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A Novel Stability Study on Multiple Time-Delay Systems (MTDS) Using the Root Clustering Paradigm

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Abstract— **A novel treatment is presented for the stability of linear time invariant (LTI) systems with rationally independent multiple time delays. The stability analysis of the time-delayed systems (TDS) is quite complex, because they are infinite dimensional. Multiplicity and 'rationally independent' feature of the delays makes the problem even more challenging compared to the TDS with commensurate time delays (where time delays are rationally related). Recently the authors introduced a new perspective, which brings a unique, exact and structured methodology for the stability analysis of *commensurate time delayed* cases. The transition from this class of TDS to those with multiple delays using a similar perspective motivates the present study. The new framework is described and the enabling propositions are proven. We show that this procedure reveals all possible stability regions exclusively in the space of independent multiple time delays. As an added strength, it does not require the MTDS under consideration to be stable for zero delays. We present an example numerical case study, which is considered prohibitively difficult to solve using the peer methodologies.**

I. INTRODUCTION AND THE PROBLEM STATEMENT

This study is on the stability of LTI multiple time delay system (LTI-MTDS). We present a new procedure for the most generic form of two time-delay and second order retarded LTI systems, with the intension of extending it to higher order systems with more delays in a later report.

First, we present some notational definitions. In the text $\mathbf{C}^+$, $\mathbf{C}^-$, $\mathbf{C}^0$ are used for right half, left half and the imaginary axis of the complex plane, respectively. Consequently, $\mathbf{C} = \mathbf{C}^+ \cup \mathbf{C}^- \cup \mathbf{C}^0$ represents the entire complex plane. Vector arrays are denoted with curly brackets, e.g. $\{\mathbf{a}\} = (a_1 , a_2 , ...)$

The most generic form of the characteristic equation for a second order dynamics with two time delays is

$$\begin{aligned} CE\,(s, \tau_1, \tau_2) = a_0(s) + a_1(s)\, e^{-\tau_1 s} + a_2(s)\, e^{-\tau_2 s} \\ + a_3(s)\, e^{-(\tau_1+\tau_2)\, s} = 0 \ , \ \ \tau_1 , \tau_2 > 0 \end{aligned} \tag{1}$$

where $a_j(s)$, $j = 0,1,2,3$ are polynomials of s with degrees of 2, 1, 1, 0, respectively. The highest degree of s in (1) is 2 and it appears in $a_0(s)$ only, where there is no time delay influence, making (1) a 'retarded system' [1-5]. The delays are rationally unrelated, i.e., $\tau_1 / \tau_2 =$ irrational number. The problem is to determine the stability mapping of this system in $\{\boldsymbol{\tau}\} = (\tau_1 , \tau_2)$ space. Notice the last term in the equation, which represents the cross-talk between the two delays.

The stability of TDS when there is only one single delay has been studied extensively. The investigators typically search for the stability switches of the dynamics in the space of the time delay, $\tau \in \Re^+$ [1, 6-10]. For the MTDS stability, we consider extending the unique perspective of [8], which is later named the "cluster treatment of characteristic roots (CTCR)" [9]. It introduces a construct, which determines all the stability intervals in $\tau \in \Re^+$, exhaustively. Furthermore it offers a closed form expression of τ for the *number of unstable roots* (*NU*) of such systems for any $\tau \in \Re^+$, first time in the literature.

The primary reason of complication in this problem is the transcendentality (infinite dimensionality) of equation (1) even when there is a single delay. Multiple time delay systems (MTDS) are clearly much more cumbersome to solve [2-5]. Therefore all the present literature on the subject is limited to tackling considerably simple dynamics. For instance, [3, 4] solve the stability problem with no cross-talk between the delays and $a_0(s)$ containing no damping term, simplifying the involved mathematics substantially. The extension of their procedure to cases with a damping term and delay cross-talk is prohibitively difficult, if not impossible. In this paper, we overcome both of these hurdles, using an extension to our "Cluster Treatment of Characteristic Roots (CTCR)" methodology. The new treatment constitutes the primary contribution of the work. In the meantime, since the inclusion of these two critical terms result in the most general form of the second order dynamics with two-delays, we hope to have opened a path for a treatment on

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MTDS of much higher orders and with more time delays.

By definition, (1) represents an asymptotically stable LTI-MTDS if and only if all its roots are in $\mathbf{C}^-$. Infinitely many such roots exist and finite number of them may be in $\mathbf{C}^+$ [11]. They have to be tracked for a complete stability assessment. This is obviously a very cumbersome task because of the multiple transcendentality (i.e. multiple independent delays), and their cross-talk in the dynamics, There is very limited literature on the stability of this class of systems [2-5]. [2] confines the problem to cases with linear $a_0(s)$ term. [5] suggests a geometric procedure, which is inherently inaccurate and incomplete in determining the stability mapping in the domain of $\{\boldsymbol{\tau}\} = (\tau_1, \tau_2)$. For cases with a damping term in $a_0(s)$, none of these procedures can be deployed due to the added analytical complications. To the best knowledge of the authors, there is no evidence of prior investigation on MTDS where multiple delays have cross-talk characteristics. All of these limitations are removed in this work, albeit for a modest second order dynamics (1). The treatment we present is extendable to more complicated systems, as we intend to document in the near future.

The above investigations question the conditions imparting purely imaginary roots first, in order to assess the stability. These roots display some interesting behavior, which are critical to the new methodology. The following two propositions summarize these peculiar features, which are also the key elements in the CTCR framework [8, 9]. They lead to an exhaustive stability study of the system in the space of single time delay. The aim of the present work is to achieve the same, except one important difference; the multiplicity of time delays naturally with cross-talk among them. And this transition from single to multiple delays is not trivial.

The text is organized as follows. Section II and III present the two underlying propositions with proofs. Section IV entails the CTCR perspective. In Section V we give an example case study.

II. SOME PRELIMINARIES AND PROPOSITION I

We present some observations, definitions and properties first:

Observation 1. If there exists an imaginary root of equation (1) at $s = \mp\omega_c i$ ('c' for crossing) for a given set of time delays $\{\boldsymbol{\tau}_0\} = (\tau_{10}, \tau_{20})$ the same imaginary root will also exist at all the countably infinite grid points of

$$\{\boldsymbol{\tau}\} = (\tau_{10} + \frac{2\pi}{\omega_c} j \ , \tau_{20} + \frac{2\pi}{\omega_c} k) \ , \ j = 1, 2, ..., k = 1, 2, ...$$

$$\tau_{10} - \frac{2\pi}{\omega_c} \le 0 \ , \ \tau_{20} - \frac{2\pi}{\omega_c} \le 0 \qquad (2)$$

This signifies that τ_{i0} is the smallest positive τ_{ij}, $\tau_{i0} = \min(\tau_{ij})$, $i = 1, 2$, $j = 0, 1, 2, ...$, ($\tau_{ij} > 0$). Notice that for a fixed ω_c the distinct points of (2) generate a grid in $\{\boldsymbol{\tau}\} \in \Re^{2+}$ space with equal grid size, $2\pi / \omega_c$ in both dimensions.

Observation 2. The grid points of (2) are all earmarked with a parameter, ω_c. One can show that these grid points will form into infinitely many curves in (τ_1, τ_2) space for continuously varying ω_c. That is, for $\omega_c \to \omega_c + \varepsilon_c$, one can find the corresponding $\tau_1 \to \tau_1 + \varepsilon_1$, and $\tau_2 \to \tau_2 + \varepsilon_2$, where $|\varepsilon_\bullet| << 1$. The existence and the correspondence of ε_c, ε_1 and ε_2 can be determined from the variational form of (1):

$$\left.\frac{\partial CE}{\partial s}\right|_{s=\omega_c i} i\,\varepsilon_c + \left.\frac{\partial CE}{\partial \tau_1}\right|_{s=\omega_c i} \varepsilon_1 + \left.\frac{\partial CE}{\partial \tau_2}\right|_{s=\omega_c i} \varepsilon_2 = 0 \qquad (3)$$

If one considers all possible variations of $\omega \in \Re^+$, one obtains the continuous curves in (τ_1, τ_2) space passing through the grid points (2). Again, notice that they are all derived from a fundamental curve traced by $(\tau_{10}, \tau_{20})|_{\omega_c}$ where the notation denotes the imaginary characteristic root of $\omega_c i$ corresponding to the minimum delays of (τ_{10}, τ_{20}).

Definition "kernel curves": Assume that the set of $(\tau_{10}, \tau_{20})|_{\omega_c}$ is determined exhaustively in $\{\boldsymbol{\tau}\} = (\tau_1, \tau_2)$ space for all possible ω_c values satisfying (1) and (2). These curves as a group are called the "kernel curves" of system described by the characteristic equation (1). We denote these curves by $\wp_0(\tau_1, \tau_2)$.

Definition "offspring curves": All the curves which are obtained using the point-wise non-linear mapping given by equation (2) on the complete set of "kernel curves" are defined as the "offspring curves" of the dynamics described in (1). The "offspring curves" carry the identical frequency signatures of the "kernel curves", as (2) preserves ω_c from the kernel to offspring. They are represented by $\wp_{jk}(\tau_1, \tau_2)$, $j = 1, 2, ...$, $k = 1, 2, ...$.

The "kernel curve" and their "offspring curves", by definition, contain **all** possible points in $\{\boldsymbol{\tau}\}$ space, which render an imaginary characteristic root. These curves need to be exhaustively determined.

These definitions and the proposition I constitute our first root-clustering feature: We study the "kernel curve" as the generating core of the infinitely many "offspring curves" with their common clustering identifier, ω_c. Note that ω_c remains unchanged as one moves from a point on the "kernel curve" to its projections on the "offspring curves" according to (2).

Determination of "Kernel curves" exhaustively. A crucial question is the determination of all the "kernel curves" exhaustively for all possible $\omega_c \in \Re^+$. In order to achieve this, an interesting mathematical manipulation, so called the Rekasius substitution is adopted [10]. It suggests the replacement of the transcendental terms with rational expressions,

$$e^{-\tau_i s} = \frac{1 - T_i\, s}{1 + T_i\, s}\ ,\ T_i \in \Re\ ,\ i = 1,\, 2 \qquad (4)$$

We next present two properties relevant to this operation.

Property 1. This Rekasius substitution holds for $s \in \mathbf{C}^0$ (say $s = \omega_c\, i$) exactly, with a companion condition relating τ_i and T_i:

$$\tau_i = \frac{2}{\omega_c}[\, \tan^{-1}(\omega_c\, T_i) + j\pi]\ ,\ j = 0, 1, ... \qquad (5)$$

This equation describes an asymmetric mapping in which one T_i is mapped into countably infinite τ_i's which are distributed with a periodicity of $2\pi/\omega_c$. Proof is simply by checking the magnitude and phase equations arising from (4) ♦.

The substitution of (4) into (1) results in a new equation

$$CE\,(s, \tau_1, \tau_2)\Big|_{via(4)} \Rightarrow CE\,(s, T_1, T_2) \qquad (6)$$

which consists of some fractional polynomial expressions. Multiplying (6) with $(1 + T_1\, s)(1 + T_2\, s)$ the polynomial form of the characteristic equation is reached:

$$\begin{aligned} &CE\,(s, T_1, T_2)\,(1 + T_1\, s)\,(1 + T_2\, s) \\ &= \overline{CE}\,(s, T_1, T_2) = \sum_{k=0}^{4} b_k\,(T_1, T_2)\, s^k \end{aligned} \qquad (7)$$

This expression is a polynomial in s of which the coefficients are parametric functions of T_1 and T_2. We define four sets next:

$\mathbf{\Omega}_1 = \{ s, CE\,(s, \tau_1, \tau_2) = 0\}$, a set with countably infinite members for a given point $(\tau_1, \tau_2) \in \Re^{2+}$ (8)

$\overline{\mathbf{\Omega}}_1 = \{ \mathbf{\Omega}_1, (\tau_1, \tau_2) \in \Re^{2+}\}$, the complete topology of $\mathbf{\Omega}_1$'s for the entire 2-D quadrant $(\tau_1, \tau_2) \in \Re^{2+}$

$\mathbf{\Omega}_2 = \{ s, \overline{CE}\,(s, T_1, T_2) = 0\}$, a set of four members for a given point $(T_1, T_2) \in \Re^2$ (9)

$\overline{\mathbf{\Omega}}_2 = \{ \mathbf{\Omega}_2, (T_1, T_2) \in \Re^2\}$, the complete topology of $\mathbf{\Omega}_2$'s for the entire 2-D space $(T_1, T_2) \in \Re^2$

Property 2. All purely imaginary members of the topologies $\overline{\mathbf{\Omega}}_1$ for $\{\boldsymbol{\tau}\} \in \Re^{2+}$ and $\overline{\mathbf{\Omega}}_2$ for $\{\mathbf{T}\} \in \Re^2$ are identical. That is,

$$\overline{\mathbf{\Omega}}_1 \cap \mathbf{C}^0 \equiv \overline{\mathbf{\Omega}}_2 \cap \mathbf{C}^0 \qquad (10)$$

and for such a root, $s = \omega_c\, i$, (5) is automatically satisfied.

Proof. Due to the identity expressed in Property 1, if there is a root of (1), $s = \mp\omega_c\, i \in \overline{\mathbf{\Omega}}_1 \cap \mathbf{C}^0$ for a certain $(\tau_1, \tau_2) \in \Re^{2+}$, one can determine the corresponding (T_1, T_2) from (5). For this (T_1, T_2) and $s = \omega_c\, i$, equation (4) holds and therefore $\overline{CE}\,(s, T_1, T_2) = 0$, which implies that $s = \omega_c\, i \in \overline{\mathbf{\Omega}}_2$. Same premise can be proven from the opposite direction as well. If there exists a root, $s = \omega_c\, i \in \overline{\mathbf{\Omega}}_2 \cap \mathbf{C}^0$ for a given $(T_1, T_2) \in \Re^2$, one can find the corresponding kernel $(\tau_{10}, \tau_{20}) \in \Re^{2+}$ and its offspring $(\tau_{1j}, \tau_{2k}) \in \Re^{2+}$, $j = 1, 2, ...$, $k = 1, 2, ...$ according to (2) for which the same root $s = \omega_c\, i \in \overline{\mathbf{\Omega}}_1 \cap \mathbf{C}^0$. Therefore the property indicated by (10) holds.♦

Proposition I: The characteristic equation (1) can have an imaginary root only along countably infinite number of curves $\wp(\tau_1, \tau_2)$; $(\tau_1, \tau_2) \in \Re^{2+}$. These curves are, in fact, generated from a manageably small number (≤ 3 as proven later in a lemma) of curves, which we will call the 'kernel curves', $\wp_0(\tau_1, \tau_2)$. All the remaining curves $\wp(\tau_1, \tau_2)$ are derived from the kernel curves $\wp_0(\tau_1, \tau_2)$ via a nonlinear transformation.

Proof: The Rekasius substitution preserves the imaginary roots while reducing the infinite dimensional $CE\,(s, \tau_1, \tau_2)$ to four dimensional $CE\,(s, T_1, T_2)$. It is obvious that determining $\overline{\mathbf{\Omega}}_2 \cap \mathbf{C}^0$ for $(T_1, T_2) \in \Re^2$ is considerably easier mission than determining $\overline{\mathbf{\Omega}}_1 \cap \mathbf{C}^0$ for $(\tau_1, \tau_2) \in \Re^{2+}$. This is the crucial reason of the proposed Rekasius substitution. The question we try to answer becomes much simpler: Determine the locus (or loci) of $(T_1, T_2) \in \Re^2$ which result in $s \in \overline{\mathbf{\Omega}}_2 \cap \mathbf{C}^0$. To respond to this question we use Routh's array over the characteristic equation (7) in parametric form. The resulting array looks like Table 1. We take advantage of some subtle features of this array [12]. It is known that any $s \in \overline{\mathbf{\Omega}}_2 \cap \mathbf{C}^0$ requires that the corresponding (T_1, T_2) satisfy

$$R_1\,(T_1, T_2) = 0 \qquad (11)$$

This equation represents a well-defined curve in (T_1, T_2) space. One important nuance is that equation (11) is always satisfied if there is a "symmetric pair" of roots in $\overline{\mathbf{\Omega}}_2$ with respect to the origin ($s = 0$). In order for this condition to yield imaginary pair (as opposed to symmetrical real roots), so called, the "auxiliary equation"

$$R_{21}\,(T_1, T_2)\, s^2 + R_{22}\,(T_1, T_2) = 0 \qquad (12)$$

should have sign agreement between its coefficients, i.e.

$$R_{21}\,(T_1, T_2)\, R_{22}\,(T_1, T_2) > 0 \qquad (13)$$

Statements (11) and (13) define a curve in (T_1, T_2) which constitute the complete locus (or loci) containing all (T_1, T_2) points yielding the entire set of imaginary characteristic roots $s \in \overline{\mathbf{\Omega}}_2 \cap \mathbf{C}^0$. And the corresponding crossing frequencies are found from (12). Let us denote the resultant curve by

$$\overline{R}_1(T_1, T_2) = \{[R_1 = 0] \cap [R_{21}\ R_{22} > 0]\} \quad (14)$$

s^4	$b_4(T_1, T_2)$	$b_2(T_1, T_2)$	b_0
s^3	$b_3(T_1, T_2)$	$b_1(T_1, T_2)$	
s^2	$R_{21}(T_1,T_2)$	$R_{22}(T_1,T_2) = b_0$	
s^1	$R_1(T_1,T_2)$		
s^0	$R_0(T_1,T_2) = b_0$		

Table 1. Routh's array for $\overline{CE}(s, T_1, T_2)$

Every point on $\overline{R}_1(T_1, T_2)$ corresponds to an $s = \omega_c\, i$ as per equation (2) and there is no point $(T_1, T_2) \notin \overline{R}_1(T_1, T_2)$ for which an imaginary characteristic root may exist. Thus $\overline{R}_1(T_1, T_2)$ is exhaustive. In fact, $\overline{R}_1(T_1, T_2)$ represents the projection of all $\wp_{jk}(\tau_1, \tau_2)$ on the (T_1, T_2) space. Using these ω_c's and equation (2) point by point along $\overline{R}_1(T_1, T_2)$, one can construct the respective "kernel curve" $\wp_0(\tau_1, \tau_2)$ in (τ_1, τ_2) space. It is easy to determine for every point $(T_1, T_2)|_{\omega_c}$ the corresponding unique point of $(\tau_{10}, \tau_{20})|_{\omega_c} \in \Re^{2+}$ using (2), as discussed earlier. Note that one should also check if $T_1 \to \mp\infty$ (and T2 finite), $T_2 \to \mp\infty$ (and T1 finite), and $T_1, T_2 \to \mp\infty$ cause a crossing, $s = \omega_c\, i$. The limiting cases imply $e^{-\tau_i s} \to -1$ as per (4). Then one can detect the respective delays using (5) as $\tau_i = (2j+1)\pi / \omega_c$. As a result, these operations yields a unique $\wp_0(\tau_1, \tau_2)$ 'kernel curve' for the system given by (1). Departing from this 'kernel curve', one can generate the complete set of 'offspring curves', $\wp_{jk}(\tau_1, \tau_2)$, $j = 1, 2, \ldots$, $k = 1, 2, \ldots$ using the point-wise feature stated in (2). The given system has imaginary characteristic roots only on this set of $\wp_{jk}(\tau_1, \tau_2)$, $j = 0, 1, 2, \ldots$, $k = 0, 1, 2, \ldots$. The proposition holds. ♦

Lemma: There can be maximum three curves forming $\wp_0(\tau_1, \tau_2)$.

Proof: $\wp_0(\tau_1, \tau_2)$ has point-wise one-to-one correspondence to the curve $\overline{R}_1(T_1, T_2)$ as per (2). If one looks at the formation of the Routh's array it becomes obvious that $R_1(T_1, T_2)$ is a multinomial of maximum degree 3 in T_1 (or T_2). Thus $R_1(T_1, T_2) = 0$ represents at the most 3 individual curves in (T_1, T_2) space. ♦

Notice that the above proof and the procedures involved (e.g., Eq(4), Table 1) are equally valid for the most general *n* dimensional (*n* delays) system. The reduced dimensional representation of (1) in this paper is for ease of conveyance only.

III. PROPOSITION II

Definition: The root tendency associated with each purely imaginary characteristic root, $\omega_c\, i$, with respect to one of the time delays, τ_i, $i = 1, 2$, is defined as

$$\text{Root Tendency} = RT\big|_{s=\omega_c i}^{\tau_j} = \text{sgn}\,[\text{Re}(S_{\tau_j}^{S}\big|_{s=\omega_c i})] \quad (15)$$

where $S_{\tau_j}^{s}\big|_{s=\omega_c i} = \left.\dfrac{\partial s}{\partial \tau_j}\right|_{s=\omega_c i}$. This property indicates the direction of transition of the imaginary root at $\omega_c\, i$ as one of the delays increase by ε, $0 < \varepsilon << 1$, while the other one remains fixed.

Proposition II. Take a crossing frequency, ω_c, an imaginary root caused by infinitely many grid points in $\{\tau_1, \tau_2\}$ defined by $(\tau_{10} + \frac{2\pi}{\omega_c} j, \tau_{20} + \frac{2\pi}{\omega_c} k)$, $j = 0, 1, \ldots$, $k = 0, 1, \ldots$. The root tendency $RT\big|_{s=\omega_c i}^{\tau_1}$ (or $RT\big|_{s=\omega_c i}^{\tau_2}$) remains invariant so long as the grid points are selected on different 'offspring curves' keeping τ_{2k} (or τ_{1j}) fixed.

Proof. Without loss of generality we prove the proposition for τ_1 keeping τ_2 fixed. We look at the root sensitivity given by (15) with respect to τ_1 at $(\tau_{10} + \frac{2\pi}{\omega_c} j, \tau_2)$, $j = 0, 1, 2, \ldots$. It can be shown that it is in the form of:

$$S_{\tau_1}^{s}\big|_{s=\omega_c i} = \omega_c\, i \left(H(s, \tau_2) - \tau_{10}\right)^{-1}\big|_{s=\omega_c i} \quad (16)$$

where $H(s, \tau_2)$ is a self-evident quasi-polynomial which is invariant with respect to the actual values of τ_1.

Taking $\omega_c > 0$, we can use (16) to write (15):

$$RT\big|_{s=\omega_c i}^{\tau_{ij}} = \text{sgn}\{\text{Im}[H(s,\tau_2)]\}\big|_{s=\omega_c i} \quad (17)$$

which is invariant for all τ_{1j}, $j = 0, 1, 2, \ldots$.

In other words, the imaginary root always crosses either to $\mathbf{C}^+$ (for $RT = +1$) or to $\mathbf{C}^-$ (for $RT = -1$), when one of the delays is kept fixed independent of the actual values of the second delay.♦

This proposition helps identifying certain sections of $\wp_0$ and the 'offspring curves' to be marked as stabilizing transitions along the τ_1 (or τ_2) axis or vice versa. Once we detect completely $\wp_0(\tau_1, \tau_2)$, $\wp_{jk}(\tau_1, \tau_2)$

'offspring curves' and the invariant root sensitivities, we can determine all possible stability regions in the parametric space of time delays $\{\boldsymbol{\tau}\}$ using the well-known D-Subdivision methodology [13]. This implies the exhaustiveness of our methodology because it covers the complete set of stability regions in the entire semi-infinite time delay space entirely. Over an example case we will also show the exactness of the methodology; determining the precise boundaries of time delays where stability switches (i.e., from stable to unstable or vice versa) occur.

In the next section we present the steps of the extended CTCR procedure, which is the first structured methodology, to the best knowledge of the authors that can declare the complete stability mapping of multiple time-delayed systems in the domain of the delays.

IV. CTCR METHODOLOGY

The cluster treatment of the characteristic roots (CTCR) framework consists of the following steps:

i) The CTCR routine requires an exhaustive detection of all those loci in $\{\boldsymbol{\tau}\}$, which yield $s = \omega_c\, i$. That is, the "kernel curve" and the "offspring curves". This is the **first** "clustering" operation. As per the earlier lemma, the "kernel curve" is analytically well defined and numerically manageable. This $\wp_0(\tau_1, \tau_2)$ is the generator of the infinitely many and complete set of crossing hyperplanes $\wp_{jk}(\tau_1, \tau_2)$ as defined earlier.

ii) A **second** "clustering" is done within the kernel hyperplane, $\wp_0(\tau_1, \tau_2)$, utilizing the concept of RT (root tendency). Certain segments of the kernel hyperplane $\wp_0(\tau_1, \tau_2)$ exhibit certain RT's along τ_1 and τ_2 axes and their reflections on the offspring exhibit the same features according to Proposition II.

These steps of CTCR ultimately generate a simple $\wp_0(\tau_1, \tau_2)$ and its infinitely many offspring $\wp_{jk}(\tau_1, \tau_2)$ with the earmarking of $RT\big|^{\tau_1}$ and $RT\big|^{\tau_2}$ on certain segments of each offspring. This exclusive tableau renders completely the possible root crossing points and their contributions to the stability picture. The rest of the procedures are following the 2D extensions of our earlier described methodology [8]. In essence we deploy the D-Subdivision method along any path made of segments parallel to τ_1 and τ_2 axes, which connects $(\tau_1 = 0, \tau_2 = 0)$ point to the operating point of interest $\{\bar{\boldsymbol{\tau}}\} = (\bar{\tau}_1, \bar{\tau}_2)$. As the path crosses the $\wp_{jk}(\tau_1, \tau_2)$'s we build a quantity called the $NU(\tau_1, \tau_2)$. Starting with $NU(\tau_1 = 0, \tau_2 = 0)$ which is the number obtained for the non-delayed systems, we add 2 if at a crossing of $\wp_{jk}(\tau_1, \tau_2)$ the $RT = +1$ and subtract 2 if the $RT = -1$.

Obviously the regions in (τ_1, τ_2) space, which result $NU(\tau_1, \tau_2) = 0$, are declared stable and the others with $NU \geq 2$ unstable. This finalizes the complete stability analysis of the dynamics with two independent time delays. Let's now give an example case, of which the solution is prohibitive using peer methodologies [1-4].

V. EXAMPLE CASE STUDY

Take the system of (1) as:

$$CE(s,\tau_1,\tau_2) = s^2 + 7.1s + 21.1425 + (6s+14.80)e^{-\tau_1 s} + (2s+7.3)e^{-\tau_1 s} + 8e^{-(\tau_1+\tau_2)s} = 0 \quad (18)$$

Non-delayed system has two roots at -9.95 and -5.15, thus $NU(0, 0) = 0$. As one performs (4), (18) becomes

$$\overline{CE}(s,T_1,T_2) = (\{s^4 - 0.9s^3 + 7.0425s^2\}T_2 + s^3 + 3.1s^2 + 5.6425s)T_1 + \{s^3 + 11.1s^2 + 20.6425s\}T_2 + s^2 + 15.1s + 51.2425 \quad (19)$$

The parameterized Routh's array is performed next on (19) and $R_1(T_1, T_2)$ term is generated as per Table 1. It is

$$R_1(T_1,T_2) = (5.7221721\ T_2^2 + 1.254892\ T_2 - 2.79868)\ T_1^3 + (20.9339721\ T_2^3 + 47.842184\ T_2^2 - 20.01324\ T_2 - 0.1936)\ T_1^2 + (77.913052\ T_2^3 + 38.15956\ T_2^2 + 16.5408\ T_2 - 2.416)\ T_1 - 36.66108\ T_2^3 - 21.9216\ T_2^2 - 2.416\ T_2 \quad (20)$$

As claimed by the earlier lemma, it is of degree 3 in either T_1 or T_2. Thus, there can be at most three curves represented by (20) in (T_1, T_2) space. The constraint (13) forms as:

$$R_{21}(T_1, T_2)\ R_{22}(T_1, T_2) = \frac{N(T_1, T_2)}{D(T_1, T_2)} > 0 \quad (21)$$

where;

$$N = 5560\ T_1^2 T_2 + 25353\, T_1^2 T_2^2 + 7200\, T_1 T_2 + 94360\, T_1 T_2^2 - 12400\ T_1^2 - 4000\ T_1 - 4000\ T_2 - 44400\, T_2^2$$

$$D = 9T_1\ T_2 - 10\ T_1 - 10\ T_2$$

$R_1(T_1, T_2) = 0$ and the inequality constraint of (21) are displayed in Fig 1. A numerical procedure is followed along the segments of $\overline{R_1}$, which is displayed by (14) to generate the corresponding 'kernel curve' $\wp_0(\tau_1, \tau_2)$. This 'kernel curve' and its 'offspring curves' are depicted in Fig. 2. The kernel, $\wp_0(\tau_1, \tau_2)$, happens to be a closed loop oval, confined within a small range of τ_1 and a small range of τ_2. This is, however, an exhaustive representation of all possible imaginary roots of the system. That is, any imaginary characteristic root of (18) can be generated by one of the points either on this 'kernel curve' or one of its 'offspring curves'. No other possible root crossing can exist. This is the first step of the operation, which reveals the imaginary crossings in its entirety.

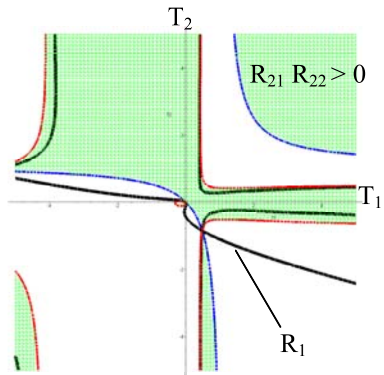

Figure 1. R_1 and the constraint of (21)

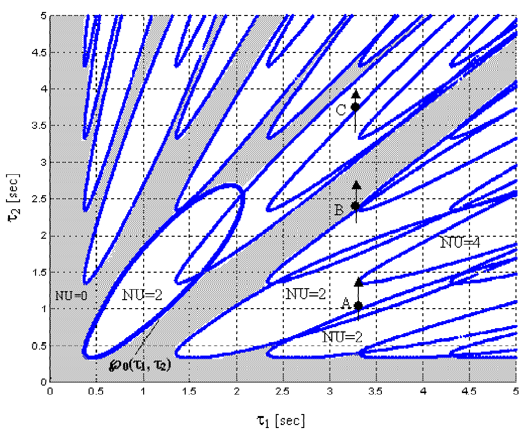

Figure 2. The complete stability picture of the example system

The second step is to look at the root tendencies at these crossings. It is very simple to mark the obvious stable region using the D-Subdivision reasoning. As $NU\,(0\,,\,0) = 0$, at $(\tau_1\,,\,\tau_2) = (0,\,0)$ point, i.e., the origin is stable, as is the shaded regions, in Fig 2.

The dendrite like formation of the stable region is interesting. One of these dendrites is interrupted by the kernel $\wp_0(\tau_1\,,\,\tau_2)$. Another interesting point is that all the offspring $\wp(\tau_1\,,\,\tau_2)$ are also oval shaped except their points of closure are outside the ranges given on Fig 2. This indicates the concentration of low frequencies around the upper ebb of the oval. The root tendency invariance property with respect to τ_2 is tested at points A, B and C;

$$A(\tau_{1A}\,,\,\tau_{2A})\,,\qquad B(\tau_{1A}\,,\,\tau_{2A}+\frac{2\pi}{\omega_A})\,,\qquad C(\tau_{1A}\,,\,\tau_{2A}+\frac{4\pi}{\omega_A})\,.$$

$RT\big|_A^{\tau_2} = RT\big|_B^{\tau_2} = RT\big|_C^{\tau_2} = -1$ are verified. Notice that the crossings at these three points take place at the same $\omega_A\, i$ and with $RT\big|^{\tau_2} = -1$, i.e. increasing τ_2 at any one of these crossings yields two less unstable roots. This property enables us to mark certain regions with the corresponding *NU* (see figure).

VI. Conclusion

A new procedure is described based on the Cluster Treatment of Characteristic Roots (CTCR) perspective in analyzing the stability of multiple time-delayed LTI dynamics. For simplicity of conveyance we take a two-delay case into account in this work. It is evident from the literature that the stability assessment of this class of dynamics remains unsolved yet. The proposed methodology detects all the stability regions precisely, in the space of the time delays. This is due to the exhaustive nature of our approach. To achieve this, we introduce the concepts of 'kernel' and 'offspring' curves, which impart the complete portrait of the possible imaginary roots for this system. Then we suggest an interesting invariance property of the crossings of the roots. Using these two *clustering* properties we cover the entire semi-infinite time delay space completely for an exhaustive stability analysis.

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