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Majorizations for the Dominant Eigenvector of a Nonnegative Matrix

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***Abstract*— Motivated by network controller design applications, we develop several majorization results for the dominant eigenvector of an irreducible nonnegative matrix.**

I. INTRODUCTION

Recent efforts on network control and design have made clear that graph-theoretic characterization of *eigenvector* components is critical, for both undirected and directed graphs (which correspond to symmetric and asymmetric topology matrices). However, graph-theoretic characterization of eigenvalues and especially eigenvectors of asymmetric matrices is very limited. Here, we study the structure of the eigenvector associated with the dominant eigenvalue, for the broad class of irreducible nonnegative matrices (see e.g. [1]). More precisely, we characterize the dependence of the dominant-eigenvector components on individual entries in the matrix (equivalently, edge weights in an associated graph), as well as on row scalings (Section 2). Motivated by network design tasks in particular, we also briefly study the dependence of dominant eigenvector components on simultaneous modifications in multiple rows (Section 3).

Our particular motivation for studying the dominant eigenvectors of positive matrices stems from our efforts in decentralized controller design [2], [3]. From another viewpoint, this work also contributes to the extensive research on positive dynamical systems and the associated nonnegative matrices (see e.g [1], [4]), by further characterizing the eigenvector associated with the dominant eigenvalue. We ask the reader to see the extended document [6] for further development of these motivations.

II. MAJORIZATIONS FOR SINGLE-ROW INCREMENTATIONS

We are concerned in this section with understanding how the dominant-eigenvector components of a nonnegative matrix depend on individual matrix entries, and on scalings of single rows in the matrix. In fact, we find it most convenient to study the dependence in the case that a single row of the matrix is incremented in an arbitrary fashion, and hence to obtain results for single-entry changes and row scalings as special cases.

Precisely, let us consider an $n \times n$ real irreducible nonnegative matrix $G \triangleq [g_{ij}]$. We consider incrementing a single row of G, say (WLOG) the first row, by a vector $\mathbf{a}^T \triangleq \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}$, where each $a_i \geq 0$ and $\mathbf{a} \neq \mathbf{0}$. That is, we study $\widehat{G} = G + \mathbf{e}_1\mathbf{a}^T$, where $\mathbf{e}_1$ is an 0–1 indicator vector with first entry equal to 1.

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We notice that $\widehat{G}$ is also an irreducible nonnegative matrix. Thus, G and $\widehat{G}$ each has a real positive eigenvalue (denoted λ and $\widehat{\lambda}$, respectively) that is non-repeated and has magnitude at least as large as each of its other eigenvalues. The eigenvector $\mathbf{v}$ (respectively $\widehat{\mathbf{v}}$) associated with G ($\widehat{G}$) is strictly positive entrywise. For the purpose of this note, we refer to $\mathbf{v}$ ($\widehat{\mathbf{v}}$) as the **dominant eigenvector** of G ($\widehat{G}$). Also, we use the notation v_i (respectively, $\widehat{v}_i$) for the ith component of $\mathbf{v}$ (respectively $\widehat{\mathbf{v}}$).

Our purpose is to compare the components of the dominant eigenvectors $\mathbf{v}$ and $\widehat{\mathbf{v}}$. We find that incrementing the first row increases the first eigenvector component relative to each remaining component:

Theorem 1: Consider the dominant eigenvectors $\mathbf{v}$ *and* $\widehat{\mathbf{v}}$. *Then* $\frac{\widehat{v}_1}{\widehat{v}_j} > \frac{v_1}{v_j}$ *and for* $j = 2, \dots, n$. *We thus recover that, when the eigenvector is normalized to unit length,* $\widehat{v}_1 > v_1$.

Proof: It is well known, see e.g. [1], that incrementing entries of an irreducible nonnegative matrix strictly increases its dominant eigenvalue, so in our case $\widehat{\lambda} = \lambda + \Delta$ for some $\Delta > 0$. Also, from the positivity of the dominant eigenvectors of G and $\widehat{G}$, we note that the dominant eigenvector of $\widehat{G}$ can be scaled so that its first entry is equal to the first entry of $\mathbf{v}$. Thus, the dominant eigenvector for $\widehat{G}$ can be written (for some normalization) in the form $\tilde{\mathbf{v}} = \begin{bmatrix} v_1 \\ \mathbf{v}_{2:n} + \mathbf{q} \end{bmatrix}$, where $\mathbf{v}_{2:n}^T = \begin{bmatrix} v_2 & \dots & v_n \end{bmatrix}$, and $\mathbf{q}$ is a $(n-1)$-component vector (and where we have used $\tilde{\mathbf{v}}$ for the eigenvector to indicate its particular normalization).

We shall prove that the vector $\mathbf{q}$ is (elementwise) strictly positive. To do so, let us simply work from the eigenvector equation $\widehat{G}\tilde{\mathbf{v}} = \widehat{\lambda}\tilde{\mathbf{v}}$. Let us consider the last $n-1$ equations in the system of equations. Specifically, substituting for $\widehat{\lambda}$ and $\tilde{\mathbf{v}}$, and using the eigenvector equation $G\mathbf{v} = \lambda\mathbf{v}$, we find that

$$G_{2:n,2:n}\mathbf{q} = \widehat{\lambda}\mathbf{q} + \Delta\mathbf{v}_{2:n},$$

where $G_{2:n,2:n} = \begin{bmatrix} g_{22} & \cdots & g_{2n} \\ \vdots & & \vdots \\ g_{n2} & \cdots & g_{nn} \end{bmatrix}$. Rearranging, we obtain $(\widehat{\lambda}I - G_{2:n,2:n})\mathbf{q} = -\Delta\mathbf{v}_{2:n}$. From classical results, we have that the dominant eigenvalue of the nonnegative matrix $G_{2:n,2:n}$ is less than or equal to λ, and hence strictly less that $\widehat{\lambda}$ [1]. Thus, $\widehat{\lambda}I - G_{2:n,2:n}$ is a nonsingular M-matrix [1]. It is thus automatic that its inverse is elementwise nonnegative. In fact, noting that each irreducible submatrix of $G_{2:n,2:n}$ has a strictly postive inverse, we obtain that $(\widehat{\lambda}I - G_{2:n,2:n})^{-1}$ has at least one strictly positive entry on each row. Noting that $\mathbf{v}_{2:n}$ is entrywise strictly positive, we thus find that $\mathbf{q}$ is entrywise strictly negative. The majorizations in the theorem statement follow immediately. ∎

The above theorem shows that, when a particular row of a positive matrix is incremented (in an arbitrary way), the associated component of the dominant eigenvector increases relative to the other components. This results automatically specializes to two cases of particular interest, namely 1) the incrementation of a single entry in the matrix (which corresponds to incrementing an edge weight in an associated graph), and 2) the scaling of a row in the matrix (which corresponds to scaling the "influence" of a node in the graph, e.g. by changing a controller gain):

Corollary 1: Consider incrementing the entry at row i and column j of an irreducible nonnegative matrix G. Then the ratio of the ith component of the dominant right eigenvector of G to each other component strictly increases. Similarly, the ratio of the jth component of the dominant left eigenvector to each other component strictly increases.

Corollary 2: Consider scaling the ith row of an irreducible nonnegative matrix G by a factor $\alpha > 1$ (respectively $0 < \alpha < 1$). Then the ratio of the ith component of the dominant right eigenvector of G to each other component strictly increases (respectively, strictly decreases).

The result for row-scaling of positive matrices also permits us characterize the eigenvectors of nonsingular irreducible M *matrices* (see [1]) upon row-scaling, using the fact that inverses of irreducible M matrices are nonnegative (in fact, strictly positive) matrices:

Corollary 3: Consider scaling the ith row of a nonsingular and irreducible M matrix by a constant $\alpha > 1$ (respectively, $\alpha < 1$), and consider the left eigenvector associated with the eigenvalue of minimum magnitude that is real. The ratio of the ith component of this eigenvector to each other component strictly decreases (respectively, increases) upon row-scaling.

Let us take a moment to briefly interpret the above results for a couple applications, to illustrate their use. In the interest of space, we shall only discuss these examples at a conceptual level.

1) Velocity-control problems in autonomous-vehicle-coordination applications can often be abstracted that of designing a diagonal gain matrix K so as to optimize the dynamics $\dot{\mathbf{x}} = -KH\mathbf{x}$, where H is an M-matrix (see e.g. [2]). The above analysis clarifies that increasing the gain for a particular vehicle not only speeds up the slow mode of the system, but reduces the excitation caused by the initial conditions of that vehicle (since the *left* eigenvector component is depressed).

2) In virus-spreading-control applications, reducing the flow of infectives from one region i to another j has the effect of reducing the impact of the infectives in region i on other regions and reducing the size of the infected population in j (in addition to slowing the spread of the infection in general).

III. Majorizations for Multiple-Row Incrementations

In decentralized controller design tasks, it turns out that understanding the dependence of eigenvector components upon scaling of multiple rows or incrementation of multiple diagonal entries is important [2], [3]. With these applications in mind, here we briefly characterize the dependence of dominant eigenvector components of nonnegative matrices on diagonal entries of the matrices (noting that similar results hold for other multi-row incrementations). We develop the majorization results in two steps: first, we consider *designing* incrementations of multiple diagonal entries to achieve certain ratios among the dominant eigenvector components. Second, we use this result to study arbitrary incrementations of multiple diagonal entries. We exclude the proofs in the interest of space, see [6] for further details.

First, here is the design result:

Theorem 2: Consider an irreducible nonnegative matrix G, and say (WLOG) that we can increment the first $m < n$ diagonal entries by amounts $k_1, \ldots, k_m$, respectively, to obtain $\widehat{G}$. Then, for each $k_1 > 0$, we can find $k_2 > 0, \ldots, k_m > 0$ so that 1) $\frac{\widehat{v}_i}{\widehat{v}_j} = \frac{v_i}{v_j}$ for $i = 1, \ldots, m$, $j = 1, \ldots, m$, and 2) $\frac{\widehat{v}_i}{\widehat{v}_j} > \frac{v_i}{v_j}$ for $i = 1, \ldots, m$, $j = m+1, \ldots, n$. Furthermore, $k_2, \ldots, k_m$ increase monotonically with increasing k_1.

This theorem states that there is a way to increment m diagonal entries of a nonnegative matrix so that the corresponding m components of the dominant eigenvector remain the same to within a scaling, while the ratios of these components to the others increase.

Finally, the above design result yields a majorization of eigenvector components for arbitrary diagonal incrementations, as formalized in the following theorem:

Theorem 3: Consider an irreducible nonnegative matrix G, and say (WLOG) that we increment the first $m < n$ diagonal entries by amounts $k_1, \ldots, k_m$, respectively, to obtain $\widehat{G}$. Then, there is $i \in 1, \ldots, m$ such that $\frac{\widehat{v}_i}{\widehat{v}_j} > \frac{v_i}{v_j}$ for all $j = m+1, \ldots, n$.

This theorem states that, when multiple diagonal entries of an irreducible nonnegative matrix are incremented, at least one corresponding components in the dominant eigenvector becomes larger in relation to all the components corresponding to non-incremented entries.

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