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Output Feedback Control of Switched Nonlinear Systems Using Multiple Lyapunov Functions

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Abstract— **This work proposes a hybrid nonlinear output feedback control methodology for a broad class of switched nonlinear systems with input constraints. The key feature of the proposed methodology is the integrated synthesis, via multiple Lyapunov functions, of "lower-level" nonlinear output feedback controllers together with "upper-level" switching laws, based on available state estimates, that orchestrate the transitions between the constituent modes and their respective controllers. The output feedback controllers are synthesized, using a combination of bounded state feedback controllers, high-gain observers and appropriate saturation filters, to enforce asymptotic stability for the individual closed-loop modes and provide an explicit characterization of the corresponding output feedback stability regions in terms of the input constraints and the observer gain. The switching logic tracks the evolution of the state estimates generated by the observers and orchestrates switching between the stability regions of the constituent modes in a way that guarantees asymptotic stability of the overall switched closed-loop system. The differences between the state and output feedback switching strategies are discussed and a chemical process example is used to demonstrate the proposed approach.**

I. INTRODUCTION

The study of hybrid systems in control is motivated by the fundamentally hybrid nature of many modern-day control systems, which are characterized by the interaction of lower-level continuous dynamics and upper-level discrete or logical components. It is well understood that the interaction of discrete events with even simple continuous dynamical systems can lead to complex dynamics and, potentially, to undesirable outcomes, if not appropriately accounted for in the control system design. Motivated by this, and the abundance of situations where hybrid systems arise in practice, significant research work has focused on hybrid systems over the last decade, covering a broad range of problems including, for example, modeling [25], simulation [1], [10], optimization [11], and control [19], [16], [2].

A class of hybrid systems that has attracted significant attention, because it can model several practical control problems that involve the integration of supervisory logic-based control schemes and feedback control algorithms, is the class of switched systems. For this class, results have been developed for stability analysis using the tools of multiple Lyapunov functions, for linear [21] and nonlinear systems [22], [3], [26], and the concept of dwell-time [12]; the reader may refer to [17], [5] for a survey of results in this area. These results have motivated the development of methods for control of various classes of switched systems (e.g., [24], [27], [13], [6]).

In a previous work [7], we developed a hybrid nonlinear control method for a broad class of switched nonlinear systems with input constraints. These are systems that consist of a finite family of continuous nonlinear dynamical modes, subject to hard constraints on their manipulated inputs, together with a higher-level supervisor that governs the transitions between the constituent modes. The key feature of the proposed control method was the integrated synthesis, via multiple Lyapunov functions (MLFs), of: (1) lower-level feedback controllers that stabilize the constituent constrained modes and provide, simultaneously, an explicit characterization of the stability region for each mode, and (2) upper-level switching laws that orchestrate, on the basis of the stability regions, the transitions between the continuous modes and their respective controllers, in a way that ensures stability of the overall switched closed-loop system. The method was extended to switched systems with uncertainty and constraints in [9].

In addition to input constraints, another important issue that must be accounted for in the control system design is the lack of complete state measurements. For switched systems, this issue affects both the design and implementation of the lower-level controllers and the upper-level switching laws which both have to be based on state estimates. Motivated by these considerations, we present in this work a nonlinear output feedback control method for a class of switched nonlinear systems with input constraints. The key idea is the coupling between the switching logic and the stability regions arising from the limitations imposed by both input constraints and the lack of full state measurements on the dynamics of the constituent modes of the switched system. Using MLFs, the proposed method involves the integrated synthesis of bounded nonlinear controllers, high-gain observers and switching laws based on available state estimates, that orchestrate stabilizing mode transitions. The remainder of the manuscript is organized as follows. In section II, we present the class of switched systems considered and briefly review MLF stability analysis. In section III, we address the state feedback problem to provide the necessary background for the output feedback control problem which is addressed in section IV. The differences between the

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state and output feedback switching strategies are discussed. Finally, in section V, the proposed methodology is demonstrated using a chemical process example.

II. Preliminaries

A. Class of systems

We consider the class of switched nonlinear systems represented by the following state-space description:

$$\begin{array}{rcl} \dot{x}(t) & = & f_{\sigma(t)}(x(t)) + G_{\sigma(t)}(x(t))u_{\sigma(t)} \\ y_m & = & h_m(x) \\ \|u_\sigma\| & \leq & u_\sigma^{max}, \ \sigma(t) \in \mathcal{I} = \{1, \cdots, N\} \end{array} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ denotes the vector of continuous-time state variables, $u_\sigma(t) = [u_\sigma^1(t) \cdots u_\sigma^m(t)]^T$ denotes the vector of manipulated inputs taking values in the nonempty compact subset $\mathcal{U} := \{u_\sigma \in \mathbb{R}^m : \|u_\sigma\| \leq u_\sigma^{max}\}$, $y_m \in \mathbb{R}^m$ denotes the vector of measured variables, $h_m(x)$ is a sufficiently smooth function on $\mathbb{R}^m$, $\sigma : [0, \infty) \to \mathcal{I}$ is the switching signal which is assumed to be a piecewise continuous (from the right) function of time, implying that only a finite number of switches is allowed on any finite interval of time. Without loss of generality, we assume that $f_i(0) = 0$ for all $i \in \mathcal{I}$. We also assume that the state x does not jump at the switching instants.

B. Stability analysis via multiple Lyapunov functions

Preparatory for its use in control, we will briefly review in this section the main idea of multiple Lyapunov functions (MLFs) as a tool for stability analysis of switched systems. To this end, consider the switched system of Eq.1, with $u_i(t) \equiv 0, \ i \in \mathcal{I}$, and suppose that we can find a family of Lyapunov-like functions $\{V_i : i \in \mathcal{I}\}$, each associated with the vector field $f_i(x)$. A Lyapunov-like function for the system $\dot{x} = f_i(x)$, with equilibrium point $x_{eq} = 0 \in \Omega_i \subset \mathbb{R}^n$, is a real-valued function $V_i(x)$, with continuous partial derivatives, defined over the region Ω_i, satisfying the conditions: (1) $V_i(0) = 0$ and $V_i(x) > 0$ for $x \neq 0$, $x \in \Omega_i$, and (2) $\dot{V}_i = \dfrac{\partial V_i(x)}{\partial x} f_i(x) \leq 0$, for $x \in \Omega_i$. The following theorem provides sufficient conditions for stability.

Theorem 1 [5] (see also [3]): *Given the N-switched nonlinear system of Eq.1, with $u_i(t) \equiv 0, \ i \in \mathcal{I}$, suppose that each vector field f_i has an associated Lyapunov-like function V_i in the region Ω_i, each with equilibrium point $x_{eq} = 0$, and suppose $\bigcup_i \Omega_i = \mathbb{R}^n$. Let $\sigma(t)$ be a given switching sequence such that $\sigma(t)$ can take on the value of i only if $x(t) \in \Omega_i$, and in addition*

$$V_i(x(t_{i_k})) \quad \leq \quad V_i(x(t_{i_{k-1}})) \quad (2)$$

where t_{i_k} denotes the k-th time that the vector field f_i is switched in, i.e., $\sigma(t_{i_k}^-) \neq \sigma(t_{i_k}^+) = i$. Then, the equilibrium point, $x_{eq} = 0$, of the system of Eq.1, with $u_i(t) \equiv 0, \ i \in \mathcal{I}$, is Lyapunov stable.

A generalization of the MLF concept, in the context of control of switched systems, is that of multiple control Lyapunov functions (MCLFs) [7]. The idea is to use a family of CLFs to design both a family of stabilizing nonlinear controllers and a set of switching laws that ensure stability of the overall closed-loop system. For a clear presentation of the main results of this paper, we will start in the next section by reviewing the state feedback control problem (i.e., $y_m = x$) which will provides the necessary foundation for formulating and solving the output feedback control problem in section IV.

III. State feedback control of switched nonlinear systems

Consider the switched nonlinear system of Eq.1. Given that switching is controlled by some higher-level supervisor, the problem we focus on is how to orchestrate switching between the various subsystems in a way that respects the constraints and guarantees asymptotic closed-loop stability. To this end, we formulate the following control objectives. The first is to synthesize a family of N bounded nonlinear state feedback controllers of the form, $u_i = -k_i(x, u_i^{max})(L_{G_i}V_i)^T$, $i = 1, \cdots, N$, where V_i is a CLF for the i-th mode and $L_{G_i}V_i$ is a row vector of the form $[L_{g_{i,1}}V_i \cdots L_{g_{i,m}}V_i]$, that: (1) satisfy the constraints, (2) enforce asymptotic stability for the individual closed-loop subsystems, and (3) provide an explicit characterization of the set of admissible initial conditions starting from where each mode is guaranteed to be stable. The second objective is to identify a set of switching laws, $\sigma(t) = \psi(x(t))$, that orchestrate the transition between the constituent modes and their respective controllers in a way that respects the input constraints and guarantees asymptotic stability of the constrained switched closed-loop system.

Theorem 2 below describes the controller switching strategy under full state feedback, based on the idea of integrating, via MCLFs, feedback and supervisory control, introduced in [7]. The proof of this theorem is similar to the one given in [7] and will be omitted for brevity.

Theorem 2: *Consider the switched nonlinear system of Eq.1, for which a family of CLFs V_i, $i = 1, \cdots, N$ exist, under the following family of bounded nonlinear feedback controllers:*

$$u_i \quad = \quad -k_i(x, u_i^{max})(L_{G_i}V_i)^T, \ i = 1, \cdots, N \quad (3)$$

where $k(x, u_i^{max}) =$

$$\begin{cases} \dfrac{\alpha_i(x) + \sqrt{\alpha_i^2(x) + (u_i^{max}\|\beta_i^T(x)\|)^4}}{\|\beta_i^T(x)\|^2 \left[1 + \sqrt{1 + (u_i^{max}\|\beta_i^T(x)\|)^2}\right]}, & \beta_i^T(x) \neq 0 \\ 0, & \beta_i^T(x) = 0 \end{cases} \quad (4)$$

with $\alpha_i(x) = L_{f_i}V_i(x) + \rho_i V_i(x)$, $\rho_i > 0$ and $\beta_i(x) = L_{G_i}V_i(x)$. Let $\Phi_i(u_i^{max})$ be the largest set of x, containing the origin, such that:

$$L_{f_i}V_i(x) + \rho_i V_i(x) \quad \leq \quad u_i^{max}\|(L_{G_i}V_i(x))^T\| \quad (5)$$

Also, let $\Omega_i^*(u_i^{max}) := \{x \in \mathbb{R}^n : V_i(x) \leq \delta_{x,i}\}$ be a level set of V_i completely contained in Φ_i for some $\delta_{x,i}$, and assume, without loss of generality, that $x(0) \in \Omega_i^*(u_i^{max})$ for some $i \in \mathcal{I}$. If, at any given time, T, the following conditions hold:

$$x(T) \in \Omega_j^*(u_j^{max}) \quad (6)$$

$$V_j(x(T)) < V_j(x(t_{j*})) \quad (7)$$

for some $j \in \mathcal{I}$, $j \neq i$, where $t_{j*} < T$ is the time when the j-th subsystem was last switched in, i.e. $\sigma(t_{j*}^-) \neq \sigma(t_{j*}^+) = j$, then setting $\sigma(T^+) = j$ guarantees that the origin of the switched closed-loop system is asymptotically stable.

Remark 1: Note that asymptotic stability of each mode of the closed–loop system implies that there exists a family of class $\mathcal{KL}$ functions β_i, $i = 1, \cdots, N$ such that a bound of the following form holds for each closed-loop mode:

$$\|x(t)\| \leq \beta_i(\|x(0)\|, t) \quad (8)$$

This property will be used later in the design of the output feedback controllers.

Remark 2: The controllers of Eqs.3-4 are synthesized, via multiple CLFs, by reshaping the nonlinear gain $k(\cdot)$ of the bounded $L_G V$ controller design originally proposed in [18]. In particular, the addition of the term $-\rho_i V_i$ is done to enforce (local) exponential stability which will be needed in designing the appropriate output feedback controllers.

Remark 3: The use of CLF-based controllers of the form of Eqs.3-4 is motivated by the fact that this class of controllers account explicitly for input constraints and provide an explicit characterization of the constrained stability region. Specifically, the i-th inequality in Eq.5 describes a state-space region, $\Phi_i(u_i^{max})$, where the i-th control law satisfies the constraints and forces V_i to decrease monotonically along the trajectories of the i-th closed-loop subsystem. To guarantee that $\dot{V}_i$ remains negative for all the times that the i-th mode is active, we compute an invariant set, $\Omega_i^*(u_i^{max})$, within $\Phi_i(u_i^{max})$. This set is an estimate of the stability region associated with each mode.

Remark 4: The switching rules of Eqs.6-7 determine, implicitly, the times when switching from mode i to mode j is permissible. The first rule, which tracks the temporal evolution of the continuous state, x, requires that, at the desired switching time, the continuous state reside within the stability region of the subsystem to be activated. This ensures that, once this subsystem is activated, its constraints are satisfied and its Lyapunov function continues to decay for as long as that mode remains active. Note that this condition applies at every time that the supervisor considers switching from one mode to another. In contrast, the second switching rule, which tracks the evolution of the Lyapunov functions, applies only when the target mode j has been previously activated. In this case, Eq.7 requires that V_j at the current "switch in" be less than its value at the previous "switch in." This requirement is less conservative than the one proposed in Theorem 2 in [7] and may allow switching to take place earlier. To implement these switching rules, the supervisor needs to monitor (on-line) how x evolves in time to determine if and when the switching conditions are satisfied. If the conditions are satisfied for the desired target mode at some given time, then switching can take place safely; otherwise, the current mode is kept active.

IV. Output feedback control of switched nonlinear systems

In this section, we consider the system of Eq.1 for the case when some of the states are not available for measurement. We highlight the implications of the lack of full state measurements for the design and implementation of the switching logic.

A. Control problem formulation

Referring to the switched nonlinear system of Eq.1, our objectives for the output feedback control problem include: (a) the synthesis a family of N bounded nonlinear dynamic output feedback controller of the general form:

$$\begin{array}{rcl} \dot{\omega} & = & \mathcal{F}_i(\omega, y_m) \\ u_i & = & -p_i(\omega, y_m, u_i^{max}), i = 1, \cdots, N \end{array} \quad (9)$$

where $\omega \in \mathbb{R}^s$ is a state, $\mathcal{F}_i(\cdot)$ is a vector function, $p_i(\cdot)$ is a bounded nonlinear function, that enforce asymptotic (and local exponential) stability, for the individual closed-loop subsystems, and provide, for each mode, an explicit characterization of the constrained stability region under output feedback, and (b) the design a set of switching laws, $\sigma(t) = \psi'(\omega, y_m)$, that orchestrate, based on state estimates, stabilizing transitions between the constituent closed-loop modes.

In the remainder of this section, we first review an output feedback controller design, based on a combination of high-gain observers, saturation filters and the state feedback controllers of Eqs.3-4 and characterize the stability properties of the closed–loop system under output feedback control (see also [14], [23], [4], [8] for results on output feedback control of nonlinear systems). We then present switching laws based on available state estimates that guarantee closed-loop stability for the overall switched system.

B. Output feedback controller synthesis

In order to synthesize an output feedback controller that enforces the requested closed-loop properties for each mode, we will need to impose the following assumption on the system of Eq.1. To simplify the notation, we will focus on the case of a single measured output. The results, however, can be readily generalized to the case of multiple outputs.

Assumption 1: *For each $i \in \mathcal{I}$, there exists a set of coordinates:*

$$\left[\xi_i \right] = \begin{bmatrix} \xi_i^{(1)} \\ \xi_i^{(2)} \\ \vdots \\ \xi_i^{(n)} \end{bmatrix} = \chi_i(x) = \begin{bmatrix} h_m(x) \\ L_{f_i} h_m(x) \\ \vdots \\ L_{f_i}^{n-1} h_m(x) \end{bmatrix} \quad (10)$$

such that the system of Eq.1 takes the form:

$$\begin{array}{rcl} \dot{\xi}_i^{(1)} & = & \xi_i^{(2)} \\ & \vdots & \\ \dot{\xi}_i^{(n-1)} & = & \xi_i^{(n)} \\ \dot{\xi}_i^{(n)} & = & L_{f_i}^n h_m(\chi_i^{-1}(\xi_i)) + L_{g_i} L_{f_i}^{n-1} h_m(\chi_i^{-1}(\xi_i)) u_i \end{array} \quad (11)$$

where $L_{g_i} L_{f_i}^{n-1} h_m(x) \neq 0$ for all $x \in \mathbb{R}^n$. Also, $\xi_i \longrightarrow 0$ if and only if $x \longrightarrow 0$.

We note that the change of variables is invertible since, for every x, the variable ξ is uniquely determined by the transformation $\xi_i = \chi_i(x)$. This implies that if one can estimate the values of ξ_i for all times, using an appropriate state observer, then we automatically obtain estimates of x for all times, which can be used to implement the state feedback controller. The existence of such a transformation will facilitate the design of the high-gain observers which will be instrumental in preserving the same closed-loop stability properties achieved under full state feedback.

Proposition 1 below presents the output feedback controller used for each mode and characterizes its stability properties. The proof of the proposition, which relies on singular perturbation arguments, is a special case of the proof of Theorem 2 in [8], and is omitted for brevity. To simplify the statement of the proposition, we first introduce the following notation: we define $\bar{\alpha}_i(\cdot)$ as a class $\mathcal{K}$ function that satisfies $\bar{\alpha}_i(\|x\|) \leq V_i(x)$. We also define the set $\Omega_{b,i} := \{x \in \mathbb{R}^n : V_i(x) \leq \delta_{b,i}\}$, where $\delta_{b,i} < \delta_{s,i}$ is chosen such that $\beta_i(\bar{\alpha}_i^{-1}(\delta_{b,i}), 0) < \bar{\alpha}_i^{-1}(\delta_{x,i})$, where $\beta_i(\cdot,\cdot)$ is a class $\mathcal{KL}$ function defined in Eq.8 and $\delta_{x,i}$ is a positive real number defined in Theorem 1.

Proposition 1: *Consider the nonlinear system of Eq.1, for a fixed mode, $\sigma(t) = i$, under the output feedback controller:*

$$\begin{array}{rcl} \dot{\tilde{y}} & = & \begin{bmatrix} -L_i a_1^{(i)} & 1 & 0 & \cdots & 0 \\ -L_i^2 a_2^{(i)} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -L_i^n a_n^{(i)} & 0 & 0 & \cdots & 0 \end{bmatrix} \tilde{y} + \begin{bmatrix} L_i a_1^{(i)} \\ L_i^2 a_2^{(i)} \\ \vdots \\ L_i^n a_n^{(i)} \end{bmatrix} y_m \\ u_i & = & -k_i(\hat{x}, u_i^{max})(L_{G_i} V_i(\hat{x}))^T \end{array} \tag{12}$$

where the parameters, $a_1^{(i)}, \cdots, a_n^{(i)}$ are chosen such that the polynomial $s^n + a_1^{(i)} s^{n-1} + a_2^{(i)} s^{n-2} + \cdots + a_n^{(i)} = 0$ is Hurwitz, $\hat{x} = \chi_i^{-1}(sat(\tilde{y}))$, $sat(\cdot) = \min\{1, \zeta_{max,i}/|\cdot|\}(\cdot)$, with $\zeta_{max,i} = \beta_\zeta(\delta_{\zeta,i}, 0)$ where β_ζ is a class $\mathcal{KL}$ function and $\delta_{\zeta,i}$ is the maximum value of the norm of the vector $[h_m(x)\ L_{f_i} h_m(x) \cdots L_{f_i}^{n-1} h_m(x)]$ for $V_i(x) \leq \delta_{x,i}$ and let $\epsilon_i = 1/L_i$. Then, given $\Omega_{b,i}$, there exists $\epsilon_i^* > 0$ such that if $\epsilon_i \in (0, \epsilon_i^*]$, $x(0) \in \Omega_{b,i}$, and $\|\tilde{y}(0)\| \leq \delta_{\zeta,i}$, the origin of the closed–loop system is asymptotically (and locally exponentially) stable. Furthermore, given $\epsilon_i \in (0, \epsilon_i^*]$ and some real number $e_{m,i} > 0$, there exists a real number $T_i^b > 0$ such that $\|x(t) - \hat{x}(t)\| \leq e_{m,i}$ for all $t \geq T_i^b$.

Remark 5: The output feedback controller of Eq.12 consists of a high-gain observer which provides estimates of the derivatives of the output y_m up to order $n-1$, denoted as $\tilde{y}_0, \tilde{y}_1, \cdots, \tilde{y}_{n-1}$, and thus estimates of the variables $\xi_i^{(1)}, \cdots, \xi_i^{(n)}$ (note from Assumption 1 that $\xi_i^{(k)} = \frac{d^{k-1} y_m}{dt^{k-1}}, k = 1, \cdots, n$), and a static state feedback controller that enforces closed-loop stability. To eliminate the peaking phenomenon associated with the high-gain observer, we use a standard saturation function, sat, to eliminate wrong estimates of the output derivatives for short times. The use of a high-gain observer (together with the saturation filter) allows us to practically preserve the closed-loop stability region obtained under state feedback. Specifically, starting from any compact subset of initial conditions within the state feedback region ($\Omega_{b,i} \subset \Omega_i^*$), the output feedback controller of Eqs.12 continues to enforce asymptotic stability in the closed-loop system provided that the observer gain is chosen sufficiently large. As expected, the nature of this semi-regional result is consistent with the semi-global result obtained for the unconstrained case. It should be noted, however, that while the output feedback stability region can, in principle, be chosen as close as desired to its state feedback counterpart by increasing the observer gain L_i, it is well known that large observer gains can amplify measurement noise and induce poor performance. This points to a fundamental trade-off that cannot be resolved by simply changing the estimation scheme. For example, although one could replace the high-gain observer design with other observer designs (for example, a moving horizon estimator) to gain a better handle on measurement noise, it is difficult in such schemes to obtain an explicit relationship between the observer tuning parameters and the output feedback stability region.

Remark 6: Owing to the presence of the fast (high-gain) observer in the dynamical system of Eq.12, the closed-loop system for the i-th mode can be cast as a two time-scale system and, therefore, represented in the following singularly perturbed form, where $\epsilon_i = 1/L_i$ is the singular perturbation parameter:

$$\begin{array}{rcl} \epsilon_i \dot{e}_o & = & A e_o + \epsilon_i b \Delta_i(x, \hat{x}) \\ \dot{x} & = & f_i(x) + g_i(x) p_i(\hat{x}, u_i^{max}) \end{array} \tag{13}$$

where e_o is a vector of the auxiliary error variables $\hat{e}_k = L^{n-k}(y^{(k-1)} - \tilde{y}_k)$, A is an $n \times n$ matrix, $b = [0 \cdots 0\ 1]^T$ is a $n \times 1$ vector, and Δ_i is a Lipschitz function of its argument. It is clear from the above representation that, within the singular perturbation formulation, the observer error states, e_o, which are directly related to the estimates of the output and its derivatives up to order $n-1$, constitute the fast states of the singularly perturbed system of Eq.13, while the states of the original system of Eq.1 under the static component of the controller represent the slow states.

Remark 7: Note that asymptotic stability of each mode of the closed–loop system under the output feedback controller of Eq.12 implies that there exists a Lyapunov function $V_i^c, i = 1, \cdots, N$, for each mode of the closed–loop system (the existence of which is ascertained via a converse Lyapunov theorem argument; see, for example, Theorem 3.14 in [15]), such that $\dot{V}_i^c(x_f) < 0$, where $x_f = [x^T\ e_o^T]^T$ is the state of the full closed-loop system of Eq.13. We will use these Lyapunov functions in the next section to design the appropriate switching rules under output feedback.

C. Output feedback switching logic

Owing to the lack of full state measurements, the supervisor can rely only on the available state estimates to decide whether switching at any given time is permissible, and, therefore, needs to make reliable inferences regarding the position of the states based upon the available state estimates. Proposition 2 below establishes the existence of a set, $\Omega_{s,i}$, such that once the state estimation error has fallen below a certain value (note that decay rate can be controlled by adjusting L_i), the presence of the state trajectory in the output feedback stability region, $\Omega_{b,i}$, can be guaranteed by verifying the presence of the state estimates in the set $\Omega_{s,i}$. The proof of this proposition follows directly from the continuity of V_i with respect to its argument, and is omitted due to space limitations (see [20] for a similar proof in the linear case).

Proposition 2: *Given any positive real number $\delta_{b,i}$, there exists a positive real number $e^*_{m,i}$, and a set $\Omega_{s,i} := \{x \in \mathbb{R}^n : V_i(x) \leq \delta_{s,i}\}$ such that if $\|x - \hat{x}\| \leq e_{m,i}$, where $e_{m,i} \in (0, e^*_{m,i}]$ then $\hat{x} \in \Omega_{s,i} \Longrightarrow x \in \Omega_{b,i}$.*

We are now ready to proceed with the design of the switching logic. To this end, consider the switched nonlinear system of Eq.1, for which Assumption 1 holds and, for each mode, an output feedback controller of the form of Eq.12 has been designed and a Lyapunov function V_i^c has been determined. Given the desired output feedback stability regions $\Omega_{b,i} \subset \Omega^*_i$, $i = 1, \cdots, N$, we choose, for simplicity, $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_n \leq \min\{\epsilon^*_i\}$ (i.e., the same observer gain is used for all modes). Also assume that, for each mode, and for choices of $e_{m,i} \leq e^*_{m,i}$, the sets $\Omega_{s,i}$ (see Proposition 2) and the times $T_{b,i}$ (see Proposition 1) have been determined, and let $T_b^{max} = \max\limits_{i=1,\cdots,N}\{T_{b,i}\}$. Theorem 3 below presents the output feedback switching rules. The proof is sketched in the appendix.

Theorem 3: *Assume, without loss of generality, that $x(0) \in \Omega^*_{b,i}(u_i^{max})$ for some $i \in \mathcal{I}$ and choose $\tilde{y}(0)$ such that $\|\tilde{y}(0)\| \leq \delta_{\zeta,i}$, where $\delta_{\zeta,i}$ was defined in Proposition 1. Let T_{old} be the time that the previous switch took place. Let M_i be such that $\|z_1 - z_2\| \leq e_{m,i} \Longrightarrow |V_i^c(z_1) - V_i^c(z_2)| \leq M_i$. If, at any given time, T, the following conditions hold:*

$$T \;\; \geq \;\; T_{old} + T_b^{max}, \;\; \hat{x}(T) \; \in \; \Omega_{s,j}(u_j^{max}) \tag{14}$$

for some $j \in \mathcal{I}$, $j \neq i$, and

$$V_i^c(\hat{x}_f(T)) + 2M_i \quad < \quad V_i^c(\hat{x}_f(t_{i*})) \tag{15}$$

where $\hat{x}_f = [\hat{x}^T \; e_o^T]^T$, $e_o = [\hat{e}_1 \; \hat{e}_2 \; \cdots \; \hat{e}_n]^T$ $\hat{e}_k = L^{n-k}(y^{(k-1)} - \tilde{y}_k)$ and $t_{i*} < T$ is the time when the i-th subsystem was last switched out, i.e. $\sigma(t^+_{i*}) \neq \sigma(t^-_{i*}) = i$, then setting $\sigma(T^+) = j$ and

$$\tilde{y}(T^+) \quad = \quad \left\{ \begin{array}{cc} \tilde{y}(T) & , \|\tilde{y}(T)\| \leq \delta_{\xi,j} \\ \tilde{y}(T)\dfrac{\delta_{\xi,j}}{\|\tilde{y}(T)\|} & , \|\tilde{y}(T)\| > \delta_{\xi,j} \end{array} \right\} \tag{16}$$

guarantees that the origin of the switched closed-loop system of Eqs.1,12, 14-15 is asymptotically stable.

Remark 8: The fact that the switching rules are based on the state estimates has several important implications that distinguish the output feedback switching logic from its state feedback counterpart. First, note that the switching rules dictate that there be a time interval of at least T_b^{max} between two consecutive switches. This is done to ensure that for the given choice of the observer gain, and once a given mode is switched in, the estimation error has enough time to decrease to a sufficiently small value, such that, for all future times, the position of the state can be inferred by looking at the state estimate. Recall from Proposition 2 that the relation $\hat{x} \in \Omega_{s,j} \Longrightarrow x \in \Omega_{b,j}$ holds only when the estimation error is sufficiently small. Second, the decision to switch is not based on $\hat{x}$ entering $\Omega_{b,j}$ (under state feedback it was based on x entering Ω^*_j); rather it is based on $\hat{x}$ entering $\Omega_{s,j}$. The inference that $\hat{x} \in \Omega_{s,j} \Longrightarrow x \in \Omega_{b,j}$, however, can only be made once the error has sufficiently decreased, and this is guaranteed to happen after the closed–loop system has evolved in mode i for a time $T_b^{max} \geq T_{b,i}$. Therefore, a switch is not executed before an interval of length T_b^{max} elapses (from the last switching instance) even if $\hat{x}$ enters $\Omega_{s,j}$ at some earlier time. Furthermore, in contrast to the switching rules under state feedback, the MLF condition of Eq.15 is checked for switch-out rather than switch-in times (once again, this is to ensure that the error has decreased sufficiently). Also, in contrast to Theorem 2, the MLF condition of Eq.15 is checked using the Lyapunov function, $V_i^c(x_f)$, for the full closed-loop system of Eq.13, instead of the CLF, $V_i(x)$, used in the controller design. Finally, since the MLF condition is checked using the state estimate, Eq.15 requires that the value of V_i^c based on $\hat{x}$ decay by a margin ($2M_i$) sufficient to guarantee the decay of V_i^c based on x.

Remark 9: Note that even though the estimation error decreases while a certain mode is active, right after the switch, the transformation $\chi_i(\cdot)$ of Eq.11 changes (because the vector field, f_i, changes), while both ξ (if not re-initialized) and x evolve continuously. Therefore, unless the high-gain observer is re-initialized appropriately upon switching into the new mode, the estimate $\hat{x} = \chi^{-1}(sat(\tilde{y}))$ will evolve discontinuously, which implies that right after a switch to mode j, there is no guarantee that $\|\tilde{y}(T)\| \leq \delta_{\xi,j}$. To circumvent this problem, the values of the state estimates, generated through $\tilde{y}$, are re-initialized using Eq.16 to ensure that $\|\tilde{y}(T)\| \leq \delta_{\xi,j}$ which, from Proposition 1, is necessary for $\Omega_{b,j}$ to continue to be the output feedback stability region for mode j.

V. Application to a chemical process example

Consider a continuous stirred tank reactor where three parallel, irreversible, first-order exothermic reactions of the form $A \xrightarrow{k_1} D$, $A \xrightarrow{k_2} U$ and $A \xrightarrow{k_3} R$ take place, where A is the reactant species and D, U, R denote three product species. The reactor has two inlet streams: the first continuously feeds pure A at flow rate $F = 83.33\ L/min$, concentration $C_{A0} = 4.0\ mol/L$ and temperature $T_{A0} = 300\ K$, while the second can be turned on or off (by means of an on/off valve) during reactor operation. When turned on, the second stream feeds pure A at flow rate $F^* = 200\ L/min$, concentration $C^*_{A0} = 5.0\ mol/L$ and temperature $T^*_{A0} = 500\ K$. Under standard modeling assumptions, the mathematical model for the process takes the form:

$$\begin{aligned}\dot{C}_A &= \frac{F}{V}(C_{A0}-C_A)+(\sigma(t)-1)\frac{F^*}{V}(C^*_{A0}-C_A)\\ &- \sum_{i=1}^{3}R_i(C_A,T)\\ \dot{T} &= \frac{F}{V}(T_{A0}-T)+(\sigma(t)-1)\frac{F^*}{V}(T^*_{A0}-T)\\ &+ \sum_{i=1}^{3}G_i(C_A,T)+\frac{Q}{\rho_m c_{pm} V}\end{aligned} \tag{17}$$

where $R_i(C_A,T) = k_{0i}\exp\left(\frac{-E_i}{RT}\right)C_A$, $G_i(C_A,T) = \frac{(-\Delta H_i)}{\rho_m c_{pm}}R_i(C_A,T)$, C_A denotes the concentration of species A, T denotes the temperature of the reactor, Q denotes the rate of heat input to the reactor, V denotes the volume of the reactor, k_{0i}, E_i, ΔH_i denote the pre-exponential constants, the activation energies, and the enthalpies of the three reactions, respectively, c_{pm} and ρ_m, denote the heat capacity and density of the fluid in the reactor. The values of these process parameters can be found in Table 1 in [8]. $\sigma(t) = 1$ when the second inlet stream is turned off and $\sigma(t) = 2$ when it is turned on. Initially, it is assumed that $\sigma = 1$. During reactor operation, however, it is desired to enhance the product concentration by feeding more reactant material through the second inlet stream ($\sigma = 2$). It was verified that for $\sigma = 1$, the open-loop system (with $Q = 0$) has three equilibrium points, one of which unstable at $T_s = 388.58\ K$.

The control objectives are to: (1) stabilize the reactor temperature at the open-loop unstable steady-state of mode 1 ($T_s = 388.58\ K$), and (2) maintain the temperature at this steady-state when the reactor switches to mode 2. Note that, with this requirement, both closed-loop modes share the same steady-state temperature but have different steady-state reactant concentrations (see Figure 1 for the different equilibrium points). The control objective is to be accomplished by manipulating Q, subject to the constraint $|Q| \leq 1 \times 10^4$ KJ/min. Two quadratic, positive-definite functions of the form $\bar{V}_1 = \bar{V}_2 = \frac{1}{2}c_\delta(T-T_s)^2$, where $c_\delta = 1/T_s^2$, were used to synthesize, on the basis of the T-subsystems, two bounded controllers of the form of Eqs.3-4 (note that the $\bar{V}_i$'s are CLFs for the T-subsystem only and not for the full system of Eq.17). We will skip the details of controller synthesis due to space limitations. To estimate the stability region for each mode, the following Lyapunov functions, based on the full system, were used: $V_1 = \frac{1}{2}c_1x_1^2+\frac{1}{2}c_2x_2^2$ (for mode 1), where $x_1 = (T-T_s)/T_s$, $x_2 = (C_A - C_{As})/C_{As}$, $c_1 = 38.8$, $c_2 = 1.0$, and $V_2 = \frac{1}{2}c_3x_1^2 + \frac{1}{2}c_4x_2^2$ (for mode 2) where $c_3 = 19.4$, $c_4 = 1.0$. The invariant regions, denoted in Figure 1 by Ω_1^*, Ω_2^*, respectively, represent estimates of the stability regions for each mode.

We first present the case when both T and C_A are available for measurement. The solid lines in Figures 1 and 2 depict the temperature, concentration and heat input profiles when the reactor is initialized at $x_0 =$

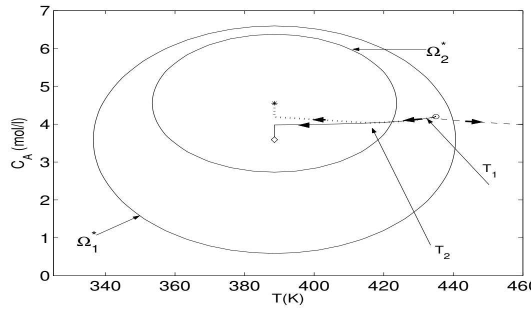

Fig. 1. A phase plot showing the stability region estimates Ω_1^*, Ω_2^*.

$[435\ K,\ 4.2\ mol/L]^T \in \Omega_1^*$ and operated in mode 1 for all times (with no switching). We observe that the controller for this mode successfully stabilizes the reactor temperature at the desired steady-state. The dashed lines in Figures 1-2 depict the result when the reactor (initialized at x_0 within Ω_1^*) switches to mode 2 (with its corresponding controller) at a randomly chosen time of $t = T_1 = 0.1\ min$. It is clear that in this case the controller is unable to stabilize the temperature at the desired steady-state. The reason is the fact that at $t = 0.1\ min$, the state of the system lies outside the stability region of mode 2 and, therefore, the available control action is insufficient to achieve stabilization as can be seen from the input profile in Figure 2 (dashed lines). To avoid this instability, we use the switching scheme proposed in Theorem 2. To this end, the reactor is initialized in mode 1 at x_0 and the closed-loop state is monitored (dotted trajectory in Figure 1). At $t = T_2 = 1.0\ min$, the state is observed to belong to Ω_2^* (i.e., the condition in Eq.6 is satisfied) and, consequently, the supervisor switches to mode 2 (note that the condition of Eq.7 is not needed here since mode 1 is never reactivated). The temperature, concentration and heat input profiles for this case are given in Figure 2 (dotted lines) which show that the controllers successfully drive the reactor temperature to the desired steady-state and maintain it there with the available control action (note that the concentration settles at a higher steady-state value than that of mode 1 thus achieving our switching objective of enhancing reactant concentration).

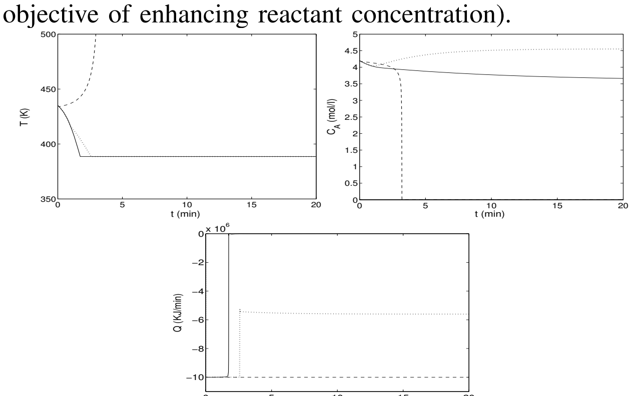

Fig. 2. Closed-loop state and input profiles when reactor is initialized within Ω_1^* and operated in mode 1 for all times (solid), when reactor switches to mode 2 at $t = 0.1\ min$ (dashed), and when the switch is executed at $t = 1.0\ min$ (dotted).

We now consider the case when only C_A is measured. For this choice of the measured output, Assumption 1 is satisfied and, therefore, an output feedback controller of the form of Eq.12 is designed for each mode. The values of the observer parameters in the state estimator design of Eq.12 are chosen as $L_1 = L_2 = 100$, $a_1^{(1)} = a_1^{(2)} = 10$ and $a_2^{(1)} = a_2^{(2)} = 20$. We fist demonstrate the implementation of the switching rule of Eq.14. As shown in Figure 3, starting from the same initial condition considered under state feedback, $x(0) = [435\ K,\ 4.2\ mol/L]^T$, and with $\hat{x}(0) = [411.3\ K,\ 4.2\ mol/L]^T$ (the state and state estimate trajectories are shown by the solid and dashed lines, respectively), implementing the output feedback controller of Eq.12 for mode 1 and switching to mode 2 (and its associated output feedback controller) at $t = 1.0$ (upon observing that the state estimates are within Ω_2^*) results in closed–loop stability. Included in Figures 3-4 also are the corresponding state (dotted lines in Figure 4) and manipulated input (dotted lines in Figure 4) trajectories under state feedback control. Notice that the manipulated input profile under output feedback control differs from that under state feedback control for a brief period of time, due to the initial error in the state estimates, but converges to that under state feedback very quickly as the estimates converge to the true state values.

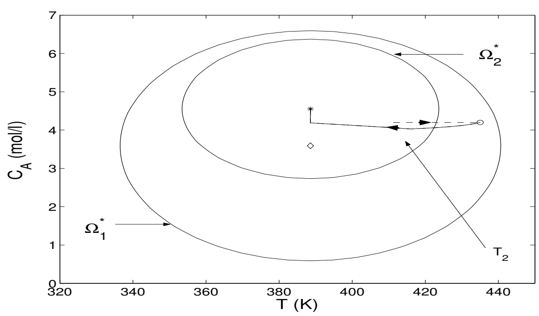

Fig. 3. Closed-loop state trajectories under state (dotted) and output (solid) feedback, and the state estimate trajectory (dashed) under output feedback are shown for the case when the reactor is initialized within Ω_1^* and the switch is executed at $t = 1.0\ min := T_2$.

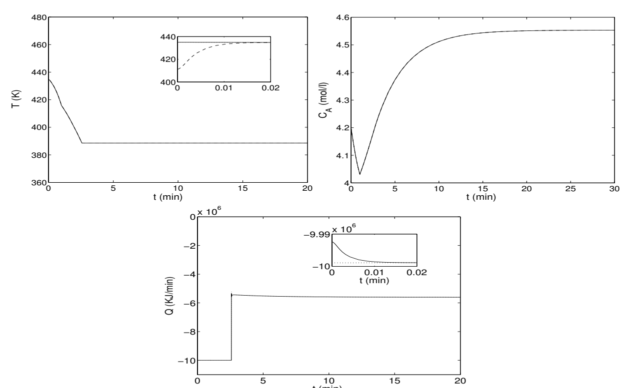

Fig. 4. Closed-loop state profiles under state (dotted) and output (solid) feedback; state estimate profiles under output feedback (dashed), and the manipulated input profiles are shown for the case when the reactor is initialized within Ω_1^* and the switch is executed at $t = 1.0\ min$.

To demonstrate the importance of using an observer gain consistent with the choice of the output feedback stability region, we present in Figures 5-6 (solid lines) a scenario, where starting from the same initial conditions in Ω_1^*, the reactor is operated in mode 1 for all times but with a lower value of the observer gain, $L = 0.1$. In this case, the error in the control action, resulting from the error in the value of the state estimates, leads to instability, demonstrating the need for appropriate choice of the observer parameters.

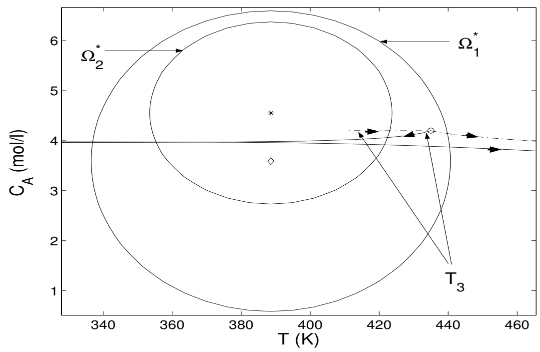

Fig. 5. The solid line shows the closed-loop state trajectory when the reactor is initialized within Ω_1^* and operated in mode 1 for all times with $L = 0.1$. The dotted and dash-dotted lines show, respectively, the closed-loop state and state estimate trajectories for $L = 100.0$ when the reactor switches to mode 2 at $t = 0.002\ min := T_3$.

Finally, we illustrate the importance of waiting for a small period of time before a decision regarding switching is made even if a sufficiently large value of the observer gain is being used (i.e., a value for which Ω_1^* and Ω_2^* closely estimate the output feedback stability region). To this end, we use $L_1 = L_2 = 100$, as in the first scenario. Starting from the same initial conditions, it is observed that at $t = 0.002\ min := T_3$ the state estimates reside in Ω_2^* (see dotted (which coincides with the solid line) and dash-dotted lines in Figure 5). Notice that while the state estimates belong to Ω_2^*, the true states are outside of Ω_2^* at this time. If the switch is executed immediately (as done under state feedback; see Theorem 2), then the closed–loop system becomes unstable. In contrast, if the decision regarding a switch is made after waiting for a sufficiently long period of time (as required by the switching rule of Eq.14) after which the state estimates have converged to their true values, closed–loop stability is achieved (see solid lines in Figures 3-4).

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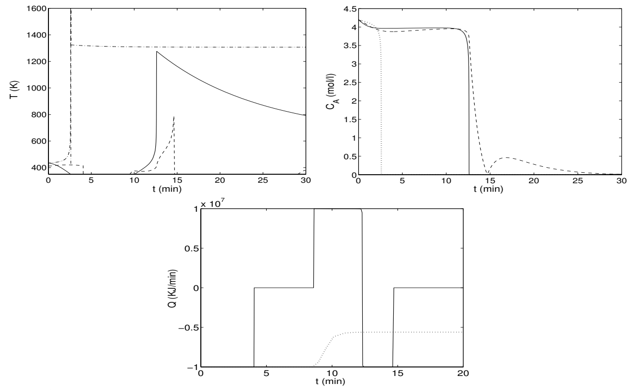

Fig. 6. Closed-loop state and input profiles: the solid lines show the case when the reactor is initialized within Ω_1^* and operated in mode 1 for all times with $L = 0.1$ (solid). The dotted (state) and dash-dotted (state estimates) lines shows the case with $L = 100.0$ when the reactor switches to mode 2 at $t = 0.002\ min := T_3$.

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VI. Appendix

Proof of Theorem 3:

Step 1: Consider the switched closed-loop system where $x(0) \in \Omega_{b,i}$, $\|\tilde{y}(0)\| \leq \delta_{\zeta,i}$ and $\epsilon_i \in (0,\ \epsilon_i^*]$, for some $i \in \mathcal{I}$. Then it follows from the result of Proposition 1 that the Lyapunov function for this mode, V_i^c, decays monotonically, along the trajectories of the closed-loop system, for as long as mode i is to remain active, i.e., for all times such that $\sigma(t) = i$. Consider now the scenario where at some given time T, we have $\hat{x}(T) \in \Omega_{s,j}$ and the system switches from mode i to mode j. Since we set $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_n \leq \min\{\epsilon_i^*\}$, $i = 1, \cdots, N$, we have $\epsilon_j \leq \epsilon_j^*$. Since the closed–loop system has been evolving in mode i for a time greater than or equal to $T_b^{max} > T_{b,i}$ (from Eq.14), it follows from Proposition 1 that $\|x(T) - \hat{x}(T)\| \leq e_{m,i}$. This, together with the fact that $\hat{x}(T) \in \Omega_{s,j}$, implies (using Proposition 2) that $x(T) \in \Omega_{b,j}$. Also, once mode j is switched in, the switching rule of Eq.16 sets $\|\tilde{y}(T)\| \leq \delta_{\zeta,j}$. All the conditions in Proposition 1 are, therefore, satisfied ($x(T) \in \Omega_{b,j}$, $\|\tilde{y}(T)\| \leq \delta_{\zeta,j}$ and $\epsilon_j \leq \epsilon_j^*$), which implies that the corresponding Lyapunov function for this mode, V_j^c, will also decay monotonically for $t > T$, and as long as we keep $\sigma(t) = j$. In this manner, we have that for all $j_k \in \mathcal{I}$, $k \in Z_+$:

$$\dot{V}^c_{\sigma(t_{j_k})} \quad < 0 \qquad \forall \ t \in [t_{j_k}, t_{j'_k}) \tag{18}$$

where t_{j_k} and $t_{j'_k}$ refer, respectively, to the times that the j-th mode is switched in and out for the k-th time, by the supervisor.

Step 2: The first part of Eq.14 ensures that the system has stayed in mode i for at least a time $T_b^{max} \geq T_{b,i}$ before switching to mode j. It follows from Proposition 1, therefore, that $\|x(T) - \hat{x}(T)\| \leq e_{m,i}$. From the continuity of the function $V^c(\cdot)$, we get that for a given $e_{m,i}$, there exists a positive real number $M_i(e_{m,i})$ such that if $\|x_f - \hat{x}_f\| \leq e_{m,i}$, $|V_i^c(x_f) - V_i^c(\hat{x}_f)| \leq M_i$. Therefore, we can write, for $x_f(t_{i'_{k-1}})$ and $x_f(t_{i'_k})$:

$$V_i^c(\hat{x}_f(t_{i'_{k-1}})) - M_i \leq V_i^c(x_f(t_{i'_{k-1}})) \tag{19}$$

$$V_i^c(\hat{x}_f(t_{i'_k})) + M_i \geq V_i^c(x_f(t_{i'_k})) \tag{20}$$

From Eq.15, we have for any admissible switching time $T = t_{i'_k}$

$$V_i^c(\hat{x}_f(t_{i'_{k-1}})) - M_i > V_i^c(\hat{x}_f(t_{i'_k})) + M_i \tag{21}$$

which, together with Eqs.19-20, implies

$$V_i^c(x_f(t_{i'_k})) < V_i^c(x_f(t_{i'_{k-1}})) \tag{22}$$

Using Eqs.18-22, one can finally show, with calculations similar to those used in the proof of Theorem 2 in [7], that the origin of the switched closed-loop system, under the switching laws of Eqs.14-15, is asymptotically stable.