtain the upper bound of Γ by Lemma 1.

Lemma 1: [2] The system (4) is robustly stable, that is, $\|\bar{S}^{1/2}G_{zw}(s)\bar{S}^{-1/2}\|_\infty < 1$ for some $\bar{S}\in\mathcal{S}$ if and only if there exist $P > 0$ and $S\in\mathcal{S}$ such that

$$\begin{bmatrix} PA_q + A_q^TP + C_{q1}^TSC_{q1} & & \text{sym.} \\ B_{q1}^TP + D_{q11}^TSC_{q1} & D_{q11}^TSD_{q11} - S & \\ C_{q2} & D_{q21} & -I \end{bmatrix} < 0 \tag{6}$$

Then the robust H_2 performance measure Γ is finite if $D_{q12} = 0$, in which case, it is bounded by

$$\Gamma < \|B_{q2}^TPB_{q2}\| \tag{7}$$

Assumption 1: Since we can always design $H(s)$ such that $D_{q12} = 0$, we assume $D_{q12} = 0$ without loss of generality.

With the above preliminaries, we consider the following problem in Section III.

Problem 1: Design a robust H_2 controller $K(s)$ for the system (4) such that it minimize $\|B_{q2}^TPB_{q2}\|$ in Lemma 1.

III. Main Result

We obtain the optimal robust H_2 controller for the system (4) in the sense of Problem 1.

Theorem 1: If there exist $X = X^T,\ Y = Y^T,\ S\in\mathcal{S}$, $V\in\mathcal{S}$ and γ that satisfy (8), then we can derive a robust H_2 controller $K(s)$ with the order of n_k for the system (4) such that $\Gamma < \gamma$.

$$N_X^T\begin{bmatrix} \bar{A}X+X\bar{A}^T+\bar{B}_1V\bar{B}_1^T & X\bar{C}_e^T+\bar{B}_1V\bar{D}_e \\ \bar{C}_eX+\bar{D}_eV\bar{B}_1^T & \bar{D}_eV\bar{D}_e^T-V_e \end{bmatrix}N_X < 0 \tag{8a}$$

$$N_Y^T\begin{bmatrix} Y\bar{A}+\bar{A}^TY+\bar{C}_e^TS_e\bar{C}_e & Y\bar{B}_1+\bar{C}_e^TS_e\bar{D}_e \\ \bar{B}_1^TY+\bar{D}_e^TS_e\bar{C}_e & \bar{D}_e^TS_e\bar{D}_e-S \end{bmatrix}N_Y < 0 \tag{8b}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \ge 0,\quad \operatorname{rank}(I - XY) \le n_k \tag{8c}$$

$$VS = I \tag{8d}$$

$$B_r^TYB_r \le \gamma I \tag{8e}$$

where $\bar{C}_e := \left[\bar{C}_1^T, C_r^T\right]^T$, $\bar{D}_e := \left[D_{zw}^T,\ D_w^T\right]^T$, $V_e := \operatorname{diag}\{V, I\}$, $S_e := \operatorname{diag}\{S, I\}$, $N_X := \left[\bar{B}_2^T\ \ D_u^T\ \ 0_{m\times l}\right]^\perp$ and $N_Y := \left[\bar{C}_2\ \ D_w\right]^\perp$.

Proof: Since, by the preceding affine relation with respect to $\mathcal{K}$ in (5), (6) can be reduced to a scaled H_∞ synthesis problem [1] with additional matrices C_r and D_w contained in $\bar{C}_e$ and $\bar{D}_e$, respectively, we obtain (8a)-(8d). Furthermore, (7) can be reduced to (8e) since the structure of B_{q2} yields $B_{q2}^TPB_{q2} = B_r^TYB_r$, from which the desired result follows. ■

Theorem 1 is equivalent to a scaled H_∞ synthesis problem [1] with the additional constraint (8e). So we obtain Theorem 2 as the special case of Theorem 1.

Theorem 2: Let $\Delta(t)$ be one full block and n_k be $n_k = \bar{n}$. The optimal robust H_2 controller $K(s)$ for the system (4) is given by solving the convex optimization problem of minimizing γ that is described by the following LMI conditions with respect to $X = X^T,\ Y = Y^T$ and γ.

$$\begin{bmatrix} N_X^T & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \bar{A}X + X\bar{A}^T & X\bar{C}_e^T & \bar{B}_1 \\ \bar{C}_eX & -I & \bar{D}_e \\ \bar{B}_1^T & \bar{D}_e^T & -I \end{bmatrix}\begin{bmatrix} N_X & 0 \\ 0 & I \end{bmatrix} < 0 \tag{9a}$$

$$\begin{bmatrix} N_Y^T & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} Y\bar{A} + \bar{A}^TY & Y\bar{B}_1 & \bar{C}_e^T \\ \bar{B}_1^TY & -I & \bar{D}_e^T \\ \bar{C}_e & \bar{D}_e & -I \end{bmatrix}\begin{bmatrix} N_Y & 0 \\ 0 & I \end{bmatrix} < 0 \tag{9b}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \ge 0 \tag{9c}$$

$$B_r^TYB_r \le \gamma I \tag{9d}$$

Then the robust H_2 performance measure Γ is bounded by γ, namely $\Gamma < \gamma$.

Proof: It is clear that Theorem 1 for the case of $V = S = I$ and $n_k = \bar{n}$ yields Theorem 2. ■

Theorem 2 is equivalent to a H_∞ synthesis problem [3] with the additional constraint (9d).

Remark 1: In general, in the case of one degree of freedom control systems, $\|B_{q2}^TPB_{q2}\|$ in (7) cannot be expressed as an LMI condition. In 2DOF control systems, however, as we have shown above, it can be described as the LMI condition (8e) due to existence of the feedforward controller, which leads to the the LMI conditions (9).

IV. Conclusion

We have shown that a robust H_2 control problem for two-degree of freedom control systems can be reduced to a scaled H_∞ synthesis problem with one additional LMI condition. Particularly it can be reduced to an LMI optimization problem if the uncertainty is one full block, namely unstructured. As a result, we can obtain the optimal controller.

Finally we mention another performance measure [2], that is, robust L_∞ performance measure for bounded energy. The control problem using this measure can be also obtained as our dual result since its analysis problem has the dual relation [2].

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