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Stable Adaptive Model Predictive Control for Nonlinear Systems

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Abstract— **This research investigates the development of a stable adaptive model predictive control approach for a constrained nonlinear system. The method is well- known as multistep Newton-type control strategies however, the formulation here differs from the original one. The nonlinear physical equations of the system are extracted considering all possible effective forces. The nonlinear model is adaptively linearized during prediction procedure. The linearization not only takes place at each sampling instant of the control system, but also at each instant of the prediction horizon. The first step of this research is devoted to develop linearized models in the operating points, which are unknown and desired. Developing the equations to form a linear quadratic objective function with constraints is then carried out. Finally, the stability of the control system is provided using terminal equality constraints. To show the effectiveness of the proposed method, it is applied on a constrained highly nonlinear aerodynamic test bed, twin rotor MIMO system (TRMS).**

I. INTRODUCTION

MODEL predictive control (MPC) was introduced in the late seventies and early eighties [1-2] and has considerably been developed since then. It is a class of control approach that uses an explicit model of the plant and tries to calculate the manipulated variables through an optimization method. Linear MPC has become popular with the publication of some papers on model predictive heuristic control [1] and dynamics matrix control (DMC) [2].

In recent years MPC, as one of the modern computer optimization control techniques, has achieved a significant level of acceptability and great development in control theory and applications. Despite most of the practical processes being nonlinear, the majority of the MPC techniques implemented on the industrial processes are based on linear models. One of the main reasons for this is that a linear model is easy and fast to develop compared to a nonlinear one. Another reason refers to the stability, and more generally robustness problem that is really difficult to be provided in a nonlinear case. Some of the nonlinear models and/or constraints lead to non-convex nonlinear optimization problems that are relatively complex to solve. Last but not least, in some cases a linear model provides satisfactory results. Due to the mentioned difficulties related to nonlinear model predictive control the application of this method in the practical situations is still very limited but its potential is really great [3].

Note that in the case of a severe nonlinear system a single linear model cannot provide acceptable results in all operating regions. In other words, a highly nonlinear system cannot be linearly modelled to be adequate in all operating regions, unless the process always works in the neighbourhood of the point of interest. In the case of the TRMS a linear model predictive control is insufficient to obtain satisfactory performance in all operating regions. As mentioned before depending on the degrees of nonlinearity of a system it is sometimes possible to find a linear model to be valid in some specific operating points, but generally for a highly nonlinear system such as the TRMS, it is hardly possible to find a linear model to be adequate in all operating regions. Note that a nonlinear plant can be modelled using multiple linear modelling [4-5] or adaptive linear modelling [6-7] approaches. In the case of multiple-modelling approach the operating region of a nonlinear system is divided into several sub-regions and for each of sub-regions a linear model is developed. Therefore, according to the current operating point of the system the appropriate linear model is used to predict the output of the process. Note that in relatively high order systems with multiple inputs, it is a cumbersome task to find these linear models to cover all operating regions. Also, multiple-model MPC uses a linear model during the prediction horizon that cannot be adequate in highly nonlinear systems with large prediction horizon.

On the other hand, adaptive linear modelling method updates the linear model according to measurement data or linearization of a nonlinear plant model. For example, adaptive linear model predictive control has been presented to update linear model online based on measurement data to handle model uncertainties [6]. Zhang et al. [7] have proposed a method that uses pseudo-partial derivative to dynamically linearize a nonlinear system at each step of predictive functional control in order to have the benefits of linear quadratic optimization methods. Li and Biegler [8] have proposed multistep Newton-type control strategies for constrained nonlinear processes in which the nonlinear model is linearized around a nominal trajectory and solve a quadratic problem over the horizon. The extension of the mentioned approach can be found in [9] that put the performance index into augmented form and performed some modifications such as extending the output prediction horizon to infinity.

Note that the closed loop stability of a generic MPC with finite-horizon cost function cannot be guaranteed [10] and therefore further action should be carried out, e.g. modifying the cost function and/or constraints. Terminal equality constraints have been widely proposed to stabilize linear/nonlinear and discrete/continuous model predictive control systems [11-14]. Rawlings et al. [15] have proposed an infinite horizon controller to guarantee the stability of both stable and unstable linear plants. A quasi-infinite horizon scheme has been proposed to stabilize stable and unstable nonlinear model predictive controllers with input constraints [16]. Scokaert et al. [17] have investigated

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conditions under which suboptimal model predictive controllers are stabilizing. Robustness properties of nonlinear receding horizon controller with terminal equality constraints have been investigated with respect to gain and additive perturbations [18]. Michalska et al. [19] have proposed a robust dual mode, receding horizon controller for a wide class of nonlinear systems with state and control constraints and model error. A complete survey on both linear and nonlinear MPC with focusing on sufficient conditions to guarantee stability and robustness can be found in [20]. Comprehensive review on stability and robustness of nonlinear MPC can be found in [21].

In this work it is assumed that all state variables of the system are accessible, and therefore state feedback MPC is taken into consideration. However, an output feedback nonlinear MPC can be obtained using a combination of state feedback nonlinear MPC and a state observer [22].

In this investigation a multistep Newton-type MPC is used to control a constrained nonlinear MIMO system, TRMS. As mentioned before, the idea has been originated from the work of Li and Biegler [8]. The main difference between the present research and the original one is related to the way of formulation. In the original one, the objective function variables have been considered the difference between the current and the nominal input trajectories however, here they are input changes during the control horizon as it is more common [23].

II. Adaptive MPC Method

Conventional adaptive MPC is a linear MPC method that uses a nonlinear model to update the linear model only at each sample time, k. This method has no satisfactory performance for a severe nonlinear system, since the method uses only a single linear model at each iteration and then updates it in the new step according to the current operating point. Specially, if the prediction horizon is assumed to be large, the error between the linearized model and nonlinear model gradually increases as approaches to the end of the prediction horizon. This research is originated from the work of Li and Biegler [8] in which the linear model is updated during the prediction horizon as well. The main problem for updating the linear model during the prediction horizon is that, the operating points during the prediction horizon are unknown and the linearization can be carried out only on the basis of known operating points. In this situation one approach is using all the control efforts from previous optimization to linearize during the prediction horizon.

A. Methodology

A nonlinear system with n_u inputs, n_y outputs and n_x states can be adaptively linearized at each *real* sample time, k, as the following discrete state space equations:

$$\begin{cases}\mathbf{x}(k+1)=f\left(\mathbf{x}(k),\mathbf{u}(k)\right)\\ \mathbf{y}(k)=g\left(\mathbf{x}(k)\right)\end{cases}\Rightarrow\begin{cases}\mathbf{x}(k+1)=\mathbf{A}(k)\,\mathbf{x}(k)+\mathbf{B}(k)\,\mathbf{u}(k)\\ \mathbf{y}(k)=\mathbf{C}(k)\,\mathbf{x}(k)\end{cases}\tag{1}$$

where,

$\mathbf{x}(k)$: State vector at instant k

$\mathbf{u}(k)$: Input vector at instant k

$\mathbf{y}(k)$: Output vector at instant k

Note that the state variables and inputs related to the previous instant are used as initial conditions to linearize the nonlinear system at each time. Now we need to linearize the nonlinear system N_P times at each sampling instant according to the previous optimization results.

$$\begin{cases}\mathbf{x}(k+1)=f\left(\mathbf{x}(k),\mathbf{u}(k)\right)\\ \mathbf{y}(k)=g\left(\mathbf{x}(k)\right)\end{cases}\Rightarrow$$

$$\begin{cases}\hat{\mathbf{x}}(k+i+1|k)=\mathbf{A}(k+i|k)\,\hat{\mathbf{x}}(k+i|k)+\mathbf{B}(k+i|k)\,\hat{\mathbf{u}}(k+i|k)\\ \hat{\mathbf{y}}(k+i|k)=\mathbf{C}(k+i|k)\,\hat{\mathbf{x}}(k+i|k)\end{cases}$$

$$i=0,1,\ldots,N_P-1\tag{2}$$

In order to solve the optimization problem of an MPC, one needs to obtain the relationship between the internal model outputs during the prediction horizon interval, $1\le i\le N_P$, and the internal model inputs during the control horizon interval, $0\le i\le N_C-1$, where N_P and N_C are the prediction and control horizons, respectively, see Fig. 1. If this relationship is linear and the constraints are also linear then the optimization problem is a linear quadratic problem.

Assume that all state variables of the system are available. The model state variables in the prediction horizon interval with respect to the current state variables and future inputs can be expressed as,

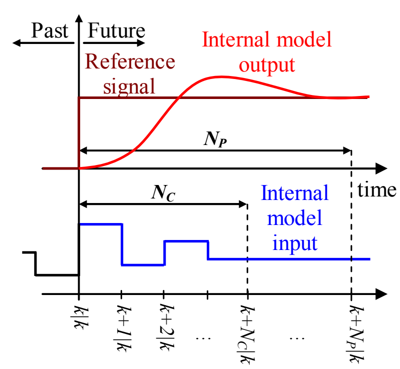

Fig. 1. The MPC approach of a single-input-single-output plant

$$\hat{\mathbf{x}}(k+1|k)=\mathbf{A}(k|k)\mathbf{x}(k)+\mathbf{B}(k|k)\hat{\mathbf{u}}(k|k)\tag{3}$$

$$\begin{aligned}\hat{\mathbf{x}}(k+2|k)&=\mathbf{A}(k+1|k)\hat{\mathbf{x}}(k+1|k)+\mathbf{B}(k+1|k)\hat{\mathbf{u}}(k+1|k)\\ &=\mathbf{A}(k+1|k)\mathbf{A}(k|k)\mathbf{x}(k)+\mathbf{A}(k+1|k)\mathbf{B}(k|k)\hat{\mathbf{u}}(k|k)\\ &\quad+\mathbf{B}(k+1|k)\hat{\mathbf{u}}(k+1|k)\end{aligned}\tag{4}$$

$$\vdots$$

$$\begin{aligned}\hat{\mathbf{x}}(k+N_P|k)&=\mathbf{A}(k+N_P-1|k)\hat{\mathbf{x}}(k+N_P-1|k)\\ &\quad+\mathbf{B}(k+N_P-1|k)\hat{\mathbf{u}}(k+N_P-1|k)\\ &=\mathbf{A}(k+N_P-1|k)\cdots\mathbf{A}(k+1|k)\mathbf{A}(k|k)\mathbf{x}(k)\\ &\quad+\mathbf{A}(k+N_P-1|k)\cdots\mathbf{A}(k+1|k)\mathbf{B}(k|k)\hat{\mathbf{u}}(k|k)+\cdots\\ &\quad+\mathbf{A}(k+N_P-1|k)\mathbf{B}(k+N_P-2|k)\hat{\mathbf{u}}(k+N_P-2|k)\\ &\quad+\mathbf{B}(k+N_P-1|k)\hat{\mathbf{u}}(k+N_P-1|k)\end{aligned}\tag{5}$$

It is common to use the change of input, $\Delta\hat{\mathbf{u}}(k+i|k)$, instead of input itself, $\hat{\mathbf{u}}(k+i|k)$, where $\Delta\hat{\mathbf{u}}(k+i|k) = \hat{\mathbf{u}}(k+i|k) - \hat{\mathbf{u}}(k+i-1|k)$ [23]. Note that the inputs only change during the control horizon interval and remain constant after that, i.e. $\hat{\mathbf{u}}(k+i|k) = \hat{\mathbf{u}}(k+N_C-1|k)$ or $\Delta\hat{\mathbf{u}}(k+i|k) = 0$ for $N_C \le i \le N_P - 1$. The relationship between inputs and changes of inputs are as follows,

$$\hat{\mathbf{u}}(k+j|k) = \mathbf{u}(k-1) + \sum_{i=0}^{j} \Delta\hat{\mathbf{u}}(k+i|k) \qquad j = 0,1,\ldots,N_C-1 \tag{6}$$

By substituting equation (6) into equations (3) to (5) the following equation can be written,

$$\begin{bmatrix} \hat{\mathbf{x}}(k+1|k) \\ \hat{\mathbf{x}}(k+2|k) \\ \vdots \\ \hat{\mathbf{x}}(k+N_C|k) \\ \vdots \\ \hat{\mathbf{x}}(k+N_P|k) \end{bmatrix} = \begin{bmatrix} \mathbf{A}(k|k) \\ \mathbf{A}(k+1|k)\mathbf{A}(k|k) \\ \vdots \\ \prod_{i=1}^{N_C} \mathbf{A}(k+N_C-i|k) \\ \vdots \\ \prod_{i=1}^{N_P} \mathbf{A}(k+N_P-i|k) \end{bmatrix} \mathbf{x}(k)$$

$$+ \begin{bmatrix} \mathbf{M}_{1,1}^T(k) & \mathbf{M}_{2,1}^T(k) & \cdots & \mathbf{M}_{N_C,1}^T(k) & \cdots & \mathbf{M}_{N_P,1}^T(k) \end{bmatrix}^T \mathbf{u}(k-1)$$

$$+ \begin{bmatrix} \mathbf{M}_{1,1}(k) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{M}_{2,1}(k) & \mathbf{M}_{2,2}(k) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{N_C,1}(k) & \mathbf{M}_{N_C,2}(k) & \cdots & \mathbf{M}_{N_C,N_C}(k) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{N_P,1}(k) & \mathbf{M}_{N_P,2}(k) & \cdots & \mathbf{M}_{N_P,N_C}(k) \end{bmatrix} \begin{bmatrix} \Delta\hat{\mathbf{u}}(k|k) \\ \Delta\hat{\mathbf{u}}(k+1|k) \\ \vdots \\ \Delta\hat{\mathbf{u}}(k+N_C-1|k) \end{bmatrix} \tag{7}$$

where,

$\mathbf{M}_{1,1}(k) = \mathbf{B}(k|k)$, $\mathbf{M}_{2,1}(k) = \mathbf{A}(k+1|k)\mathbf{B}(k|k) + \mathbf{B}(k+1|k)$

$\mathbf{M}_{2,2}(k) = \mathbf{B}(k+1|k)$

$$\mathbf{M}_{N_C,1}(k) = \sum_{j=0}^{N_C-2} \left(\prod_{i=1}^{N_C-1-j} \mathbf{A}(k+N_C-i|k) \right) \mathbf{B}(k+j|k) + \mathbf{B}(k+N_C-1|k)$$

$$\mathbf{M}_{N_C,N_C}(k) = \mathbf{B}(k+N_C-1|k)$$

$$\mathbf{M}_{N_P,1}(k) = \sum_{j=0}^{N_P-2} \left(\prod_{i=1}^{N_P-1-j} \mathbf{A}(k+N_P-i|k) \right) \mathbf{B}(k+j|k) + \mathbf{B}(k+N_P-1|k)$$

$$\mathbf{M}_{N_P,2}(k) = \sum_{j=1}^{N_P-2} \left(\prod_{i=1}^{N_P-1-j} \mathbf{A}(k+N_P-i|k) \right) \mathbf{B}(k+j|k) + \mathbf{B}(k+N_P-1|k)$$

$$\mathbf{M}_{N_P,N_C}(k) = \sum_{j=N_C-1}^{N_P-2} \left(\prod_{i=1}^{N_P-1-j} \mathbf{A}(k+N_P-i|k) \right) \mathbf{B}(k+j|k) + \mathbf{B}(k+N_P-1|k)$$

Note that in equation (7) the first two terms are related to the past variables that are known and the last term is associated with the future signals that should be optimally calculated using an optimization technique. The output predictions can be obtained as,

$$\begin{bmatrix} \hat{\mathbf{y}}(k+1|k) \\ \hat{\mathbf{y}}(k+2|k) \\ \vdots \\ \hat{\mathbf{y}}(k+N_P|k) \end{bmatrix} = \begin{bmatrix} \mathbf{C}(k+1|k) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{C}(k+2|k) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{C}(k+N_P|k) \end{bmatrix}$$

$$\times \begin{bmatrix} \hat{\mathbf{x}}(k+1|k) \\ \hat{\mathbf{x}}(k+2|k) \\ \vdots \\ \hat{\mathbf{x}}(k+N_P|k) \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{d}}(k+1|k) \\ \hat{\mathbf{d}}(k+2|k) \\ \vdots \\ \hat{\mathbf{d}}(k+N_P|k) \end{bmatrix} \tag{8}$$

where $\hat{\mathbf{d}}(k+i|k)$ is the disturbance that can be considered as constant value for all i or can be estimated as the difference between real and estimated output. Substitution (7) into (8) leads to following equation,

$$\begin{aligned} \mathbf{Y}(k) = \mathbf{M_C}(k)\mathbf{M_A}(k)\mathbf{x}(k) + \mathbf{M_C}(k)\mathbf{M_B}(k)\mathbf{u}(k-1) \\ + \mathbf{M_C}(k)\mathbf{M_U}(k)\Delta\mathbf{U}(k) + \mathbf{M_d}(k) \end{aligned} \tag{9}$$

where,

$$\mathbf{Y}(k) = \begin{bmatrix} \hat{\mathbf{y}}^T(k+1|k) & \hat{\mathbf{y}}^T(k+2|k) & \cdots & \hat{\mathbf{y}}^T(k+N_P|k) \end{bmatrix}^T$$

$$\mathbf{M_C}(k) = \begin{bmatrix} \mathbf{C}(k+1|k) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{C}(k+2|k) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{C}(k+N_P|k) \end{bmatrix}$$

$$\mathbf{M_A}(k) = \begin{bmatrix} \mathbf{A}(k|k) \\ \mathbf{A}(k+1|k)\mathbf{A}(k|k) \\ \vdots \\ \prod_{i=1}^{N_P} \mathbf{A}(k+N_P-i|k) \end{bmatrix}, \quad \mathbf{M_B}(k) = \begin{bmatrix} \mathbf{M}_{1,1}(k) \\ \mathbf{M}_{2,1}(k) \\ \vdots \\ \mathbf{M}_{N_P,1}(k) \end{bmatrix}$$

$$\mathbf{M_U}(k) = \begin{bmatrix} \mathbf{M}_{1,1}(k) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{M}_{2,1}(k) & \mathbf{M}_{2,2}(k) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{N_P,1}(k) & \mathbf{M}_{N_P,2}(k) & \cdots & \mathbf{M}_{N_P,N_C}(k) \end{bmatrix}$$

$$\Delta\mathbf{U}(k) = \begin{bmatrix} \Delta\hat{\mathbf{u}}^T(k|k) & \Delta\hat{\mathbf{u}}^T(k+1|k) & \cdots & \Delta\hat{\mathbf{u}}^T(k+N_C-1|k) \end{bmatrix}^T$$

$$\mathbf{M_d}(k) = \begin{bmatrix} \hat{\mathbf{d}}^T(k+1|k) & \hat{\mathbf{d}}^T(k+2|k) & & \hat{\mathbf{d}}^T(k+N_P|k) \end{bmatrix}^T$$

B. Objective Function

Assume that the following objective function should be minimized according to the mentioned constraints,

$$\begin{aligned} J(k) = \sum_{i=1}^{N_P} \left[\mathbf{r}(k+i) - \hat{\mathbf{y}}(k+i|k) \right]^T \boldsymbol{\delta}(i) \left[\mathbf{r}(k+i) - \hat{\mathbf{y}}(k+i|k) \right] + \\ \sum_{i=1}^{N_C} \left[\Delta\hat{\mathbf{u}}(k+i-1|k) \right]^T \boldsymbol{\lambda}(i) \left[\Delta\hat{\mathbf{u}}(k+i-1|k) \right] \end{aligned} \tag{10}$$

$$\mathbf{y}_{\min} \le \hat{\mathbf{y}}(k+i|k) \le \mathbf{y}_{\max} \qquad i = 1,2,\ldots,N_P \tag{11}$$

$$\mathbf{u}_{\min} \le \hat{\mathbf{u}}(k+i-1|k) \le \mathbf{u}_{\max} \qquad i = 1,2,\ldots,N_C \tag{12}$$

$$\Delta\mathbf{u}_{\min} \le \Delta\hat{\mathbf{u}}(k+i-1|k) \le \Delta\mathbf{u}_{\max} \quad i = 1,2,\ldots,N_C \tag{13}$$

where,

$\mathbf{r}$: Reference trajectory

$\boldsymbol{\delta}$: Weighting matrix of tracking error

$\boldsymbol{\lambda}$: Weighting matrix of control effort

The indices min and max highlight the lower and upper bounds respectively.

The objective function can be rewritten as follows,

$$J(k) = \left[\mathbf{M_r}(k) - \mathbf{Y}(k) \right]^T \mathbf{Q} \left[\mathbf{M_r}(k) - \mathbf{Y}(k) \right] + \Delta\mathbf{U}^T(k)\,\mathbf{R}\,\Delta\mathbf{U}(k) \tag{14}$$

where, $\mathbf{M_r} = \begin{bmatrix} \mathbf{r}^T(k+1) & \mathbf{r}^T(k+2) & \cdots & \mathbf{r}^T(k+N_P) \end{bmatrix}^T$

$$\mathbf{Q} = \text{blockdiag}\begin{bmatrix} \boldsymbol{\delta}(1) & \boldsymbol{\delta}(2) & \cdots & \boldsymbol{\delta}(N_P) \end{bmatrix}$$

$$\mathbf{R} = \text{blockdiag}\begin{bmatrix}\boldsymbol{\lambda}(1) & \boldsymbol{\lambda}(2) & \cdots & \boldsymbol{\lambda}(N_C)\end{bmatrix}$$

By substituting equation (9) into (14) the following linear quadratic function is obtained,

$$J(k) = \tfrac{1}{2}\Delta\mathbf{U}^T(k)\mathbf{H}(k)\Delta\mathbf{U}(k) + \Delta\mathbf{U}^T(k)\mathbf{G}(k) + c(k) \quad (15)$$

where, $\mathbf{H}(k) = 2\left(\mathbf{M}_{\mathbf{U}}^T(k)\mathbf{M}_{\mathbf{C}}^T(k)\mathbf{Q}\mathbf{M}_{\mathbf{C}}(k)\mathbf{M}_{\mathbf{U}}(k) + \mathbf{R}\right)$

$$\mathbf{G}(k) = -2\mathbf{M}_{\mathbf{U}}^T(k)\mathbf{M}_{\mathbf{C}}^T(k)\mathbf{Q}\mathbf{E}(k)$$

$$c(k) = \mathbf{E}^T(k)\mathbf{Q}\mathbf{E}(k)$$

$$\mathbf{E}(k) = \mathbf{M}_{\mathbf{r}}(k) - \mathbf{M}_{\mathbf{C}}(k)\mathbf{M}_{\mathbf{A}}(k)\mathbf{x}(k) - \mathbf{M}_{\mathbf{C}}(k)\mathbf{M}_{\mathbf{B}}(k)\mathbf{u}(k-1) - \mathbf{M}_{\mathbf{d}}(k)$$

Note that all the constraints should be transferred in the standard form of $\boldsymbol{\Lambda}(k)\Delta\mathbf{U}(k) \le \mathbf{b}(k)$.

C. Stability

Terminal equality constraints method is a way to provide stability [11]. The aim is to add a set of terminal state constraints to force the state variables to take particular equilibrium values at the end of prediction horizon. The stability can be proved using Lyapunov function even in a general case [11].

As the proposed adaptive MPC is based on discrete model, a continuous model can be approximately transferred into a discrete counterpart using,

$$d\mathbf{x}/dt = f_c\left(\mathbf{x}(t), \mathbf{u}(t)\right) \quad \Rightarrow$$

$$\mathbf{x}(k+1) = \mathbf{x}(k) + T_s f_c\left(\mathbf{x}(k), \mathbf{u}(k)\right) = f\left(\mathbf{x}(k), \mathbf{u}(k)\right) \quad (16)$$

where T_s is the sampling time, and f_c and f are the right hand sides of continuous and discrete nonlinear state space equations, respectively. Note that in an equilibrium point $(\mathbf{x}_0, \mathbf{u}_0)$ the following equations can be written in the case of continuous and discrete, respectively,

$$f_c\left(\mathbf{x}_0, \mathbf{u}_0\right) = \mathbf{0} \quad (17)$$

$$f\left(\mathbf{x}_0, \mathbf{u}_0\right) = \mathbf{x}_0 \quad (18)$$

As mentioned before, a set of following terminal equality constraints is added to the inequality constrained objective function to guarantee the stability,

$$\hat{\mathbf{x}}(k + N_P | k) = \mathbf{x}_0(k) \quad (19)$$

where $\mathbf{x}_0$ is an equilibrium point that satisfies the reference signals as well. The equality state constraints in (19), can be transferred into the equality control signals using (7) as follows,

$$\mathbf{A}_{eq}(k)\Delta\mathbf{U}(k) = \mathbf{B}_{eq}(k) \quad (20)$$

where,

$$\mathbf{A}_{eq}(k) = \begin{bmatrix}\mathbf{M}_{N_P,1}(k) & \mathbf{M}_{N_P,2}(k) & \cdots & \mathbf{M}_{N_P,N_C}(k)\end{bmatrix} \quad (21)$$

$$\mathbf{B}_{eq}(k) = \mathbf{x}_0(k) - \prod_{i=1}^{N_P}\mathbf{A}(k + N_P - i\,|k)\mathbf{x}(k) + \mathbf{M}_{N_P,1}(k)\mathbf{u}(k-1) \quad (22)$$

III. TRMS

This work is focused on an aerodynamic test rig, twin rotor multiple-input-multiple-output system (TRMS), shown in Fig. 2. The TRMS is a laboratory platform designed for control experiments by Feedback Instruments Ltd [24]. It is a highly nonlinear system which can be considered as an experimental model of a complex air vehicle. The control objective is to make the beam of the TRMS move quickly and accurately to the desired positions, i.e., the pitch and the yaw angles. Developing controller for this type of system is challenging due to the coupling effects between two axes and also due to its highly nonlinear characteristics. In the case of the TRMS, state variables, inputs and outputs are as follows:

$$\mathbf{x}(k) = \begin{bmatrix}\omega_h(k) & S_h(k) & \alpha_h(k) & \omega_v(k) & S_v(k) & \alpha_v(k)\end{bmatrix}^T \quad (23)$$

$$\mathbf{u}(k) = \begin{bmatrix}U_h(k) & U_v(k)\end{bmatrix}^T \quad (24)$$

$$\mathbf{y}(k) = \begin{bmatrix}\alpha_h(k) & \alpha_v(k)\end{bmatrix}^T \quad (25)$$

where,

ω_h : Rotational speed of the tail rotor

S_h : Angular velocity of TRMS beam in the horizontal plane without the effect of the main rotor [rad/s]

α_h : Horizontal position of the TRMS beam

ω_v : Rotational speed of the main rotor

S_v : Angular velocity of TRMS beam in the vertical plane without the effect of the tail rotor [rad/s]

α_v : Vertical position of TRMS beam

U_h : Input voltage signal of the tail motor

U_v : Input voltage signal of the main motor

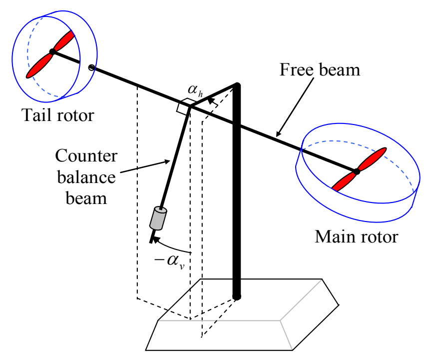

Fig. 2. The twin rotor MIMO system

The nonlinear continuous state space equations of the TRMS can be summarized as,

$$\frac{d}{dt}\begin{bmatrix}\omega_h \\ S_h \\ \alpha_h \\ \omega_v \\ S_v \\ \alpha_v\end{bmatrix} = \begin{bmatrix} -\dfrac{(k_{ah}\varphi_h)^2}{J_{tr}R_{ah}}\omega_h - \dfrac{B_{tr}}{J_{tr}}\omega_h - \dfrac{k_{thp/n}}{J_{tr}}|\omega_h|\omega_h + \dfrac{k_{ah}\varphi_h}{J_{tr}R_{ah}}f_1(U_h) \\ \dfrac{l_t\, k_{fhp/n}|\omega_h|\,\omega_h\cos\alpha_v - f_2(\Omega_h) - f_3(\alpha_h)}{D\cos^2\alpha_v + E\sin^2\alpha_v + F} \\ S_h + \dfrac{k_m\omega_v\cos\alpha_v}{D\cos^2\alpha_v + E\sin^2\alpha_v + F} \\ -\dfrac{(k_{av}\varphi_v)^2}{J_{mr}R_{av}}\omega_v - \dfrac{B_{mr}}{J_{mr}}\omega_v - \dfrac{k_{tvp/n}}{J_{mr}}|\omega_v|\omega_v + \dfrac{k_{av}\varphi_v}{J_{mr}R_{av}}f_4(U_v) \\ f_{Sv} \\ S_v + \dfrac{k_t}{J_v}\omega_h \end{bmatrix} \quad (26)$$

where,

$$f_{Sv} = k_{fvp/n}|\omega_v|\omega_v\left(l_m + k_g\Omega_h\cos\alpha_v\right) - f_5(\Omega_v)\big/J_v + g\left[(A-B)\cos\alpha_v - C\sin\alpha_v\right] - 0.5\Omega_h^2 H\sin 2\alpha_v\big/J_v$$

and R_{ah}, $k_{ah}\varphi_h$, J_{tr}, B_{tr}, $k_{thp/n}$, l_t, $k_{fhp/n}$, D, E, F, k_m,

R_{av}, $k_{av}\varphi_v$, J_{mr}, B_{mr}, $k_{tvp/n}$, $k_{fvp/n}$, l_m, k_g, g, A, B, C, H, J_v, and k_t are positive constant values, and angular velocities of the TRMS beam in horizontal and vertical planes, Ω_h and Ω_v, are defined as,

$$\Omega_h = S_h + \frac{k_m \omega_v \cos\alpha_v}{D\cos^2\alpha_v + E\sin^2\alpha_v + F} \tag{27}$$

$$\Omega_v = S_v + k_t\,\omega_h / J_v \tag{28}$$

also $f_1(U_h)$, $f_2(\Omega_h)$, $f_3(\alpha_h)$, $f_4(U_v)$, and $f_5(\Omega_v)$ are nonlinear functions. For more details on TRMS see [25].

In the case of the TRMS $\mathbf{x}_0(k)$ is defined as,

$$\mathbf{x}_0(k) = [\omega_{h0} \quad S_{h0} \quad \alpha_{h0} \quad \omega_{v0} \quad S_{v0} \quad \alpha_{v0}]^T \tag{29}$$

where each element is found according to the current reference signals, $\alpha_{href}(k)$ and $\alpha_{vref}(k)$, and equations (17) and (26),

$$\alpha_{h0} = \alpha_{href}(k), \quad \alpha_{v0} = \alpha_{vref}(k) \tag{30}$$

$$\omega_{h0} = \begin{cases} \sqrt{\left|f_3(\alpha_{h0}) / (k_{fhp} l_t \cos\alpha_{v0})\right|} & \text{for} \quad \alpha_{h0} \ge 0 \\ -\sqrt{\left|f_3(\alpha_{h0}) / (k_{fhn} l_t \cos\alpha_{v0})\right|} & \text{for} \quad \alpha_{h0} < 0 \end{cases} \tag{31}$$

$$S_{h0} = -\frac{k_m \omega_{v0} \cos\alpha_{v0}}{D\cos^2\alpha_{v0} + E\sin^2\alpha_{v0} + F} \tag{32}$$

$$\omega_{v0} = \begin{cases} \sqrt{\left|F_{v0} / k_{fvp}\right|} & \text{for} \quad F_{v0} \ge 0 \\ -\sqrt{\left|F_{v0} / k_{fvn}\right|} & \text{for} \quad F_{v0} < 0 \end{cases} \tag{33}$$

where,

$$F_{v0} = -g\left[(A-B)\cos\alpha_{v0} - C\sin\alpha_{v0}\right] / l_m$$
$$S_{v0} = -k_t \omega_{h0} / J_v \tag{34}$$

IV. Results

Based on the nonlinear model, N_P linear models have been developed to model the nonlinear system during the prediction horizon at each instant, k. The objective function and constraints have been formed based on these linear models to have a constrained linear quadratic optimization problem. A set of terminal equality state constraints is formed to force the state to an equilibrium point at the end of prediction horizon in order to guarantee the stability of the closed loop system. At each iteration the first set of optimum input vector, $\Delta\hat{\mathbf{u}}(k\,|\,k)$, is added to the previous control signal, $\mathbf{u}(k-1)$, and the result is sent to the plant and also the linear model at $k\,|\,k$ instant. The others optimum values are kept for the next sample time, $k+1$, calculations. The block diagram of the adaptive MPC is shown in Fig. 3. Note that the nonlinear model and linearization operator have not been shown in Fig. 3. Although it is assumed that all state variables are measurable in this research, a state observer is shown in Fig. 3 for the case of output feedback MPC methods. It is noted that the proposed adaptive MPC has been able to produce very fast and precise response to various reference signals for highly nonlinear systems. As mentioned before, the plant has 2 inputs, 2 outputs and 6 states. Note that the sampling time of the model predictive controller is set to be 0.2 seconds and the optimization approach is chosen to be an active set method. The adaptive MPC developed for the TRMS has been tested with a variety of reference signals and the results obtained demonstrates that the controller has a high performance and reliability in the various operating regions. The controller has been proven to be reliable under disturbances and various reference signals used. Figs. 4 and 5 show the results with square references as horizontal (yaw) and vertical (pitch) angles of the beam, respectively. It is clear from Figs. 4 and 5 that two channels have significant effect on each other. For instance, at time 25 seconds the yaw angle reference signal has been changed from 0.6 to -0.6 rad and subsequently, yaw angle has followed the reference signal. On the other hand the pitch angle, due to the mentioned coupling, has been affected by yaw angle however the controller has regained the control and forced the pitch angle to follow the reference trajectory. The overshoots at instants 50 and 150 seconds of yaw angle response have been caused by the step change of pitch angle reference signal at those instants. Figs. 6 and 7 illustrate the responses of the controller according to another reference signals.

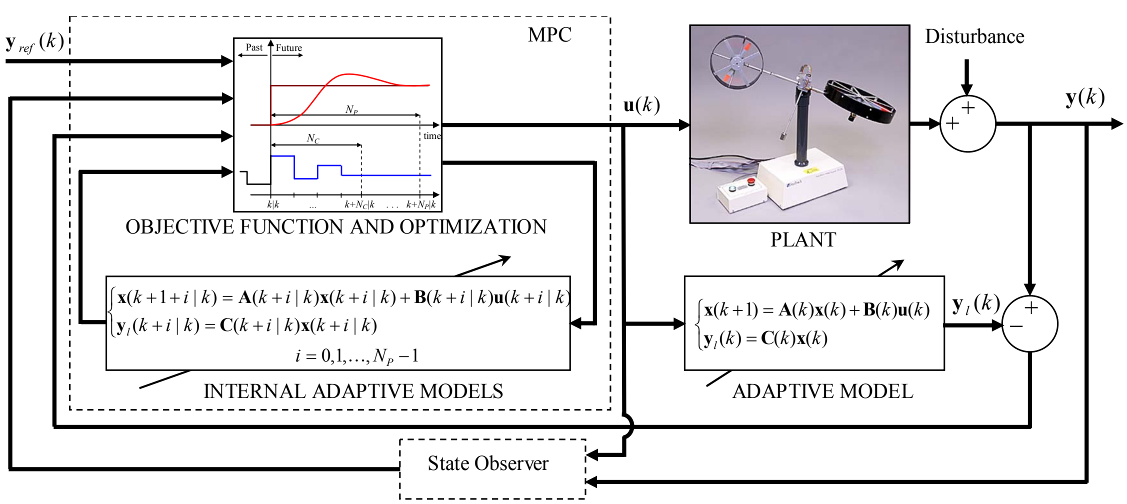

Fig. 3. Block diagram of the proposed adaptive MPC approach

V. CONCLUSION

In this investigation an efficient adaptive MPC has been presented for a highly nonlinear system, TRMS. The proposed method has used a nonlinear model to adaptively find a set of N_P linear models at each instant. These linear models have been utilized to form a linear quadratic objective function according to MPC approach. Terminal equality constraints have been imposed to the problem to guarantee the closed loop stability. A TRMS has been selected as a test bed to validate the control technique. The results of the controller have proved the effectiveness and reliability of the control system in following the trajectories of the yaw and pitch angles of the beam.

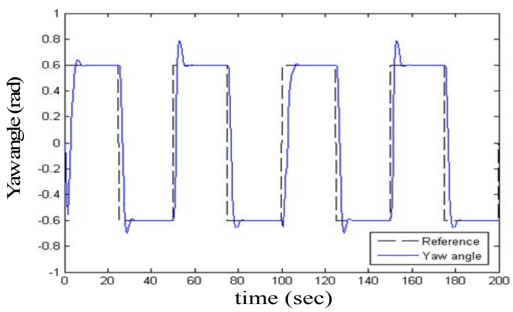

Fig. 4. Square wave response of the horizontal angle (case 1)

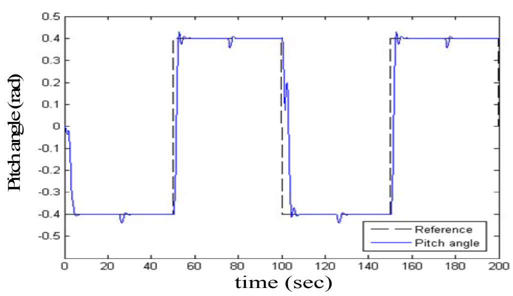

Fig. 5. Square wave response of the vertical angle (case 1)

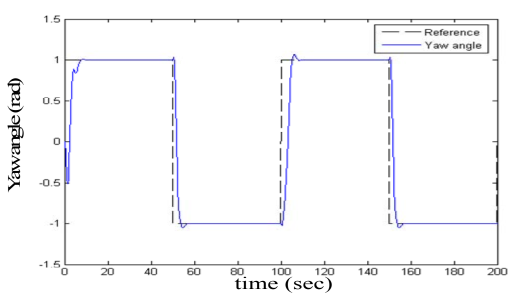

Fig. 6. Square wave response of the horizontal angle (case 2)

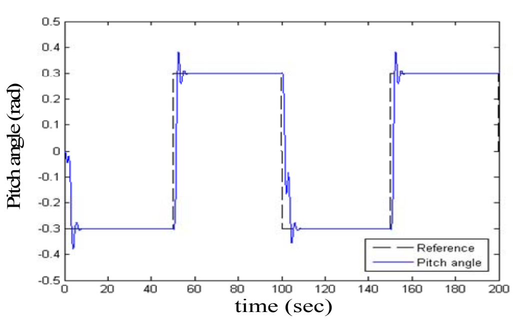

Fig. 7. Square wave response of the vertical angle (case 2)

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