Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Robust Stability Limit of Time-Delay Systems

Cs. Bányász and L. Keviczky, *Senior Member, IEEE*

***Abstract*—Robust stability of time-delay systems are discussed. A new stability condition is introduced. Influence of uncertainty in time-delay is also investigated for the performance and robustness of a two-degree-of-freedom control system.**

I. INTRODUCTION

An important area of research in control theory is the design of feedback controllers for systems which have significant uncertainties in the plant and the explicit incorporation of model uncertainties in the design of high performance control systems. This leads to methods for designing robust stability and performance. Most of the discrete-time identification, control even adaptive algorithms assume the apriori knowledge of the process time-delay. This apriori knowledge is sometimes very uncertain and the uncertainty can result from a lack of precision in mathematical modeling of the plant and/or changes in the plant parameters with time. It would be desirable to know how the time-delay mismatch influences the basic robustness and performance behaviors of the closed-loop control.

Some controller design methodology, mostly for discrete-time systems, include the time-delay of the plant also into the parameters. Unfortunately relatively few papers (e.g., [1-4]) can be found dealing with the influence of the accuracy of the apriori knowledge or estimate of the time-delay, which is sometimes called the time-delay mismatch problem. Our paper investigates the influence of the time-delay uncertainty on the robust stability and performance.

The framework how this issue will be discussed is the *generic two-degree of freedom* (*GTDOF*) system topology [5], which is based on the *Youla-parametrization* [9] providing all realizable stabilizing regulators (*ARS*) for open-loop stable plants and capable to handle the plant time-delay. The advantage of this approach is that it is easy to calculate the "best" reachable optimal regulator depending on the applied $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ norms as criteria. The drawback is that this methodology can be applied only for open-loop stable plants.

A *GTDOF* control system is shown in Fig. 1, where y_r, u, y and w are the reference, process input, output and

y_r, u, y and w are the reference, process input, output and disturbance signals, respectively. The optimal *ARS* regulator of the *GTDOF* scheme [6] is given by

$$R_o = \frac{P_w K_w}{1 - P_w K_w S} = \frac{Q_o}{1 - Q_o S} = \frac{P_w G_w S_+^{-1}}{1 - P_w G_w S_- z^{-d}} \qquad (1)$$

where

$$Q_o = Q_w = P_w K_w = P_w G_w S_+^{-1} \qquad (2)$$

is the associated optimal *Y-parameter* [8] furthermore

$$Q_r = P_r K_r = P_r G_r S_+^{-1} \ ; \ K_w = G_w S_+^{-1} \ ; \ K_r = G_r S_+^{-1} \qquad (3)$$

assuming that the process is factorable as

$$S = S_+ \bar{S}_- = S_+ S_- z^{-d} \qquad (4)$$

where S_+ means the inverse stable (*IS*) and S_- the inverse unstable (*IU*) factors, respectively. z^{-d} corresponds to the discrete time-delay, where d is the integer multiple of the sampling time. Here P_r and P_w are assumed stable and proper transfer functions (reference models). An interesting result was [7] that the optimization of the *GTDOF* scheme can be performed in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm spaces by the proper selection of the serial G_r and G_w embedded filters.

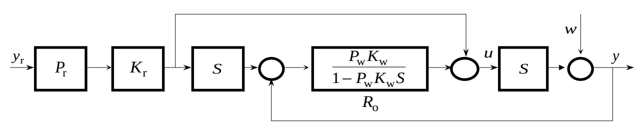

Fig. 1. The *generic TDOF* (*GTDOF*) control system

II. ROBUST STABILITY OF *GTDOF* SYSTEMS

Be M the model of the process. Assume that the process and its model are factorizable as

$$S = S_+ \bar{S}_- = S_+ S_- z^{-d} \quad ; \quad M = M_+ \bar{M}_- = M_+ M_- z^{-d_m} \qquad (5)$$

where S_+ and M_+ mean the inverse stable (*IS*), $\bar{S}_-$ and $\bar{M}_-$ the inverse unstable (*IU*) factors, respectively. z^{-d} and z^{-d_m} correspond to discrete time delays, where d and d_m are the integer multiple of the sampling time, usually $d = d_m$ is assumed. (To get a unique factorization it is

---

Authors are with Computer and Automation Research Institute of the H.Ac.Sc. and with Control Engineering Research Group of the H.Ac.Sc. at the Department of Automation and Informatics, TU of Budapest, (banyasz@sztaki.hu; keviczky @sztaki.hu)
This work was supported in part by the Hungarian NSF (OTKA) and the Control Engineering Research Group of the H.Ac.Sc.

reasonable to ensure that $\bar{S}_-$ and $\overline{M}_-$ are monic, i.e., $\bar{S}_-(1)=\overline{M}_-(1)=1$, having unity gain.) It is important that the inverse of the term z^{-d} is not realizable, because it would mean an ideal predictor z^d. These assumptions mean that $S_-=\bar{S}_- z^{-d}$ and $M_-=\overline{M}_- z^{-d_\mathrm{m}}$ are uncancelable invariant factors for any design procedure. Introduce the additive

$$\Delta = S - M \quad ; \quad \Delta_+ = S_+ - M_+ \quad ; \quad \Delta_- = S_- - M_- \tag{6}$$

and relative

$$\ell = \frac{\Delta}{M} = \frac{S-M}{M} \quad ; \quad \ell_+ = \frac{\Delta_+}{M_+} \quad ; \quad \ell_- = \frac{\Delta_-}{M_-} \tag{7}$$

model errors. It is easy to show that the characteristic equation using the *ARS* regulator is (for $d = d_\mathrm{m} = 0$)

$$M_+ M_- = 0 \tag{8}$$

if a $Q = \tilde{Q}\left(M_+ M_-\right)^{-1}$ parameter is applied, i.e., if someone tries to cancel both factors. This means that the zeros of the *IU* factor will appear in the characteristic equation and cause unstability. This is why these zeros (and the time delay itself) are called invariant uncancelable factors.

Introducing the model based, nominal complementary sensitivity function

$$\hat{Z} = \frac{\hat{R}M}{1+\hat{R}M} = \hat{Q}M \tag{9}$$

the well known robust stability condition $\left\| \hat{Z}\,\ell \right\|_\infty < 1$ for the *ARS* regulator gives $\left\| \hat{Q}M\,\ell \right\|_\infty < 1$, i.e.,

$$\left|\hat{Q}M\right| < \frac{1}{|\ell|} \quad \text{or} \quad |\ell| < \frac{1}{\left|\hat{Q}M\right|} \quad \forall\omega \tag{10}$$

Thus the robust stability strongly depends on the model M and how the model-based *Y-parameter* $\hat{Q}$ is selected.

Consider the practical form of the optimal regulator (using M in (1)) of the *GTDOF* system based on the available model M of the process

$$\hat{R}_\mathrm{o} = \frac{P_\mathrm{w} G_\mathrm{w} M_+^{-1}}{1 - P_\mathrm{w} G_\mathrm{w} M_- z^{-d_\mathrm{m}}} = \\ = \frac{\left(P_\mathrm{w} G_\mathrm{w} M_+^{-1}\right)}{1-\left(P_\mathrm{w} G_\mathrm{w} M_+^{-1}\right)\left(M_+ M_- z^{-d_\mathrm{m}}\right)} = \frac{\hat{Q}}{1-\hat{Q}M} \tag{11}$$

where

$$\hat{Q} = P_\mathrm{w} G_\mathrm{w} M_+^{-1} \tag{12}$$

is the nominal *Y-parameter* depending on the model of the plant, which gives back (2) as $\hat{Q}\big|_{M=S} = Q_\mathrm{o} = P_\mathrm{w} G_\mathrm{w} S_+^{-1}$. The dependence on the inverse stable part is direct and visible, however, G_w generally depends on the inverse unstable part. We can now state that $\hat{R}_\mathrm{o}$ is also an *ARS* controller (but do not forget that only for the model M and not for the true process S).

Analyze the basic robust stability condition (10) obtained for *ARS* regulators in case of the *generic scheme*, where the optimal regulator is given by (10) and $\hat{Q} = P_\mathrm{w} G_\mathrm{w} M_+^{-1}$ from (11). We get

$$\left|\hat{Q}M\ell\right| = \left|P_\mathrm{w} G_\mathrm{w} M_+^{-1} M \ell\right| = \left|P_\mathrm{w} G_\mathrm{w} M_- z^{-d}\,\ell\right| = \left|P_\mathrm{w}\,\ell\right| \tag{13}$$

where $\left|G_\mathrm{w} M_-\right| = 1$, (because M_- is monic by definition and G_w is monic by construction), furthermore $\left|z^{-d}\right| = 1$ (which is well known) were used, thus finally

$$\sup_\omega |\ell| \le 1/\left|P_\mathrm{w}\right| \quad \text{or} \quad \|\ell\|_\infty \le 1/\left\|P_\mathrm{w}\right\|_\infty \tag{14}$$

Because the right hand side of this inequality depends only on P_w, which is the reference model for the regulatory property of the *GTDOF* system, this means that this is a special controller structure, where the performance of the closed-loop is directly influenced by the robustness limit (via the selected P_w).

III. RELATIVE MODEL ERROR CAUSED BY TIME-DELAY UNCERTAINTY

Let us compute the relative model error ℓ for an *IS* plant, where the model uncertainty comes only from a time-delay mismatch. The delay-free term is assumed to be known exactly, so $\overline{M}_- = 1$ and $M_+ = S_+$. In this case

$$\ell = \ell_\mathrm{d} = \frac{\Delta}{M} = \frac{S-M}{M} = \frac{S_+ z^{-d} - S_+ z^{-d_\mathrm{m}}}{S_+ z^{-d_\mathrm{m}}} = z^{-(d-d_\mathrm{m})} - 1 \tag{15}$$

Assume an equivalent continuous time plant with time-delay τ and a model with time-delay τ_m. The analogous equivalence means

$$\ell = \ell_\tau = e^{-\Delta\tau s} - 1 \tag{16}$$

where $\Delta\tau = \tau - \tau_\mathrm{m}$. The robust stability condition (14) for the continuous time case is now

$$\sup_\omega \left|\ell_\tau\right| = \sup_\omega \left|e^{-j\Delta\tau\omega} - 1\right| \le 1/\left|P_\mathrm{w}(j\omega)\right| \tag{17}$$

For the sake of simplicity assume a first order reference model now

$$P_\mathrm{w} = \frac{1}{1+s\mathrm{T}_\mathrm{w}} \quad ; \quad P_\mathrm{w}(j\omega) = \frac{1}{1+j\omega\mathrm{T}_\mathrm{w}} \tag{18}$$

which means an $1/T_{\mathrm{w}}$ bandwidth design goal for the resulting closed-loop. Using the first order reference model (18) the inequality to be solved for $\Delta\tau$ is

$$\sup_{\omega}\left|e^{-j\Delta\tau\omega}-1\right|\le\left|1+j\omega T_{\mathrm{w}}\right| \tag{19}$$

which has the solution as a robust stability (*RS*) condition

$$\boxed{\ell_{\tau}=\left|\frac{\Delta\tau}{\tau}\right|=\left|1-\frac{\tau_{\mathrm{m}}}{\tau}\right|<\frac{\pi}{\sqrt{3}}\frac{T_{\mathrm{w}}}{\tau}=1.82\frac{T_{\mathrm{w}}}{\tau}} \tag{20}$$

This inequality is one of our major result. The solution of the inequality (19) can be easily followed on Fig. 2.

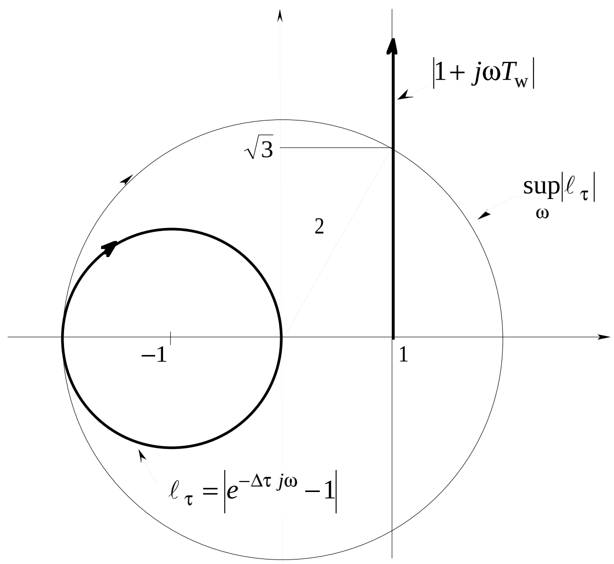

Fig. 2. Simple graphics helping to understand the solution of inequality (19)

It is interesting to mention that using the first order Taylor expansion of the exponential term one can get a good approximation of (19) and a sufficient but not necessary condition for small deviations

$$\ell_{\tau}=\left|\frac{\Delta\tau}{\tau}\right|=\left|1-\frac{\tau_{\mathrm{m}}}{\tau}\right|<\frac{T_{\mathrm{w}}}{\tau} \tag{21}$$

The interpretation of (20) and (21) is very simple: for small T_{w}, which means high closed-loop performance, the model time delay τ_{m} must be close to the true delay τ. So it is obtained that the admissible time-delay mismatch is limited by the inverse of the performance. It could be furthermore very interesting how this limit influences the robustness of the loop, see the next section.

There is a simple, however, a somewhat virtual way to increase the robust stability limit (20) by a higher order cutting filter form of the reference model

$$P_{\mathrm{w}}=\frac{1}{\left(1+sT_{\mathrm{w}}\right)^{n}}\quad;\quad P_{\mathrm{w}}(j\omega)=\frac{1}{\left(1+j\omega T_{\mathrm{w}}\right)^{n}} \tag{22}$$

Following the same procedure how (20) was obtained from (19), a more general *RS* form can be derived

$$\ell_{\tau}=\left|\frac{\Delta\tau}{\tau}\right|=\left|1-\frac{\tau_{\mathrm{m}}}{\tau}\right|<a(n)\frac{T_{\mathrm{w}}}{\tau} \tag{23}$$

where the increasing coefficient $a(n)$ is plotted in Fig. 3.

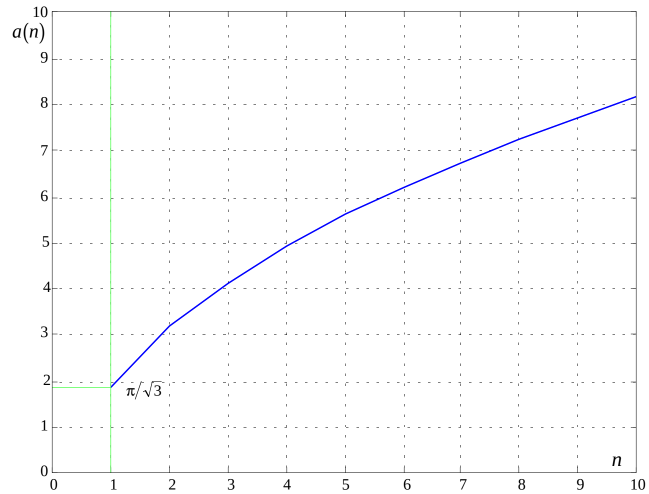

Fig. 3. The $a(n)$ function

IV. ROBUSTNESS, PERFORMANCE AND UNCERTAINTY IN TIME-DELAY

Detailed investigation of the above mentioned limiting behavior needs further numerical computations. Simple calculations give that the sensitivity function of the *GTDOF* system with *IS* plant, having time-delay mismatch for the discrete-time case is (assuming $G_{\mathrm{w}}=1$)

$$E=\frac{1-P_{\mathrm{w}}z^{-d_{\mathrm{m}}}}{1+\ell P_{\mathrm{w}}z^{-d_{\mathrm{m}}}}=\frac{1-P_{\mathrm{w}}z^{-d_{\mathrm{m}}}}{1+\ell_{\mathrm{d}}P_{\mathrm{w}}z^{-d_{\mathrm{m}}}} \tag{24}$$

and the continuous time equivalent follows as

$$E=\frac{1-P_{\mathrm{w}}e^{-s\tau_{\mathrm{m}}}}{1+\ell P_{\mathrm{w}}e^{-s\tau_{\mathrm{m}}}}=\frac{1-P_{\mathrm{w}}e^{-s\tau_{\mathrm{m}}}}{1+\ell_{\tau}P_{\mathrm{w}}e^{-s\tau_{\mathrm{m}}}} \tag{25}$$

For P_{w} given by (18) the sensitivity function (25) becomes

$$E=\frac{1+sT_{\mathrm{w}}-e^{-s\tau_{\mathrm{m}}}}{1+sT_{\mathrm{w}}+P_{\mathrm{w}}\left(e^{-s\tau}-e^{-s\tau_{\mathrm{m}}}\right)} \tag{26}$$

The well-known Nyquist stability margin (the simplest robustness measure) is defined by

$$\begin{aligned}\rho_m &= \rho_{min}(R) = \min_\omega |\rho(\omega, R)| = \min_\omega |1 + RS| = \\ &= \min_\omega |1 + Y(j\omega)| = \frac{1}{\|E\|_\infty}\end{aligned} \tag{27}$$

which is the distance between the point $(-1+0j)$ and the closest point of the open-loop transfer function $Y(j\omega)$. The reciprocal value of the norm is $\|E\|_\infty$. Unfortunately there is no simple analytical solution to obtain how the closed-loop robustness depends on the time-delay mismatch and on the performance. It is, however, possible to compute the graphical plot of a complex functional relationship $\rho_m = \rho_{min}(\tau_m/\tau, T_w/\tau)$ with the help of MATLAB.

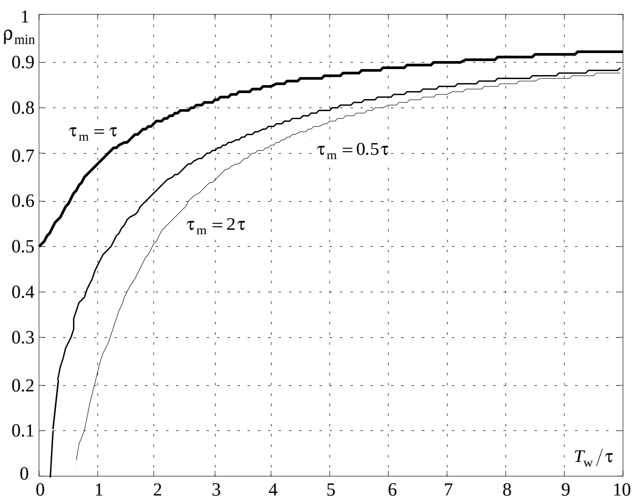

Fig. 4. Function $\rho_{min}(T_w/\tau)$ for $\tau_m = 0.5\tau, \tau, 2\tau$

As a result Fig. 4 shows the function $\rho_{min}(T_w/\tau)$ for $\tau_m = 0.5\tau, \tau, 2\tau$. For the ideal $\tau_m = \tau$ (no mismatch) case ρ_{min} depends only on our design goal (T_w) and on the plant time-delay (τ), more exactly on their relative value T_w/τ. The best robustness measure is $\rho_{min}(0) = 0.5$ for cases when the reference model P_w requires a very fast transient response from the time-delay process and the measure is $\rho_{min}(\infty) = 1$, if τ is negligible comparing to the time lag of P_w. It can be well seen that either under- or over-estimation of the time-delay causes considerable decrease of the robustness. Virtually ρ_{min} is more sensitive for over-estimation. (The left ends of the plots correspond to the robust stability limit.) While the no mismatch case provides an all stabilizing property for any performance requirement, in case of a non zero time-delay mismatch one can always expect the violation of the robustness stability limit for higher performance design.

It may be more reasonable to plot the function $\rho_{min}(\tau_m/\tau)$ parametrized by T_w/τ as Fig. 5 shows (our second major result). One can see how the robustness is extremely sensitive for high performance requirement, when T_w/τ is small and how this sensitivity decreases when T_w/τ is large for low performance design. It is also interesting to observe, that for small mismatch the over-estimation of the delay gives higher ρ_{min}, however, for large mismatch ρ_{min} is somewhat more sensitive, as it is shown in Fig. 2.

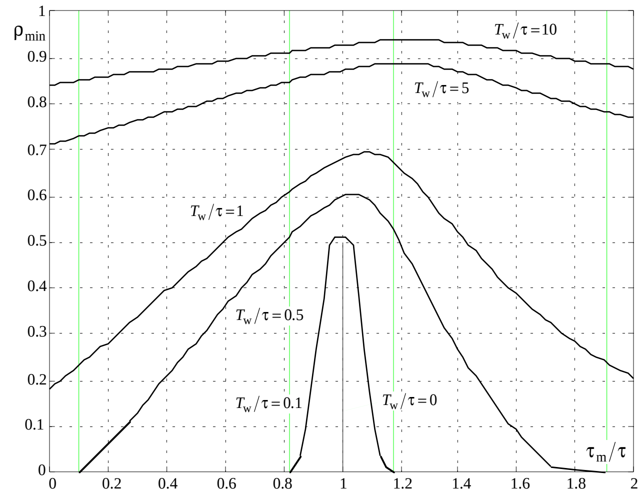

Fig. 5. The function $\rho_{min}(\tau_m/\tau)$ parametrized by T_w/τ

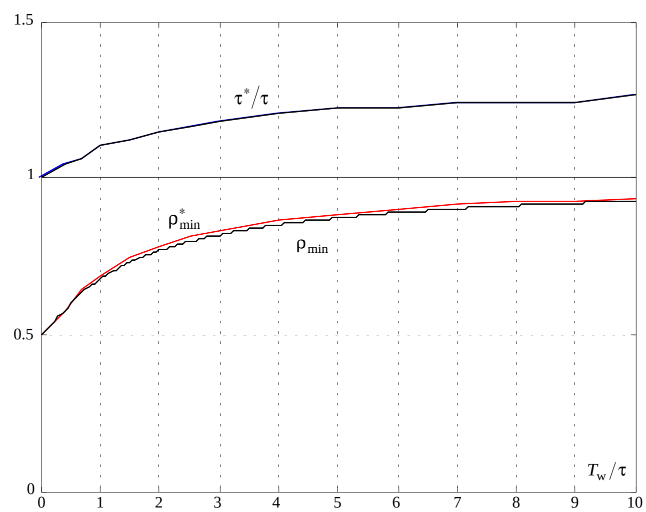

Fig. 6. Influence of the time-delay over-estimation

In a relatively wide range of T_w/τ, the over-estimation of the time-delay by τ^*/τ improves (i.e. increases) the ρ_{min} to ρ^*_{min} according to the maxima of the curves observable in Fig. 5. The over-estimation is less than 25% and the improvement is marginal, less than 5% as Fig. 6 shows.

If we assume that the time-delay mismatch is less than 20% in a practical case, the robustness degradation is always less than 10% for $T_w/\tau \geq 0.5$, which can be well seen in Fig. 6. So if we want to speed up the open-loop process to a time constant, which is considerable less than the delay, then it can only be done using a quite accurate knowledge of the time-delay. Contrary, if someone can expect a considerable variation in the time delay then only a less demanding (slower) design is more reliable and robust.

(The jags of both figures origin from the relative accuracy of the numerical computations. Do not forget that the Nyquist plot of a time-delay process has infinite number of winds around the origin and sometimes even the radius of the external wind is quite small. So it is not easy to find such frequency scaling which allows to determine both $\rho_{\min}$ (i.e. $\|E\|_\infty$) and the robust stability limit at the same time within a proper accuracy.)

The above results strengthen the conservative practical design experience that the time-delay is practically equivalent to an *IU* zero, i.e. invariant.

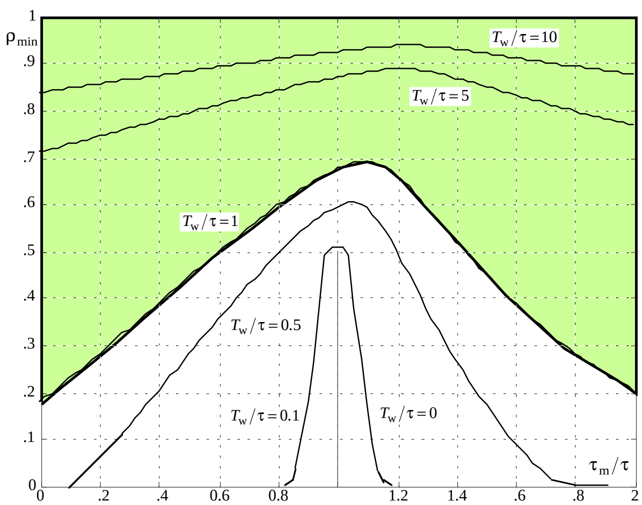

Fig. 7. The shaded area is suggested for acceptable good deal between performance, robustness and time-delay mismatch

It is interesting to summarize the complex relationship between performance, robustness and time-delay uncertainty and designate an acceptable area as Fig. 7 shows.

V. CONCLUSIONS

Most of the widely applied identification and adaptive control methods assume an apriori known time-delay. It is not easy (although possible) to incorporate the iterative or adaptive estimation of the delay into the recursive methods. Therefore one can always assume a time-delay uncertainty or mismatch at all practical applications. It was discussed here how this mismatch influences the robustness degradation and the reachable closed-loop performance.

A new necessary and sufficient inequality for *RS* is derived for the maximum allowable time-delay mismatch and a simpler sufficient condition is also given.

The complex relationship of robustness, performance and time-delay uncertainty is represented by a special new graphical plot helping the understanding and selection of an acceptable deal between these contradictory criteria.

The investigations show that bandwidth higher than the bandwidth of the delay term ($T_w < \tau$) can be reached only for a considerable lower robustness and at the same time a much more accurate knowledge of the time-delay is necessary. This corresponds to the practical design experience that the corner frequency of a delay term corresponds to an unstable zero, i.e., similarly invariant. So the acceptable performance domain means $T_w \geq \tau$.

We found that a certain slight overestimation of the time-delay improves the robustness, however, a higher overestimation causes considerable robustness degradation again. This observation can be used for model predictive algorithms, too.

REFERENCES

[1] Hocken, R.D., S.V. Salehi and J.E. Marshall (1983). Time-delay mismatch and the performance of predictor control schemes. *Int. J. Control*, **38,** 2, 433-447.

[2] De Paor, A. (1985). A modified Smith predictor and controller for unstable processes with time delay. *Int. J. Control*, **41,** 4, 1025-1036.

[3] Yamanaka, K. and E. Shimemura (1987). Effects of mismatched Smith controller on system response. *Proc. IFAC Congress*, 316-321, Munich, Germany.

[4] Tzypkin, Ya.Z. and M. Fu (1993). Robust stability of time-delay systems with an uncertain time-delay constant. *Int. J. Control*, **57,** 4, 865-879.

[5] Keviczky, L. (1995). Combined identification and control: another way. (Invited plenary paper.) *5th IFAC Symp. on Adaptive Control and Signal Processing*, *ACASP'95*, 13-30, Budapest, H.

[6] Keviczky, L. and Cs. Bányász (1997). An iterative redesign technique of reference models: How to reach the maximal bandwidth? *11th IFAC Symposium on System Identification SYSID'97*, Fukuoka, Japan, 619-624.

[7] Keviczky, L. and Cs. Bányász (1999). Optimality of two-degree of freedom controllers in $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norm space, their robustness and minimal sensitivity. *14th IFAC World Congress*, **F**, 331-336, Beijing, PRC.

[8] Keviczky, L. and Cs. Bányász (2002). Direct relationships of performance, robustness measures and amplitude constraint. CDC'2002 Las Vegas, USA.

[9] Maciejowski, J.M. (1989). *Multivariable Feedback Design*, Addison Wesley.