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# Frequency Domain Multiloop Control Synthesis for Unstable Systems: An Approach Based on $\mu$ Interaction Measure

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*Abstract*— **This paper presents a new practical framework for multiloop controller design in which controllers are designed independently, i.e., a controller in one loop is designed without exploiting information of controllers used in other loops. The method is based on the (block) diagonal approximation of a system that is different from its (block) diagonal elements. The focus of this work is on unstable systems and the approximated systems are obtained by minimizing a scaled $\mathcal{L}_\infty$ norm for the error systems. This extends the applicability of conventional $\mu$-interaction measure to a more general scenario. The validity of the proposed approach is demonstrated through numerical simulation and application to an industrial boiler system.**

## I. Introduction

In the chemical process industry and large scale systems (power systems, vehicle platooning), it is common to avoid centralized architectures in favor of simple decentralized or block decentralized controllers [1], [2], [3], [4]. Although, it is not always possible to respect this desideratum, the reasons for this preference include: ease of understanding by control engineers, tuning of fewer parameters than the multivariable controllers and loop failure tolerance of the resulting control system, to mention just a few. The design technique of a decentralized control system usually involves two steps [5]: 1) choice of suitable pairings (control structure selection) and 2) design of single input single output (SISO) or block decentralized controllers. So far in process industries, the relative gain array (RGA) [6] and Rijnsdrop's interaction measure [7] are found to be efficient tools for eliminating undesirable pairings in step 1. For the case of control design in step 2, all the past research efforts can be grouped into the following three categories [8], [9], [5], [10], [11], [12]

- Simultaneous design using parametric search methods: In this approach, the controller is assumed to have a fixed structure (like state space form, PID, PI, etc.) with unknown parameters to be designed. By using direct or indirect search methods, these parameters are then obtained by minimizing the appropriate norm of the closed loop system. A shortcoming of this approach is its relative complexity and, in some cases, the non-convexity of the resulting optimization problem.
- Sequential loop closing: In this case, the method proceeds by designing each element of the controller (or block) sequentially. The controller corresponding to a fast loop (inner loop) or lowest level is designed first and the loop is closed. Then, the controller corresponding to the next loop is designed based on this partially closed loop system. Due to its simplicity, this method is now being widely used by industries. However, when the lower level loops fail, the failure tolerance of the remaining loops cannot be guaranteed.
- Independent design: In this method, the control design is based on the (block) diagonal approximation that is usually done by taking the (block) diagonal elements of the system. The controller in each loop is designed by specifying the form of each closed loop (CL) transfer function, resulting in an IMC-PID type controller. If the interaction is less than a certain bound, this method can maintain stability of the overall closed loop system. Since the information of the controllers in other loops are not used, it is conservative but the nominal stability of the remaining loops is guaranteed if *any* loop fails.

In this paper, the work is focussed on the independent design approach. Though simpler than other design techniques, one practically important question may always attract the attention of many engineers: "How much is the performance deterioration of the overall closed loop system caused by *ignoring* the off-diagonal system blocks, and what should be the upper bound on the interactions such that overall closed loop stability can be maintained?" To answer these questions, a number of interaction measures are available in the literature, which indicates under what conditions the stability of (block) diagonal loops guarantees that of the overall closed loop system. In addition to predicting closed-loop stability, they also measure the performance loss caused by a given control structure [1].

Among the different interaction measures available so far, namely, Rijnsdrop's interaction measure [7], Relative gain array [6], $\mu$-interaction measure ($\mu$-IM) [13], [1] and Direct Nyquist array [14], the $\mu$-IM is noteworthy. This is because, it offers a dynamic measure of interactions, and is also applicable to high order systems. Its equal applicability to block pairings and other elegant properties have attracted the attention of many researchers in the field of decentralized control [13], [5], [15], [16], [11], [17], [12]. Based on $\mu$-IM, the method proposed by [13] utilizes an independent design approach. The controllers are designed independently of each other based on the diagonal approximation that is usually taken as the diagonal elements of the system. Sufficient conditions for the synthesis are provided under which they can maintain nominal stability of the overall closed loop system. This approach, however, suffers from the shortcoming that it requires that the system and its diagonal part have the same number of open right half plane (RHP)

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poles. Since this condition is not satisfied by most of the systems in practice, it limits the applicability of this approach to *only* open loop stable systems.

In spite of these restrictions, [5] generalized the results of [13] by providing simple bounds which when satisfied, guarantee robust performance of the overall system. Pairing rules for unstable plants, based on $\mu-$IM are then introduced in [18] and their relationship with RGA and Niederlinski index (NI) are explicated. In [11], phase stability conditions are presented which claimed to remove some conservativeness associated with the $\mu$-IM (since it constrains only the magnitude of each SISO loop). An independent robust decentralized control design approach for unstable as well as non-square systems was carried out in [16] and [15], respectively. Many other ideas also came into picture and subsequently improved, some were really interesting and innovative, but finding a (block) diagonal approximated system that posses the same number of unstable poles as the system itself still remains an open question. In [17], [12], a step towards solving this approximation problem for unstable systems was carried out. It presents a numerical approach, where the decentralized controller is designed based on an approximated system that is different from (block) diagonal elements, but has the same number of unstable poles of the system. The method is interesting because it extended the applicability of $\mu$-IM to unstable systems and the outcome of numerical example given in the paper is also good. However, the algorithm bears some complexity and includes approximations, iterations in the frequency-wise approximation step as well as in the parametric identification step, which have no guarantee of convergence. This open problem, which remains almost unsolved during the last twenty years, constitutes the motivation of development in this paper.

In this paper, the authors utilize Smith-McMillan decompositions, properties of norms, congruence transformations and reciprocal variant of the projection lemma to provide an easily understandable and programmable approach of obtaining (block) diagonal approximated systems. By using a constant scaling matrix ($D_r$), they convert the design algorithm into an optimization problem, which can be directly solved by the available numerical software. There is no trial and error or approximations involved and, in some cases, the optimization problem involves linear matrix inequalities (LMIs) and *only* one semidefinite constraint. This quasi-convex optimization can be readily solved using *YALMIP* (Yet Another LMI Parser) [19], which is a parser, namely, the interface between different solvers (including LMILab) and Matrix Laboratory (Matlab). An upper bound on the closed loop performance due to the decentralized architecture is also derived and special attention is paid on the effect of non-minimum phase transmission zeros. It is well known that in multiloop control systems, because of the inherent interactions between different loops, closing the loop around one subsystem moves the transmission zeros of other subsystems across the imaginary axis [6], [20]. These zero crossings occur despite of the existence of non-minimum phase behavior in the open loop system, and it represents performance limitations specific to the use of decentralized controllers [20]. They are responsible for sensitivity peak as well as bandwidth limitation of the resulting closed loop system. To overcome this problem, some conditions are developed, such that these zero crossings can be prevented. Thus the algorithm has the dual feature of being also related to intrinsic system properties.

The validity of the proposed approach is demonstrated through numerical simulation and through applications to an industrial utility boiler. Throughout this work, a nonlinear simulation package of Syncrude integrated energy system called SYNSIM is used and the effectiveness of the proposed design strategy is demonstrated through simulations in SYNSIM under different perturbed conditions.

## II. A Solution to (Block) Diagonal Approximation and Controller Design

The $\mathcal{H}_\infty$ norm of a stable transfer matrix $\mathbf{G}(s)$ is defined by $\|\mathbf{G}(s)\|_\infty = \sup_{\omega\in\mathbb{R}} \bar{\sigma}[\mathbf{G}(j\omega)]$ and $\mathcal{L}_\infty$ norm is similar to the $\mathcal{H}_\infty$ norm, except that $\mathbf{G}(s)$ can be unstable. In this section, an algorithm is developed, which finds (block) diagonal approximation $\tilde{\mathbf{G}}(s)$ for a given unstable system $\mathbf{G}(s)$ by minimizing the following scaled $\mathcal{L}_\infty$ distance between the system and its approximation

$$\min_{\tilde{\mathbf{G}}(j\omega)} \bar{\sigma}[\mathbf{D}_r(\mathbf{G}(j\omega) - \tilde{\mathbf{G}}(j\omega))\mathbf{D}_r^{-1}],\ \forall\omega\in\mathbb{R} \tag{1}$$

where the number of unstable poles in $\tilde{\mathbf{G}}(s)$ and $\mathbf{G}(s)$ are the same. Clearly, as $\mathbf{G}(s)$ contains both stable and unstable poles, achieving an optimal solution to this problem is a very difficult task. However, an acceptable solution can be obtained by proceeding in the following way:

1. Separate the stable and antistable part of $\mathbf{G}(s)$ by

$$\begin{aligned}\mathbf{G}(s) = \mathbf{G}_1(s) + \mathbf{G}_2(s) \quad &= \quad \mathbf{L}^{-1}(s)\left[\mathbf{G}^{sm}\right]_s \mathbf{R}^{-1}(s) + \\ &\quad \mathbf{L}^{-1}(s)\left[\mathbf{G}^{sm}\right]_{as} \mathbf{R}^{-1}(s),\end{aligned}$$

where $\mathbf{G}^{sm}(s)$ is the Smith-McMillan form [21] of $\mathbf{G}(s)$ and $\mathbf{L}(s)$, $\mathbf{R}(s)$ are unimodular matrices. $\left[\mathbf{G}^{sm}\right]_s$ and $\left[\mathbf{G}^{sm}\right]_{as}$ contain stable and unstable poles of $\mathbf{G}(s)$, respectively.

2. Now, $\mathbf{G}_2^T(-s)$, $\mathbf{G}_1(s) \in \mathcal{RH}_\infty$ (both have poles on the left half of the s-plane, i.e., stable). Without loss of generality, $\tilde{\mathbf{G}}(s)$ can be parameterized as $\tilde{\mathbf{G}}(s) = \tilde{\mathbf{G}}_1(s) + \tilde{\mathbf{G}}_2(s)$, which gives

$$\begin{aligned}\|\mathbf{D}_r[\mathbf{G}(s) - \tilde{\mathbf{G}}(s)]\mathbf{D}_r^{-1}\|_{\mathcal{L}_\infty} \le (\|\mathbf{D}_r[\mathbf{G}_1(s) - \tilde{\mathbf{G}}_1(s)] \\ \times\mathbf{D}_r^{-1}\|_\infty) + (\|\mathbf{D}_r[\mathbf{G}_2^T(-s) - \tilde{\mathbf{G}}_2^T(-s)]\mathbf{D}_r^{-1}\|_\infty),\end{aligned} \tag{2}$$

since $\|\mathbf{D}_r[\mathbf{G}_2(s) - \tilde{\mathbf{G}}_2(s)]D_r^{-1}\|_{\mathcal{L}_\infty} = \|\mathbf{D}_r[\mathbf{G}_2^T(-s) - \tilde{\mathbf{G}}_2^T(-s)]\mathbf{D}_r^{-1}\|_\infty$ and $\tilde{\mathbf{G}}_2(s)$ is the $\mathcal{L}_\infty$ optimal approximation of $\mathbf{G}_2(s)$ with $n_p$ (say) unstable poles.

3. Solve the following optimization problem

$$\min\ \gamma$$

$$\text{subject to}\ \ \mathbf{X}_P > 0, \tag{3}$$

$$\begin{bmatrix} -\mathbf{Q} & \mathbf{A}_{cl}^T + \mathbf{M}^T & \mathbf{0} & \mathbf{C}_{cl,h}^T & \sqrt{2}\mathbf{X}_P & \mathbf{M}^T \\ \star & -\mathbf{I} & \mathbf{B}_{cl} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\gamma\mathbf{H} & \mathbf{D}_{cl,h}^T & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -\gamma\mathbf{H} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & -\mathbf{I} & \mathbf{0} \\ \star & \star & \star & \star & \star & -\mathbf{I} \end{bmatrix} < \mathbf{0}, \tag{4}$$

$$\begin{bmatrix} -\gamma\mathbf{H} & \mathbf{D}_{cl,h}^T \\ \star & -\gamma\mathbf{H} \end{bmatrix} < \mathbf{0}, \begin{bmatrix} \mathbf{Q} & \mathbf{X}_P + \mathbf{M}^T \\ \star & \mathbf{I} \end{bmatrix} \ge \mathbf{0}, \tag{5}$$

where

$$\begin{aligned}
\mathbf{A}_{cl} &= \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{A}_d & \mathbf{B}_d \\ \mathbf{C}_d & \mathbf{D}_d \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\
\mathbf{B}_{cl} &= \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{A}_d & \mathbf{B}_d \\ \mathbf{C}_d & \mathbf{D}_d \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \\
\mathbf{C}_{cl} &= \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_d & \mathbf{B}_d \\ \mathbf{C}_d & \mathbf{D}_d \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\
\mathbf{D}_{cl} &= \mathbf{D} + \begin{bmatrix} \mathbf{0} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_d & \mathbf{B}_d \\ \mathbf{C}_d & \mathbf{D}_d \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \\
\mathbf{C}_{cl,h}^T &= \mathbf{C}_{cl}^T\mathbf{H}, \;\; \mathbf{D}_{cl,h}^T = \mathbf{D}_{cl}^T\mathbf{H}, \;\; \mathbf{H} = \mathbf{D}_r^T\mathbf{D}_r.
\end{aligned}$$

Here, $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is the minimal state space realization of $\mathbf{G}_1(s)$ and $(\mathbf{A}_d, \mathbf{B}_d, \mathbf{C}_d, \mathbf{D}_d)$ is the state space realization of $\tilde{\mathbf{G}}_1(s)$. This optimization problem, which minimizes the scaled $\mathcal{H}_\infty$ norm of the error system should also be solved to obtain structured $\tilde{\mathbf{G}}_2^T(-s)$ from $\mathbf{G}_2^T(-s)$ .

**Proof:** Please see Appendix.

Since, $\mathbf{H}$ is invertible, $\mathbf{C}_d$ and $\mathbf{D}_d$ can be obtained from the decision variables $\mathbf{C}_{cl,h}$ and $\mathbf{D}_{cl,h}$, respectively. Hence, the terms are now affine in the design parameters $(\mathbf{A}_d, \mathbf{B}_d, \mathbf{C}_d, \mathbf{D}_d)$ and can be easily computed. However, in (4) and (5) there is one bilinear term $\gamma\mathbf{H}$. Therefore, by fixing $\gamma$ to a small value or by selecting $\mathbf{D}_r$ equal to identity matrix, the conditions are LMIs with only one semidefinite constraint in (5). As $\mathbf{Q} = (\mathbf{X}_P + \mathbf{M})^T(\mathbf{X}_P + \mathbf{M})$ corresponds to the boundary of the convex set in (5) or (19), the solution of the optimization problem in (3), (4) and (5) using the parser YALMIP or the Cone Complimentary Linearization (CCL) approach [22] always yields $\mathbf{Q} = (\mathbf{X}_P + \mathbf{M})^T(\mathbf{X}_P + \mathbf{M})$ (or $\mathbf{Q} = \mathbf{M}^T\mathbf{M}$ in (21)).

### *A. Controller Design*

The algorithm of finding a (block) decentralized controller $\mathbf{K}(s)$ to satisfy the $\mu$-IM condition [13], [1], $\bar{\sigma}(\tilde{\mathbf{H}}(j\omega)) < \mu^{-1}(\mathbf{E}(j\omega)), \;\; \forall\omega$ can be reduced to solving a skewed-$\mu$ problem. Here, $\tilde{\mathbf{H}}(s) = \tilde{\mathbf{G}}\mathbf{K}(s)(\mathbf{I} + \tilde{\mathbf{G}}\mathbf{K}(s))^{-1}$ and $\mathbf{E}(s) = (\mathbf{G}(s) - \tilde{\mathbf{G}}(s)\tilde{\mathbf{G}}^{-1}(s)$ represents the relative error. Assume $\tilde{\mathbf{H}}(s)$ is stable, and that $\mathbf{G}(s)$ and $\tilde{\mathbf{G}}(s)$ have the same number of unstable poles. Then the closed loop system $\mathbf{H}(s)$ is stable (all loops are closed) if

$$\bar{\sigma}\left(\tfrac{1}{c_H}\tilde{\mathbf{H}}(j\omega)\right) \leq 1, \;\; \forall\omega \in \mathbb{R} \tag{6}$$

where at each frequency $c_H$ solves $\mu_{\hat{\mathbf{\Delta}}}\begin{bmatrix} \mathbf{0} & \mathbf{E}(j\omega) \\ c_H\mathbf{I} & \mathbf{0} \end{bmatrix} = 1$. Here, $\mu$ is computed w.r.t the structure $\hat{\mathbf{\Delta}} = \text{diag}(\tilde{\mathbf{H}}(j\omega), \tilde{\mathbf{H}}(j\omega))$. It should be noted that (6) is just a restatement of the $\mu$-IM condition [13] and can be easily derived by using properties of singular values in [5], [21].

In the following proposition, it is shown that when the $\mu$-IM condition is satisfied, an upper bound on the closed loop performance is always minimized.

*Proposition 2.1:* Assume that $\mathbf{G}(s)$ and $\tilde{\mathbf{G}}(s)$ have the same number of RHP poles, and $\mu$-IM holds. Then,

$$\bar{\sigma}(\mathbf{H}(j\omega)) \leq \frac{\kappa(\mathbf{D}(\omega))\;\bar{\sigma}(\mathbf{G}\tilde{\mathbf{G}}^{-1}(j\omega))}{\bar{\sigma}^{-1}(\tilde{\mathbf{H}}(j\omega)) - \mu(\mathbf{E}(j\omega))}, \forall\omega \in \mathbb{R}$$

where $\kappa(\mathbf{D})$ is the euclidean condition number and $\mathbf{D}(\omega)$ is the frequency dependent scaling matrix.

**Proof**: It is clear that

$$\begin{aligned}
&(\mathbf{I} + \mathbf{G}\mathbf{K}(s))\mathbf{K}^{-1}\tilde{\mathbf{G}}^{-1}(s) = (\mathbf{I} + \tilde{\mathbf{G}}\mathbf{K}(s))\mathbf{K}^{-1}\tilde{\mathbf{G}}^{-1}(s) \\
&+(\mathbf{G}(s) - \tilde{\mathbf{G}}(s))\tilde{\mathbf{G}}^{-1}(s) = \tilde{\mathbf{S}}^{-1}\mathbf{K}^{-1}\tilde{\mathbf{G}}^{-1}(s) + \mathbf{E}(s),
\end{aligned}$$

where $\tilde{\mathbf{S}}(s) = (\mathbf{I} + \tilde{\mathbf{G}}\mathbf{K}(s))^{-1}$ is the sensitivity function of the approximated system $\tilde{\mathbf{G}}(s)$. Pre- and post- multiplying by $\mathbf{D}(\omega)$ and $\mathbf{D}^{-1}(\omega)$, respectively and using the properties of the singular values [21], [12]

$$\begin{aligned}
&\underline{\sigma}\left(\mathbf{D}(\omega)\mathbf{S}^{-1}\mathbf{K}^{-1}\tilde{\mathbf{G}}^{-1}(j\omega)\mathbf{D}^{-1}(\omega)\right) \geq \underline{\sigma}(\mathbf{D}(\omega) \\
&\times\tilde{\mathbf{H}}^{-1}(j\omega)\mathbf{D}^{-1}(\omega)) - \overline{\sigma}\left(\mathbf{D}(\omega)\mathbf{E}(j\omega)\mathbf{D}^{-1}(\omega)\right), \quad (7)
\end{aligned}$$

where $\mathbf{S}(s) = (\mathbf{I} + \mathbf{G}\mathbf{K}(s))^{-1}$ is the sensitivity function of the overall closed loop system and $\tilde{\mathbf{H}}^{-1}(j\omega) = \tilde{\mathbf{S}}^{-1}\mathbf{K}^{-1}\tilde{\mathbf{G}}^{-1}(j\omega)$. Now,

$$\begin{aligned}
&\mathbf{D}(\omega)\mathbf{S}^{-1}\mathbf{K}^{-1}\tilde{\mathbf{G}}^{-1}(j\omega)\mathbf{D}^{-1}(\omega) = \mathbf{D}(\omega)\mathbf{S}^{-1}\mathbf{K}^{-1}\mathbf{G}^{-1} \\
&\times\mathbf{G}\tilde{\mathbf{G}}^{-1}(j\omega)\mathbf{D}^{-1}(\omega) = \mathbf{D}(\omega)\mathbf{H}^{-1}(j\omega)(\mathbf{I}+\mathbf{E}(j\omega))\mathbf{D}^{-1}(\omega),
\end{aligned}$$

where $\mathbf{H}(s)$ is the closed loop transfer function and

$$\begin{aligned}
&\underline{\sigma}[\mathbf{D}(\omega)\mathbf{H}^{-1}(j\omega)(\mathbf{I} + \mathbf{E}(j\omega))\mathbf{D}^{-1}(\omega)] \\
\leq\;& \overline{\sigma}(\mathbf{D}(\omega))\underline{\sigma}(\mathbf{H}^{-1}(j\omega))\overline{\sigma}(\mathbf{I} + \mathbf{E}(j\omega))\overline{\sigma}(\mathbf{D}^{-1}(\omega)) \\
=\;& \kappa(\mathbf{D}(\omega))\overline{\sigma}(\mathbf{I} + \mathbf{E}(j\omega))\underline{\sigma}(\mathbf{H}^{-1}(j\omega)). \qquad (8)
\end{aligned}$$

Here, $\kappa(\mathbf{D}(\omega)) = \overline{\sigma}(\mathbf{D}(\omega))\overline{\sigma}(\mathbf{D}^{-1}(\omega))$ is the euclidean condition number. Since, $\mathbf{D}(\omega)\tilde{\mathbf{H}}^{-1}(j\omega)\mathbf{D}^{-1}(\omega) = \tilde{\mathbf{H}}^{-1}(j\omega)$, and suppose that $\mathbf{D}(\omega)$ is chosen to minimize $\overline{\sigma}\left(\mathbf{D}(\omega)\mathbf{E}(j\omega)\mathbf{D}^{-1}(j\omega)\right)$, then from (7) and (8)

$$\overline{\sigma}(\mathbf{H}(j\omega)) \leq \frac{\kappa(\mathbf{D}(\omega))\overline{\sigma}(\mathbf{I} + \mathbf{E}(j\omega))}{\overline{\sigma}^{-1}(\tilde{\mathbf{H}}(j\omega)) - \mu(\mathbf{E}(j\omega))}. \quad \blacksquare$$

Hence, when interactions are not large and the decentralized controller stabilizes the closed loop system satisfying (6) then an upper bound on closed loop performance is always minimized. Generally speaking, this bound is very loose. However, the maximization of $\bar{\sigma}^{-1}(\tilde{\mathbf{H}}(j\omega)) - \mu(\mathbf{E}(j\omega))$ provides an advantage of maximizing the robustness of the closed loop system against unmodelled dynamics represented by an output multiplicative uncertainty. Similar condition in terms of control sensitivity function was also derived in [12].

### *B. Performance Limitations due to RHP Zero Crossings*

In the following, a method of multiloop controller design is presented, which prevents the movement of the transmission zeros of open loop subsystems across the imaginary axis (when some other loops are closed). The main concern is to find an upper bound for the interactions, such that zero crossings can be prevented. For sake of brevity, a stable open loop system is considered.

*Theorem 2.1:* Assume that $\mathbf{G}(s)$ is divided into two blocks and the subsystems $\mathbf{G}_{11}(s)$ and $\mathbf{G}_{22}(s)$ are minimum phase. When the first loop is closed with a (block) decentralized controller $\mathbf{K}_1(s)$, such that $\tilde{\mathbf{H}}_1(s) = \mathbf{G}_{11}\mathbf{K}_1(s)(\mathbf{I} + \mathbf{G}_{11}\mathbf{K}_1(s))^{-1}$ is stable, then the transmission zeros of the remaining subsystem $\mathbf{G}_{22}(s)$ will not cross the imaginary axis, if

$$\bar{\sigma}(\tilde{\mathbf{H}}_1(j\omega)) < \mu^{-1}\left[\mathbf{G}_{12}\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{G}_{11}^{-1}(j\omega)\right], \;\; \forall\omega \in \mathbb{R}$$

**Proof**: Consider the square multivariable system

$$\mathbf{y}_1 = \mathbf{G}_{11}(s)\mathbf{u}_1 + \mathbf{G}_{12}(s)\mathbf{u}_2,\ \mathbf{y}_2 = \mathbf{G}_{21}(s)\mathbf{u}_1 + \mathbf{G}_{22}(s)\mathbf{u}_2.$$

When a negative feedback $\mathbf{u}_1 = -\mathbf{K}_1(s)\mathbf{y}_1$ is applied around the subsystem $\mathbf{G}_{11}(s)$, then the remaining subsystem $\hat{\mathbf{G}}_{22}(s)$ with inputs $\mathbf{u}_2$ and outputs $\mathbf{y}_2$ is given by $\hat{\mathbf{G}}_{22}(s) = \mathbf{G}_{22}(s) - \mathbf{G}_{21}(s)(\mathbf{I}+\mathbf{G}_{11}\mathbf{K}_1(s))^{-1}\mathbf{K}_1\mathbf{G}_{12}(s) = \mathbf{G}_{22}(s)\left[\mathbf{I} - \mathbf{G}_{22}^{-1}\mathbf{G}_{21}(s)(\mathbf{I}+\mathbf{G}_{11}\mathbf{K}_1(s))^{-1}\mathbf{K}_1\mathbf{G}_{12}(s)\right]$.

Now, $\det(\hat{\mathbf{G}}_{22}(s))$

$$\begin{aligned} &= \det(\mathbf{G}_{22}(s))\det\left[\mathbf{I} - \mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{G}_{11}^{-1}\mathbf{G}_{11}(s)\right. \\ &\quad \left.\times\mathbf{K}_1(s)(\mathbf{I}+\mathbf{G}_{11}\mathbf{K}_1(s))^{-1}\mathbf{G}_{12}(s)\right] \\ &= \det(\mathbf{G}_{22}(s))\det\left[\mathbf{I} - \mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{G}_{11}^{-1}\tilde{\mathbf{H}}_1\mathbf{G}_{12}(s)\right] \\ &= \det(\mathbf{G}_{22}(s))\det\left[\mathbf{I} - \tilde{\mathbf{H}}_1\mathbf{G}_{12}\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{G}_{11}^{-1}(s)\right]. \end{aligned}$$

Therefore, if the overall system is stable, then the zeros of the second subsystem will not cross the imaginary axis if and only if the nyquist plot of $\det\left[\mathbf{I} - \tilde{\mathbf{H}}_1\mathbf{G}_{12}\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{G}_{11}^{-1}(j\omega)\right]$, $\forall\omega \in \mathbb{R}$ does not encircle the origin. Using the spectral radius stability condition [21], the zero crossing can be prevented if $\rho\left[\tilde{\mathbf{H}}_1\mathbf{G}_{12}\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{G}_{11}^{-1}(j\omega)\right] < 1,\ \forall\,\omega \in \mathbb{R}$. Since, $\tilde{\mathbf{H}}_1(j\omega)$ has a structure and $\rho[\tilde{\mathbf{H}}_1\mathbf{G}_{12}\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{G}_{11}^{-1}(j\omega)] \le \mu[\tilde{\mathbf{H}}_1\mathbf{G}_{12}\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{G}_{11}^{-1}(j\omega)]$, $\forall\omega \in \mathbb{R}$, the sufficient condition is given by

$$\bar{\sigma}(\tilde{\mathbf{H}}_1(j\omega)) < \mu^{-1}\left[\mathbf{G}_{12}\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{G}_{11}^{-1}(j\omega)\right],\ \forall\omega \in \mathbb{R}. \quad (9)$$

This derivation utilizes the fact that $\mu_{\mathbf{\Delta}}(\mathbf{AB}) \le \mu_{\mathbf{\Delta}}(\mathbf{A})\bar{\sigma}(\mathbf{B})$ and $\mu$ is computed w.r.t. the structure of $\tilde{\mathbf{H}}_1(j\omega)$. ■

*Remark 2.1:* For a 2×2 system with scalar loops, *Theorem 2.1* boils down to $\bar{\sigma}(\tilde{h}_1(j\omega)) < \mu^{-2}(\mathbf{E}(j\omega)),\ \forall\omega \in \mathbb{R}$. In general, for controllers designed independently

$$\begin{aligned} \bar{\sigma}(\tilde{\mathbf{H}}_1(j\omega)) &< \min\left(\mu^{-1}(\mathbf{E}(j\omega)),\right. \\ &\quad \left.\mu^{-1}\left[\mathbf{G}_{12}\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{G}_{11}^{-1}(j\omega)\right]\right) \quad (10) \\ \text{and } \bar{\sigma}(\tilde{\mathbf{H}}_2(j\omega)) &< \mu^{-1}(\mathbf{E}(j\omega)), \forall\omega \in \mathbb{R} \quad (11) \end{aligned}$$

guarantees overall closed loop stability and also prevents the movement of transmission zeros in $\mathbf{G}_{22}(s)$ across the imaginary axis, when the first loop is closed. This has an important effect on the closed loop system performance. For scalar loops, (10) can be represented by $\bar{\sigma}(\tilde{h}_1(j\omega)) < \min\left[\mu^{-1}(\mathbf{E}(j\omega)), \mu^{-2}(\mathbf{E}(j\omega))\right],\ \forall\,\omega$. If interactions are large, then designing controller based on (10) and (11) leads to low frequency performance deterioration in the first channel. This is because, $\bar{\sigma}(\tilde{\mathbf{H}}_1(j\omega))$ has to be reduced at low frequencies ($\mu^{-1}\left[\mathbf{G}_{12}\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{G}_{11}^{-1}(j\omega)\right] < \mu^{-1}(\mathbf{E}(j\omega)) < 1$). For systems with integral action in all channels, the upper bound of the interaction is given by

$$\max\left(\mu(\mathbf{E}(0),\ \mu\left[\mathbf{G}_{12}\mathbf{G}_{22}^{-1}\mathbf{G}_{21}\mathbf{G}_{11}^{-1}(0)\right]\right) < 1,$$

since $\tilde{\mathbf{H}}_1(0) = \tilde{\mathbf{H}}_2(0) = \mathbf{I}$.

## III. Simulation Results

Consider the following system [12]

$$\mathbf{G}(s) = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 0.5 & 0.5 \\ 0 & 2 & 0 & 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 3 & 0 & 0.5 & 0.5 & 1 \\ 0 & 0 & 0 & -4 & 1 & 0.4 & 0.4 \\ \hline 1 & 0.1 & 0.1 & 1 & 0 & 0 & 0 \\ 0.1 & 1 & 0.1 & 0.6 & 0 & 0 & 0 \\ 0.1 & 0.1 & 1 & 0.6 & 0 & 0 & 0 \end{array}\right].$$

The system has unstable poles at 1, 2 and 3. Application of the (block) diagonal approximation algorithm gives

$$\tilde{\mathbf{G}}(s) = \left[\begin{array}{ccc} \frac{0.7923s-6.009}{s^2-7.179s-37.82} & 0 & 0 \\ 0 & \frac{0.6135s-0.05103}{s^2-1.402s-27.43} & 0 \\ 0 & 0 & \frac{0.7911s+0.9201}{s^2-0.8216s-24.77} \end{array}\right]$$

which has poles at 10.71, −3.53, 5.98, −4.58, 5.40 and −4.58, i.e., the same number of unstable poles as the original system $\mathbf{G}(s)$. The optimum $\gamma$ is 1.3. Next, consider the controller design. Since, $\mathbf{E}(s) = (\mathbf{G}(s) - \tilde{\mathbf{G}}(s))\tilde{\mathbf{G}}^{-1}(s)$ is improper, the algorithm is slightly modified to $\bar{\sigma}\left(\frac{1}{c_H}\tilde{\mathbf{R}}(j\omega)\right) \le 1,\ \forall\omega \in \mathbb{R}$, where at each frequency $c_H$ solves $\mu_{\hat{\mathbf{\Delta}}}\left[\begin{array}{cc} \mathbf{0} & (\mathbf{G}(j\omega) - \tilde{\mathbf{G}}(j\omega)) \\ c_H\mathbf{I} & \mathbf{0} \end{array}\right] = 1$. Here, $\tilde{\mathbf{R}}(j\omega) = \mathbf{K}(j\omega)(\mathbf{I}+\tilde{\mathbf{G}}\mathbf{K}(j\omega))^{-1}$ is the control sensitivity function and $\mu$ is computed w.r.t. the structure $\hat{\mathbf{\Delta}} = \text{diag}(\tilde{\mathbf{H}}(j\omega), \tilde{\mathbf{R}}(j\omega))$. Application of this algorithm gives $c_H = 0.8026$ and the decentralized controller

$$\mathbf{K}(s) = \text{diag}\left(\tfrac{123.2s+435}{s-72.64}, \tfrac{34.94s+160.1}{s-4.881}, \tfrac{20.78s+95.24}{s-1.046}\right).$$

Hence, the design procedure here is very straightforward. When the unstable poles in the LMI optimization are kept fixed, the approximated system is given by $\tilde{\mathbf{G}}(s) = \text{diag}\left(\tfrac{1.067s+0.9654}{s^2+2.531s-3.531}, \tfrac{0.9054s+2.366}{s^2+2.583s-9.167}, \tfrac{0.9953s+2.508}{s^2+1.583s-13.75}\right)$, with $\gamma = 0.9$. Application of the design algorithm to this system gives $c_H = 1.1233$ and the stabilizing controller

$$\mathbf{K}(s) = \text{diag}\left(\tfrac{4.459s+15.75}{s+0.7735}, \tfrac{6.304s+28.89}{s+2.876}, \tfrac{8.283s+37.96}{s+2.34}\right).$$

### A. Industrial Utility Boilers

Inputs to the developed nonlinear boiler model are feedwater flow, firing rate, spray flow and the outputs are: drum level, header pressure and steam temperature. The overall model is of order 11 and it has shown good fitness at all operating regions. The linearized model has one pole at the origin (associated with water dynamics), one RHP zero at 0.0619 and decentralized controller is designed based on the block diagonal approximation ($\gamma = 0.28$) as well as $\mu$-IM condition. For implementation, the controller is then discretized with a sampling period of 6 seconds and it has the form $\mathbf{K}(z) = \text{diag}\,[K_{11}(z), K_{22}(z), K_{33}(z)]$, where

$$K_{11}(z) = \tfrac{60.64z^3-41.41z^2-91.75z+72.96}{z^3-1.875z^2+0.8786z+7.242e-018},$$
$$K_{22}(z) = \tfrac{0.0003238z^3-0.0009514z^2+0.0009314-0.0003039}{z^3-2.97z^2+2.941z-0.9708},$$
$$K_{33}(z) = \tfrac{-0.02555z^3-0.006817z^2-0.01441z+2.1e-4}{z^3-0.008143z^2+0.001135z+1.598e-024}.$$

With this controller, the $\mu$-IM [12] can be satisfied at all frequencies (Fig. 1) and RHP zero crossings can be prevented (conditions (10) and (11) are satisfied). Figs 2-4 shows the response of the system under perturbed conditions. Fig. 2 represents the measurements during a sudden load change of 100 kpph in the 6.306 MPa steam header. Fig. 3 shows the corresponding inputs that are required to overcome the load variations. The interesting feature of Fig. 4 is that when

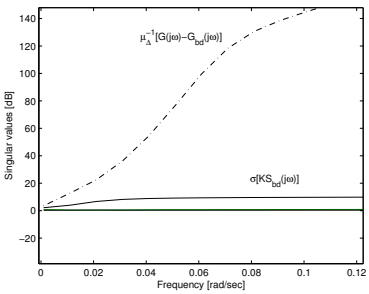


Fig. 1. Verification of $\bar{\sigma}(\mathbf{KS}_{bd}(j\omega)) < \mu_{\Delta}^{-1}(\mathbf{G}(j\omega) - \mathbf{G}_{bd}(j\omega))$

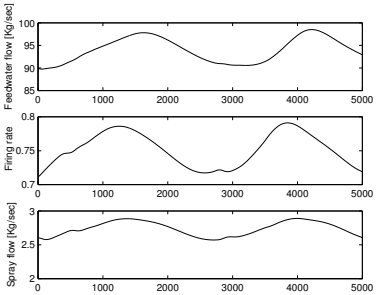


Fig. 3. Inputs during load change

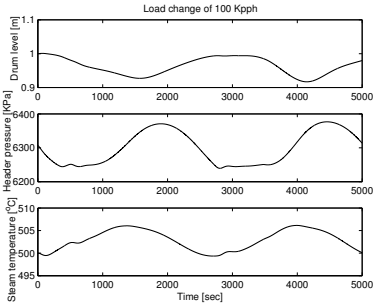


Fig. 2. Outputs during load change

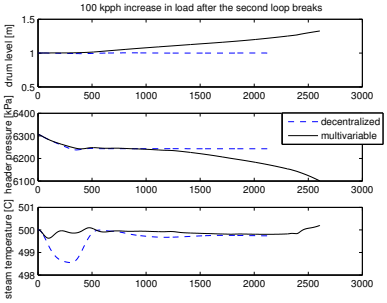


Fig. 4. System response during loop failure

the plant is controlled by a multivariable controller and if the firing rate controller fails, then overall system becomes unstable. However, the decentralized controller is capable of maintaining the stability. This is an important property of control by *independent designs*.

## IV. Conclusions

This paper extends the practical applicability of $\mu$-IM to unstable systems. Decentralized controllers are designed based on a (block) diagonal approximation that is different from the (block) diagonal elements, but has the same number of unstable poles as the system. It is shown that the (block) diagonal approximation can be obtained by solving a quasi-convex optimization problem. An upper bound of the closed loop performance due to decentralized architecture is derived and some sufficient conditions for the multiloop controller design are also presented, such that closing the loop around one subsystem does not move the transmission zeros of remaining subsystems across the imaginary axis. The validity of the proposed approach is demonstrated through application to an industrial boiler system.

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## VI. Appendix

**Proof of the block diagonal approximation**: It is straightforward to show that finding a structured $\tilde{\mathbf{G}}_1(s)$ such that $\bar{\sigma}[\mathbf{D}_r(\mathbf{G}_1(j\omega) - \tilde{\mathbf{G}}_1(j\omega))\mathbf{D}_r^{-1}] < \gamma$ is equivalent to solving the following optimization [21]

$$\mathbf{Y} > \mathbf{0}, \text{ and } \begin{bmatrix} \mathbf{Y}\mathbf{A}_{cl}^T + \mathbf{A}_{cl}\mathbf{Y} & \mathbf{B}_{cl} & \mathbf{Y}\mathbf{C}_{cl}^T\mathbf{H} \\ \star & -\gamma\mathbf{H} & \mathbf{D}_{cl}^T\mathbf{H} \\ \star & \star & -\gamma\mathbf{H} \end{bmatrix} < \mathbf{0},$$

where $\mathbf{H} = \mathbf{D}_r^T\mathbf{D}_r$. This can be expanded using the Schur's complement method [23] as

$$\begin{bmatrix} \mathbf{Y}\mathbf{A}_{cl}^T + \mathbf{A}_{cl}\mathbf{Y} + \frac{\mathbf{Y}\mathbf{C}_{cl}^T\mathbf{H}\mathbf{C}_{cl}\mathbf{Y}}{\gamma} & \mathbf{B}_{cl} + \frac{\mathbf{Y}\mathbf{C}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma} \\ \star & -\gamma\mathbf{H} + \frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma} \end{bmatrix} < \mathbf{0},$$

and further into

$$\mathbf{Y}\mathbf{A}_{cl}^T + \mathbf{A}_{cl}\mathbf{Y} + \frac{\mathbf{Y}\mathbf{C}_{cl}^T\mathbf{H}\mathbf{C}_{cl}\mathbf{Y}}{\gamma} + \left(\mathbf{B}_{cl} + \frac{\mathbf{Y}\mathbf{C}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\right) \times \left(\gamma\mathbf{H} - \frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\right)^{-1}\left(\mathbf{B}_{cl}^T + \frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{C}_{cl}\mathbf{Y}}{\gamma}\right) < \mathbf{0} \quad (12)$$

$$\left(-\gamma\mathbf{H} + \frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\right) < \mathbf{0}. \quad (13)$$

Application of the reciprocal projection lemma [24] to the inequality in (12) gives

$$\begin{bmatrix} \mathcal{F}_{11} + \mathbf{X} - (\mathbf{W} + \mathbf{W}^T) & \mathbf{Y}\mathbf{A}_{cl}^T + \mathcal{F}_{12} + \mathbf{W}^T \\ \star & -\mathbf{X} \end{bmatrix} < \mathbf{0}, \quad (14)$$

where $\mathbf{X}$ is any given positive-definite matrix, $\mathbf{W}$ is a decision variable, $\mathcal{F}_{12} = \frac{\mathbf{Y}\mathbf{C}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\left(\gamma\mathbf{H} - \frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\right)^{-1}\mathbf{B}_{cl}^T$ and $\mathcal{F}_{11} = \frac{\mathbf{Y}\mathbf{C}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\left(\gamma\mathbf{H} - \frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\right)^{-1}\frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{C}_{cl}\mathbf{Y}}{\gamma} + \frac{\mathbf{Y}\mathbf{C}_{cl}^T\mathbf{H}\mathbf{C}_{cl}\mathbf{Y}}{\gamma} + \mathbf{B}_{cl}\left(\gamma\mathbf{H} - \frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\right)^{-1}\mathbf{B}_{cl}^T$. Since, $\mathbf{X}$ can be any given positive definite matrix and $\mathbf{B}_{cl}\left(\gamma\mathbf{H} - \frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\right)^{-1}\mathbf{B}_{cl}^T$ is symmetric, in most of the cases, the parameters $(\mathbf{B}_d, \mathbf{D}_d)$, $\gamma$, $\mathbf{H}$ and $\mathbf{X}$ can be designed such that

$$\mathbf{X}+\mathbf{B}_{cl}\left(\gamma\mathbf{H}-\frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\right)^{-1}\mathbf{B}_{cl}^T=\mathbf{I}. \quad (15)$$

This selection is done to decouple the design variables from the positive definite matrix $\mathbf{Y}$. If, in some cases, this is not satisfied, then similar to [25], a large decision variable $\lambda$ can *always* be selected such that positive definiteness of

$$\mathbf{X}=\lambda\mathbf{I}-\mathbf{B}_{cl}\left(\gamma\mathbf{H}-\frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\right)^{-1}\mathbf{B}_{cl}^T \quad (16)$$

is guaranteed.

**Case I**: With the selection in (15), (14) is equivalent to

$$\begin{bmatrix} \mathbf{I}+\mathcal{Z}_{11}-(\mathbf{W}+\mathbf{W}^T) & \mathbf{Y}\mathbf{A}_{cl}^T+\mathcal{Z}_{12}+\mathbf{W}^T \\ \star & -\mathbf{I}+\mathbf{B}_{cl}\left(\gamma\mathbf{H}-\frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\right)^{-1}\mathbf{B}_{cl}^T \end{bmatrix}<\mathbf{0},$$

where $\mathcal{Z}_{11}=\frac{\mathbf{Y}\mathbf{C}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\left(\gamma\mathbf{H}-\frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\right)^{-1}\frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{C}_{cl}\mathbf{Y}}{\gamma}+\frac{\mathbf{Y}\mathbf{C}_{cl}^T\mathbf{H}\mathbf{C}_{cl}\mathbf{Y}}{\gamma}$, $\mathcal{Z}_{12}=\frac{\mathbf{Y}\mathbf{C}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\left(\gamma\mathbf{H}-\frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\right)^{-1}\mathbf{B}_{cl}^T$. Pre- and post multiplying by diag $(\mathbf{Y}^{-1},\mathbf{I})$ and diag $(\mathbf{Y}^{-1},\mathbf{I})$, respectively and expanding the above inequality

$$\begin{bmatrix} \mathbf{Y}^{-1}\mathbf{Y}^{-1}+\frac{\mathbf{C}_{cl}^T\mathbf{H}\mathbf{C}_{cl}}{\gamma}-\mathbf{Y}^{-1}\mathbf{M}-\mathbf{M}^T\mathbf{Y}^{-1} & \mathbf{A}_{cl}^T+\mathbf{M}^T \\ \star & -\mathbf{I} \end{bmatrix}-$$

$$\begin{bmatrix} \frac{\mathbf{C}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma} \\ \mathbf{B}_{cl} \end{bmatrix}\left(-\gamma\mathbf{H}+\frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\right)^{-1}\begin{bmatrix} \frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{C}_{cl}}{\gamma} & \mathbf{B}_{cl}^T \end{bmatrix}<\mathbf{0},$$

where $\mathbf{M}=\mathbf{W}\mathbf{Y}^{-1}$. Using the Schur's complement

$$\begin{bmatrix} \mathbf{J}_Y+\frac{\mathbf{C}_{cl}^T\mathbf{H}\mathbf{C}_{cl}}{\gamma} & \mathbf{A}_{cl}^T+\mathbf{M}^T & \frac{\mathbf{C}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma} \\ \star & -\mathbf{I} & \mathbf{B}_{cl} \\ \star & \star & -\gamma\mathbf{H}+\frac{\mathbf{D}_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma} \end{bmatrix}<\mathbf{0},$$

where $\mathbf{J}_Y=\mathbf{Y}^{-1}(\mathbf{Y}^{-1}-\mathbf{M})-\mathbf{M}^T\mathbf{Y}^{-1}$. This inequality is equivalent to

$$\begin{bmatrix} 2\mathbf{X}_P.\mathbf{X}_P+\mathbf{M}^T\mathbf{M}-\mathbf{Q} & \mathbf{A}_{cl}^T+\mathbf{M}^T & \mathbf{0} & \mathbf{C}_{cl}^T\mathbf{H} \\ \star & -\mathbf{I} & \mathbf{B}_{cl} & \mathbf{0} \\ \star & \star & -\gamma\mathbf{H} & \mathbf{D}_{cl}^T\mathbf{H} \\ \star & \star & \star & -\gamma\mathbf{H} \end{bmatrix}<\mathbf{0}, \quad (17)$$

where $\mathbf{X}_P=\mathbf{Y}^{-1}$ and $\mathbf{Q}=(\mathbf{X}_P+\mathbf{M})^T(\mathbf{X}_P+\mathbf{M})$. Finally, applying the Schur's complement method to the inequality in (17) and relaxing the equality constraint gives

$$\begin{bmatrix} -\mathbf{Q} & \mathbf{A}_{cl}^T+\mathbf{M}^T & \mathbf{0} & \mathbf{C}_{cl,h}^T & \sqrt{2}\mathbf{X}_P & \mathbf{M}^T \\ \star & -\mathbf{I} & \mathbf{B}_{cl} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\gamma\mathbf{H} & \mathbf{D}_{cl,h}^T & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -\gamma\mathbf{H} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & -\mathbf{I} & \mathbf{0} \\ \star & \star & \star & \star & \star & -\mathbf{I} \end{bmatrix}<\mathbf{0}, \quad (18)$$

$$\begin{bmatrix} \mathbf{Q} & \mathbf{X}_P+\mathbf{M}^T \\ \star & \mathbf{I} \end{bmatrix}\geq\mathbf{0}, \quad (19)$$

where $\mathbf{Q}=(\mathbf{X}_P+\mathbf{M})^T(\mathbf{X}_P+\mathbf{M})$ corresponds to the boundary of the convex set in (19). Moreover, the inequality $\left(-\gamma\mathbf{H}+\frac{D_{cl}^T\mathbf{H}\mathbf{D}_{cl}}{\gamma}\right)<\mathbf{0}$ in (13) is equivalent to $\begin{bmatrix} -\gamma\mathbf{H} & \mathbf{D}_{cl}^T\mathbf{H} \\ \star & -\gamma\mathbf{H} \end{bmatrix}<\mathbf{0}$.

It is interesting to note that the conditions in (18) and (19) can also be converted into (useful due to $5\times5$ structure)

$$\begin{bmatrix} -\mathbf{Q} & \mathbf{A}_{cl}^T+\mathbf{M}^T & \mathbf{0} & \mathbf{C}_{cl,h}^T & \mathbf{X}_P-\mathbf{M}^T \\ \star & -\mathbf{I} & \mathbf{B}_{cl} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\gamma\mathbf{H} & \mathbf{D}_{cl,h}^T & \mathbf{0} \\ \star & \star & \star & -\gamma\mathbf{H} & \mathbf{0} \\ \star & \star & \star & \star & -\mathbf{I} \end{bmatrix}<\mathbf{0}, \quad (20)$$

$$\begin{bmatrix} \mathbf{Q} & \mathbf{M}^T \\ \star & \mathbf{I} \end{bmatrix}\geq\mathbf{0}. \quad (21)$$

**Case II**: With the selection in (16), (14) is equivalent to

$$\begin{bmatrix} -\mathbf{Q} & \mathbf{A}_{cl}^T+\mathbf{M}^T & \mathbf{0} & \mathbf{C}_{cl,h}^T & \mathbf{X}_P & \mathbf{M}^T & \mathbf{X}_P \\ \star & -\lambda\mathbf{I} & \mathbf{B}_{cl} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\gamma\mathbf{H} & \mathbf{D}_{cl,h}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -\gamma\mathbf{H} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & -\mathbf{I} & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & -\lambda^{-1}\mathbf{I} \end{bmatrix}<\mathbf{0},$$

where $\lambda$ has a physical meaning and can be selected by the designer. ■

## REFERENCES


[1] P. Grosdidier and M. Morari, "The $\mu$ interaction measure," *Industrial & Engineering Chemistry Research*, vol. 26, pp. 1193–1202, 1987.

[2] M. Hovd and S. Skogestad, "Improved independent design of robust decentralized controllers," *JPC*, vol. 3, no. 1, pp. 43–51, 1993.

[3] A. I. Zecevic, G. Neskovic, and D. D. Siljak, "Robust decentralized exciter control with linear feedback," *IEEE Transactions on Power Systems*, vol. 19, no. 2, pp. 1096–1103, 2004.

[4] A. I. Zecevic and D. D. Siljak, "Global low-rank enhancement of decentralized control for large-scale systems," *IEEE Transactions on Automatic Control*, vol. 50, pp. 740–744, 2005.

[5] S. Skogestad and M. Morari, "Robust performance of decentralized control systems by independent designs," *Automatica*, vol. 25, pp. 119–125, 1989.

[6] E. H. Bristol, "On a new measure of interaction for multivariable process control," *IEEETAC*, vol. 11, pp. 133–134, 1966.

[7] J. E. Rijnsdorp, "Interaction in two-variable control systems for distillation columns-I," *Automatica*, vol. 1, pp. 15–28, 1965.

[8] N. Viswanadham and J. H. Taylor, "Sequential design of large-scale decentralized control systems," *IJC*, vol. 47, pp. 257–279, 1988.

[9] M. S. Chiu and Y. Arkun, "A methodology for sequential design of robust decentralized control systems," *Automatica*, vol. 28, pp. 997–1001, 1992.

[10] J. Bao, J. F. Forbes, and P. J. McLellan, "Robust multiloop PID controller design: A successive semidefinite programming approach," *Ind. Eng. Chem. Res.*, vol. 38, pp. 3407–3419, 1999.

[11] J. Lee and T. F. Edgar, "Phase conditions for stability of multi-loop control systems," *Computers and Chemical Engineering*, vol. 23, pp. 1623–1630, 2000.

[12] V. Kariwala, F. Forbes, and S. Skogestad, "$\mu$-interaction measure for unstable systems," *International Journal of Automation and Control*, vol. 1, no. 4, pp. 295–313, 2007.

[13] P. Grosdidier and M. Morari, "Interaction measures for systems under decentralized control," *Automatica*, vol. 22, pp. 309–319, 1986.

[14] H. H. Rosenbrock, *Computer-Aided Control System Design*. London: Academic Press, 1974.

[15] K. P. Loh and M. S. Chiu, "Robust decentralized control of non-square systems," *Chem. Eng. Communications*, vol. 158, pp. 157–180, 1997.

[16] ——, "Robust decentralized controller design for unstable systems," *Chemical Engineering Science*, vol. 52, no. 14, pp. 2299–2311, 1997.

[17] V. Kariwala, S. Skogestad, J. F. Forbes, and E. S. Meadows, "Input performance limitations of feedback control," in *American Control Conference*, vol. 3, Boston, MA, 2004, pp. 2063–2068.

[18] M. Hovd and S. Skogestad, "Pairing criteria for decentralized control of unstable plants," *Ind. Eng. Chem. Research*, vol. 33, 1994.

[19] J. Loefberg, "YALMIP : A toolbox for modeling and optimization in MATLAB," in *IEEE International Symposium on Computer Aided Control Systems Design*, Taipei, Taiwan, 2004, pp. 284–289.

[20] H. Cui and E. W. Jacobsen, "Performance limitations in decentralized control," *Journal of Process Control*, vol. 12, pp. 485–494, 2002.

[21] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control: Analysis and Design*. John Wiley and Sons, 2005.

[22] L. E. Ghaoui, F. Oustry, and M. A. Rami, "A cone complementary linearization algorithm for static output feedback and related problems," *IEEE Trans. on Automatic Control*, vol. 42, pp. 1171–1176, 1997.

[23] S. Boyd, L. E. Ghaoni, E. Feron, and V. Balakrishnan, *LMIs in System and Control Theory*. PA: SIAM: Philadelphia, 1994.

[24] P. Apkarian, H. D. Tuan, and J. Bernussou, "Continuous-time analysis, eigenstructure assignment, and $H_2$ synthesis with enhanced linear matrix inequalities (LMI) characterizations," *IEEE Transactions on Automatic Control*, vol. 46, pp. 1941–1946, 2001.

[25] F. Tao and Q. Zhao, "Design of stochastic fault tolerant control for $H_2$ perf." *Int. J Robust and Nonlinear Control*, vol. 17, pp. 1–24, 2007.