Active Fault Detection in Nonlinear Systems Using Auxiliary Signals

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Abstract— In recent years active approaches for fault detection using test signals have been developed. This paper reports on progress in extending one of these approaches from linear systems to nonlinear systems. Theoretical results are presented on the use of linearizations which is based on sufficiently small nonlinearities. An optimization based approach is also presented for large nonlinearities. Examples are given.

I. INTRODUCTION

Failure detection and identification are a fundamental part of many industrial processes and have been extensively studied [9], [15]. There are two general approaches. The passive approach monitors system outputs. While useful in a number of applications, system performance during normal operation can mask faults until it is too late for the fault to be detected by a passive approach in time to avoid a failure. This can happen either due to inactivity (brakes are not used until we need to stop) or due to the action of controllers [9]. In addition, it is important to maintain some functionality after damage. This happens in extended space missions and a number of other power and transportation systems. In this situation there may be a loss of sensors and system reconfiguration which results in a system for which a previously designed passive system may not be appropriate. In each of these scenarios an active approach can be useful.

Accordingly there has been increasing interest in active approaches where the system is acted upon in order to reveal the fault [5], [12]. This is not active fault tolerant control where the system is taking action to compensate for a fault [10]. Here active refers to action to determine the existence of a fault and the diagnosis of the fault. Signal injection for identification has been widely studied [8]. These signals are often persistently exciting and the identification may be ongoing. [17] introduced the use of auxiliary signals specifically in the context of fault detection. Subsequent work has developed statistical [2], [11] and robust methods [5], [14] for designing fault detection input signals. Ways to reduce the effect of the signal on the system have been investigated [5], [14]. Here we measure signals in the norm (19a), (19b). These signals are sometimes referred to as "plant-friendly" signals, that is, signals that can be introduced while the plant is in normal operation.

The work in [5] was for linear systems. However, many systems are nonlinear, especially some of the applications of

most interest to us. Here we report on progress in extending the approach of [5] to nonlinear systems. Section II briefly summarizes part of the linear time invariant approach. Space prohibits going into detail. A natural thing to try is to use linearization and apply the ideas of Section II to get a test signal that is then used on the nonlinear model. Experimental work shows that this sometimes works well [4]. Section III provides a theoretical justification for this approach. We also derive explicit bounds that can give an idea when the use of linearizations is justified. Convexity is important for switching the $\max, \min$ in the linear algortihm. For large nonlinearities we do not expect even approximate convexity. Instead we use a direct optimization approach and sophisticated optimization software in Section IV.

II. THE LINEAR CASE

We provide in this section a brief introduction to the linear algorithm. [5] contains an in depth development including a consideration of model uncertainty. We assume linear, constant coefficient models with additive uncertainty:

$$\dot{x}_i = A_i x_i + B_i v + M_i \mu_i \qquad (1a)$$
$$y_i = C_i x_i + N_i \mu_i, \qquad (1b)$$

for $i = 0, 1$. v is our input signal and μ_i is the noise associated with the properly working ($i = 0$) and the faulty ($i = 1$) system. We also assume that N_i, $i = 0, 1$, has full row rank.

$\mathcal{A}_i(v)$ is the output set of model i. Perfect fault detection would imply that $\mathcal{A}_0(v) \cap \mathcal{A}_1(v) = \emptyset$. Without a bound on the uncertainty, or noise, any output would be possible from either model and perfect model identification would be impossible. Thus, we assume that the noise in model i is bounded in the L^2 norm by the noise measure

$$\mathcal{S}_i(x_i(0), \mu_i) = x_i(0)^T Q_i x_i(0) + \int_0^T |\mu_i(t)|^2 \ dt < \gamma. \quad (2)$$

For discussion purposes, we initially take take $\gamma = 1$. Once the noise is bounded, we can consider the outputs of the models. Define $\mathcal{L}_i(f) = \int_0^T e^{A_i(t-s)} f(s) ds$ so that $\mathcal{L}_i(f)$ is the solution of $\dot{z} = A_i z + f$, $z(0) = 0$. Let ξ_i be the free initial condition for ODE (1a). Then $y_i = \bar{y}_i + (C_i \mathcal{L}_i M_i + N_i)\mu_i + C_i e^{A_i t} \xi_i$ where $\bar{y}_i = C_i \mathcal{L}_i(B_i v)$ is a vector that is linearly dependent on the detection signal, $\{(C_i \mathcal{L}_i M_i + N_i)\mu_i : \|\mu_i\| < 1\}$ is an open, convex set, and $\{C_i e^{A_i t} \xi_i : \xi_i \in \mathbf{R}^N\}$ is a finite dimensional subspace of $L^2[0, T]$.

Thus, the output sets, $\mathcal{A}_i(v)$, are translates of open sets by $\bar{y}_i$ and are affinely dependent upon the detection signal, v. The goal of the algorithms is to find the minimum proper signal, where a proper signal is one which makes the two

output sets disjoint. This signal is then input into the system during a short test period and the output is measured. Based on the measurement, a decision is made whether the system is normal or faulty.

If (x_0, μ_0, x_1, μ_1) satisfies the models (1) and the noise bound (2), then a proper detection signal, v, ensures that the output sets of the models will be distinct. Alternatively if (x_0, μ_0, x_1, μ_1) satisfy the models and v is a proper detection signal, but the outputs are still not distinct, then (x_0, μ_0, x_1, μ_1) must not satisfy the noise bound (2) for $i = 0, 1$. Thus, if (x_0, μ_0, x_1, μ_1) satisfies the models and v is a proper detection signal, but an output y is still in both output sets, it must be that

$$\max\{\mathcal{S}_0(x_0(0), \mu_0), \mathcal{S}_1(x_1(0), \mu_1)\} \geq 1. \tag{3}$$

Since this is true for all (x_0, μ_0, x_1, μ_1) that satisfy the models but do not have distinct outputs, it is sufficient to ensure it is true for the minimum of them. Thus,

$$\min_{x_i, \mu_i, y_0 = y_1} \max\{\mathcal{S}_0(x_0(0), \mu_0), \mathcal{S}_1(x_1(0), \mu_1)\} \geq 1. \tag{4}$$

The algorithm is to find the minimum proper signal v such that (4) holds. The linear algorithm utilizes the fact that the $\max, \min$ in (4) can be exchanged.

III. THEORY USING LINEARIZATIONS

We assume that both our working and faulty systems are only nonlinear in the states and are of the form

$$\dot{x} = g(x) + Bv + M\mu. \tag{5}$$

We require that $g(x)$ is locally Lipschitz, $g(0) = 0$, and linearization is around $\bar{x} = 0$ which gives the approximation

$$\dot{x} = Ax + Bv + M\mu. \tag{6}$$

A. Guaranteed Fault Detection Through Problem Scaling

When can we guarantee fault detection in the nonlinear problem using the linearization? Our first approach is to find the proper v for the linearized problem and then to scale this v, as well as μ and x_0, by β until we find a proper signal for the scaled nonlinear problem. To understand "scaling" multiply (1) by β and s (2) by β^2. Thus,

$$\beta\dot{x} = A\beta x + B\beta v + M\beta\mu \tag{7a}$$
$$\beta y = C\beta x + N\beta\mu \tag{7b}$$
$$S_i(\beta x_i(0), \beta\mu_i) = \beta x^T(0)P\beta x(0) + \int_0^T \beta\mu_i^T \beta\mu_i \, dt$$
$$< \beta^2. \tag{7c}$$

Let $\widetilde{x} = \beta x$, $\widetilde{y} = \beta y$ and $\widetilde{\mu} = \beta\mu$. Equation (7) become

$$\dot{\widetilde{x}} = A\widetilde{x} + B(\beta v) + M\widetilde{\mu} \tag{8a}$$
$$\widetilde{y} = C\widetilde{x} + N\widetilde{\mu} \tag{8b}$$
$$S_i(\widetilde{x}_i(0), \widetilde{\mu}_i) = \widetilde{x}^T(0)P\widetilde{x}(0) + \int_0^T \widetilde{\mu}_i^T \widetilde{\mu}_i \, dt < \beta^2 \tag{8c}$$

Equation (8) is the new, scaled equations. Note that (8) is the same as (1) and (2), but with a different γ. The theorem

in this section finds a bound on this β, which will guarantee separation of the nonlinear output sets by βv.

The following theory relies heavily on [16]. The notation has been slightly altered to match the notation used in this paper. We start with the ODE

$$\dot{x} = f(x, t), \ x(0) = x_0, \tag{9}$$

where $f(x, t)$ is defined on $\mathcal{I} \times \mathcal{X}$. $\mathcal{I}$ is an interval and $\mathcal{X}$ is an open set containing $x(0) = x_0$. Also, $B_\epsilon(x_0)$ is a ball of radius ϵ around x_0. We assume that 1) for each x, f is measurable in t, 2) f is locally Lipschitz in x. That is, for each $x_0 \in \mathcal{X}$, there is an ϵ and nonnegative locally integrable α such that $B_\epsilon(x_0) \subset \mathcal{X}$ and $\|f(t, x) - f(t, y)\| \leq \alpha(t)\|x - y\|$ for all $t \in \mathcal{I}$ $x, y \in B_\epsilon(x_0)$. 3) f is locally integrable on t. That is, for each x_0, there is a locally integrable Ω such that $\|f(t, x_0)\| \leq \Omega(t)$

Given these assumptions, $f(x, t)$ is locally integrable if its norm has a finite integral on each finite subinterval. Under these assumptions we have:

Theorem 3.1: Suppose that ψ is a solution of (9) on the interval $[0, \tau] \subset \mathcal{I}$. Then there are numbers $\Delta, c > 0$ so that for every $0 < \delta \leq \Delta$ the following holds: Suppose that $h(x, t)$ satisfies the above assumptions and in addition

$$\left\| \int_0^t h(\psi(s), s) \, ds \right\| \leq \delta \tag{10}$$

for all $t \in [0, \tau]$ and for all η_0 such that $\|\eta_0 - x_0\| \leq \delta$, then the solution η of $\eta' = f(t, \eta) + h(t, \eta), \eta(t_0) = \eta_0$ is defined on all of $[0, \tau]$ and $\|\psi - \eta\|_\infty \leq c\delta$.

Theorem 3.1 says, that if h satisfies (10), then the difference between the solutions to $\dot{x} = f$ and $\dot{x} = f + h$ only differ by a constant times the bound on the difference between the initial conditions of the two systems. So, if we start with sufficiently close initial conditions, then the solutions will remain close. We seek to identify faults and not failures. Our assumptions imply that there are no solutions which go to infinity during the test interval.

Our first result shows that if the uncertainty bound is small enough, then we can use a scaled linearized test signal.

Theorem 3.2: Suppose v is strictly proper for the linearized system so that the linear output sets are a positive distance ρ apart. Using the above assumptions from Theorem 3.1, for $\beta \leq \frac{\rho}{2c_3 T \xi c_1^2 \epsilon^2}$, βv separates the nonlinear output sets, with a noise bound scaled by β^2, c_1 and c_3 bounds on the unforced nonlinear equation, and the difference between the forced linear and forced nonlinear equations, with noise, respectively. ξ is the bound on the error term in the linearization and T is the final time.

The proof consists of four steps. The first two steps show that, as we scale by β, the solution to the unforced nonlinear problem and the difference between the solutions to the unforced and forced nonlinear problems are linearly bounded in the initial condition and the auxiliary signal and noise, respectively. The third step shows that the difference between the solutions to the forced nonlinear problem and the forced linear problem scales quadratically. The final step finds the bound on β for which βv will guarantee separation of the

nonlinear output sets. Figure 1 shows the effect of βv on the scaled, nonlinear output sets and how the scaling affects these output sets.
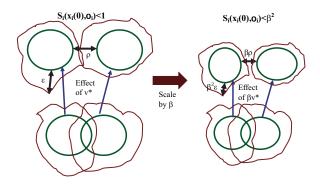

Fig. 1. Pictorial of proof of Theorem 3.2.

We seek a β which guarantees separation of the nonlinear output sets. Therefore, we need to find β such that

$$\frac{\beta\rho}{2} \leq \beta^2 c_3 T \xi c_1^2 \epsilon^2. \tag{11}$$

(11) says half the distance between the linear outputs sets is greater than the maximum distance between the linear and nonlinear output sets. Solving (11) for β, we gives the desired

$$\beta \leq \frac{\rho}{2c_3 T \xi c_1^2 \epsilon^2}. \tag{12}$$

B. Using the Original Test Signal

We now consider when the proper v from the linear system can be used on the nonlinear system. We find the optimal v^* for the linear problem and then derive conditions for which this v^* is proper for the nonlinear problem, except for possibly tighter noise bounds. Linearizations can work well, even for highly nonlinear systems [4], sometimes they work even better on the nonlinear system. Linearizations also work well for incipient faults[7]. Here we assume the noises in the ODE and the output equation are independent (the same μ_i is not present in both the ODE and the output equation).

We start with the nonlinear system

$$\dot{x} = f(x) + Bv + M\mu_1 \tag{13a}$$
$$y = g(x) + N\mu_2, \tag{13b}$$

which can be written

$$\dot{x} = Ax + E_1(x) + Bv + M\mu_1 \tag{14a}$$
$$y = Cx + E_2(x) + Dv + N\mu_2. \tag{14b}$$

The linear system is

$$\dot{x} = Ax + Bv + M\phi_1 \tag{15a}$$
$$y = Cx + Dv + N\phi_2. \tag{15b}$$

We use a noise bound $S(x(0), \phi) \leq \delta$ and then find the proper v^* for our linear system. Then, we find a noise bound for the nonlinear system (14), for which v^* is proper.

We assume M, N are invertible and rewrite (14) as

$$\dot{x} = Ax + Bv + M(M^{-1}E_1(x) + \mu_1) \tag{16a}$$
$$y = Cx + Dv + N(N^{-1}E_2(x) + \mu_2). \tag{16b}$$

We treat the nonlinearities as extra noise in the system and equate the noises from (15) to the noises in (16). Thus, $\phi_1 = M^{-1}E_1(x) + \mu_1$ and $\phi_2 = N^{-1}E_2(x) + \mu_2$. We rewrite our noise bound as

$$\|x(0)\|^2 + \int_0^T \phi_1^T \phi_1 + \phi_2^T \phi_2 \, dt \leq \delta \tag{17}$$

If the nonlinearity is bounded by $\widetilde{M}$, and the noise is bounded by $\delta - \widetilde{M} > 0$, then the linearized test signal is proper for the nonlinear problem with the noise bound $\delta - \widetilde{M}$.

Substituting $\phi_1 = M^{-1}E_1(x) + \mu_1$ and $\phi_2 = N^{-1}E_2(x) + \mu_2$ into this inequality, we can show that if $\|(M^{-1}E_1(x), N^{-1}E_2(x))\| \leq \widetilde{M} \leq \sqrt{\delta}$, then $\|(\mu_1, \mu_2)\| \leq \delta - \widetilde{M}$ guarantees that $\|(M^{-1}E_1(x), N^{-1}E_2(x)) + (\mu_1, \mu_2)\|^2 \leq \sqrt{\delta}$.

Now, we can decide on appropriate bounds for the nonlinearities and find the bounds for the μ_i's.

We have focused on the case where both models are linearized around the same set point since this is the more difficult problem for detection. The results here can be extended to other possibilities but space prohibits their discussion.

IV. DIRECT OPTIMIZATION AND LARGE NONLINEARITIES

While the preceding discussion and the tests in [4] have shown that linearizations are often useful, this is not always the case. In particular, if the system is highly nonlinear, then we must attack it directly as an optimization problem. But a number of technical issues then arise. One is that we usually no longer have the convexity that allowed us to switch the max and the min in (4). Another is that problems do not scale. A v that is proper may not be proper when the problem is scaled. We shall discuss one way to surmount many of these issues and conclude by giving two examples that illustrate several points.

For our optimization solver we will use the Sparse Optimal Control Software (SOCS) developed at the Boeing Company [1]. SOCS uses a direct transcription approach and accepts parameterized problems.

In the most general case, we have the models

$$\dot{x}_i = f_i(x_i, \mu_i, v) \tag{18a}$$
$$y = g_i(x_i, \mu_i, v) \tag{18b}$$

where the variables have the same interpretation as in the previous section. We again restrict ourselves to the additive uncertainty case with bound (2) and look for the optimal proper v^* [5] which gives minimal disturbance. The disturbance is described as

$$\dot{\psi} = f(\psi, v), \quad \psi(0) = 0. \tag{19a}$$

Often $f(\psi, v) = f_0(\psi, 0, v)$. Our measure of the auxiliary signal v thus becomes

$$\delta^2 = \psi(T)^T W \psi(T) + \int_0^T |v|^2 + \psi^T U \psi \; dt \qquad (19b)$$

where U and W are positive semidefinite matrices and δ is a norm on L^2. Notice that if $U = W = 0$, the measure becomes just a simple L^2 norm of the auxiliary signal v.

A proper signal causes an uncertainty constraints to be broken.

$$\phi(v, s) = \inf_{x_i, \mu_i, y, u, s \in [0, T]} \max(S_0, S_1) \geq \gamma \qquad (20)$$

In the linear case [5] a variable β is used to replace the max in (20) with a family of inner product norms. We refer to the β of [5] as $\widehat{\beta}$. For the model uncertainty case one cannot just fix a $\widehat{\beta}$ value since that sometimes admits too much noise. For the nonlinear problem we exploit the fact that we can approximate the max in (20) with a p-norm. Figure 2 shows the curve $x^p + y^p = 1$ in the first quadrant for p from 2 to 5. They approximate the max from the inside. $x^p + y^p = 2$ would approximate max from the outside.
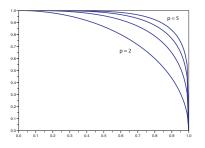

Fig. 2. Approximation of max norm by p norm for p from 2 to 5.

Thus we will consider the problem

$$\phi(v, s) = \inf_{x_i, \mu_i, y, u, s \in [0, T]} \sqrt[p]{S_0^p + S_1^p} \geq \gamma \qquad (21)$$

with relative error bound $\sqrt[p]{2}$ since

$$\max\{S_0, S_1\} \leq \sqrt[p]{S_0^p + S_1^p} \leq \sqrt[p]{2} \max\{S_0, S_1\}. \qquad (22)$$

Combining all of the above, our problem is to minimize $\psi(T)^T W \psi(T) + \int_0^T |v|^2 + \psi^T U \psi \; dt$ subject to

$$\dot{x}_i = f_i(x_i, \mu_i, v), \quad i = 0, 1 \qquad (23a)$$
$$\dot{\psi} = f(\psi, v) \qquad (23b)$$
$$0 = g_0(x_0, \mu_0, v) - g_1(x_1, \mu_1, v) \qquad (23c)$$
$$\inf_{x_i, \mu_i, y, s \in [0, T]} S_0^p + S_1^p \geq \gamma^p \qquad (23d)$$

In order to get a single optimization problem that is easier to implement we replace the inf in (23d) with the necessary condition for this minimization. We refer to this as the inner optimization problem. We have more general formulations but for this paper in order to simplify the presentation we consider the case of linear noise, nonlinear states, and additive uncertainty. Without loss of generality, we can take $\gamma = 1$ since we can adjust the weight on the noise μ_i within

the definition of f_i for a given problem. The optimization problem constraints are

$$\dot{x}_i = f_i(x_i, v) + M_i \mu_i, \quad i = 0, 1 \qquad (24a)$$
$$\dot{\psi} = f(\psi, v) \qquad (24b)$$
$$0 = g_0(x_0, v) + N_0 \mu_0 - g_1(x_1, v) - N_1 \mu_1 \qquad (24c)$$
$$\inf_{x_i, \mu_i, y} S_0^p + S_1^p \geq 1 \qquad (24d)$$
$$S_i = x_i(0)^T P_i x_i(0) + \int_0^T \mu_i^T \mu_i dt. \qquad (24e)$$

Since the N_i have full row rank. there exists representations $N_i = [\bar{N}_i 0]$ with $\bar{N}_i$ invertible, achievable by successive orthogonal changes of coordinates. Applying the same decomposition on M_i and μ_i, we have $M_i = [\bar{M}_i \tilde{M}_i]$ and $\mu_i = [\bar{\mu}_i^T \tilde{\mu}_i^T]^T$. Now we can use (24c) to solve for $\bar{\mu}_0$:

$$\bar{\mu}_0 = \bar{N}_0^{-1} g_1(x_1, v) + \bar{N}_0^{-1} \bar{N}_1 \bar{\mu}_1 - \bar{N}_0^{-1} g_0(x_0, v)$$
$$= \widetilde{g}(x_0, x_1, v, \bar{\mu}_1) \qquad (25)$$

and eliminate $\bar{\mu}_0$ from the other equations to obtain

$$\dot{x}_0 = f_0(x_0, v) + \bar{M}_0(\widetilde{g}(x_0, x_1, v, \bar{\mu}_1) + \tilde{M}_0 \tilde{\mu}_0 \qquad (26a)$$
$$\dot{x}_1 = f_1(x_1, v) + \bar{M}_1 \bar{\mu}_1 + \tilde{M}_1 \tilde{\mu}_1 \qquad (26b)$$
$$\dot{\psi} = f(\psi, v) \qquad (26c)$$
$$\inf_{x_i, \mu_i, y} S_0^p + S_1^p \geq 1 \qquad (26d)$$
$$S_0 = x_0(0)^T P_0 x_0(0) + \int_0^T \|\widetilde{g}\|_2^2 + \|\tilde{\mu}_0\|_2^2 \; dt \qquad (26e)$$
$$S_1 = x_1(0)^T P_1 x_1(0) + \int_0^T \|\bar{\mu}_1\|_2^2 + \|\tilde{\mu}_1\|_2^2 \; dt. \qquad (26f)$$

To solve the inner optimization problem we define two new differential variables:

$$\dot{z}_0 = \|\widetilde{g}(x_0, x_1, v, \bar{\mu}_1)\|_2^2 + \|\tilde{\mu}_0\|_2^2 \qquad (27a)$$
$$\dot{z}_1 = \|\bar{\mu}_1\|_2^2 + \|\tilde{\mu}_1\|_2^2 \qquad (27b)$$

satisfying boundary conditions:

$$z_i(0) = x_i(0)^T P_0 x_i(0, \quad i = 0, 1. \qquad (28a)$$

The inner minimization problem is to minimize $z_0(T)^p + z_1(T)^p$ with constraints

$$\dot{x}_0 = f_0(x_0, v) + \bar{M}_0 \widetilde{g}(x_0, x_1, v, \bar{\mu}_1) + \tilde{M}_0 \tilde{\mu}_0 \qquad (29a)$$
$$\dot{x}_1 = f_1(x_1, v) + \bar{M}_1 \bar{\mu}_1 + \tilde{M}_1 \tilde{\mu}_1 \qquad (29b)$$
$$\dot{z}_0 = \|\widetilde{g}(x_0, x_1, v, \bar{\mu}_1)\|_2^2 + \|\tilde{\mu}_0\|_2^2 \qquad (29c)$$
$$\dot{z}_1 = \|\bar{\mu}_1\|_2^2 + \|\tilde{\mu}_1\|_2^2. \qquad (29d)$$

We derive the necessary conditions for the inner minimization problem which is to minimize $J = \phi(x(0), x(T)) + \int_0^T L(x(t), u(t), t) dt$, with constraints

$$\dot{x} = f(x, u, t) \qquad (30)$$

where $x = [x_0^T, x_1^T, z_0^T, z_1^T]^T$ and $u = [\tilde{\mu}_0^T, \bar{\mu}_1^T, \tilde{\mu}_1^T]^T$:

$$\dot{x} = f(x, u, t) \qquad (31a)$$
$$\dot{\lambda} = -f_x^T \lambda - L_x \qquad (31b)$$
$$0 = L_u + f_u^T \lambda \qquad (31c)$$
$$0 = (\phi_x - \lambda)^T dx|_{t=T} \qquad (31d)$$
$$0 = (\phi_x + \lambda)^T dx|_{t=0} \qquad (31e)$$

Combining those equations with our original optimization problem we obtain our final definition of the problem.

A. Examples

We now turn to two illustrative academic examples.

1) Example 1: Find the proper v that minimizes $\int_0^T |v|^2 \, dt$ given

$$\dot{x_0} = -x_0^2 - x_0 + v + n_0\tilde{\mu}_0 \tag{32a}$$
$$\dot{x_1} = -x_1^2 - 2x_1 + v + n_1\tilde{\mu}_1 \tag{32b}$$
$$0 = x_1 - x_0 + \bar{\mu}_1 - \bar{\mu}_0 \tag{32c}$$

For this specific example, after eliminating $\bar{\mu}_0$ to get rid of the algebraic constraint (32c), and using Lagrange-Euler form, and keeping in mind that $x_2 = z_0$ and $x_3 = z_1$, the optimization problem given to SOCS is formulated as

$$\dot{x_0} = -x_0^2 - x_0 + v + n_0\tilde{\mu}_0 \tag{33a}$$
$$\dot{x_1} = -x_1^2 - 2x_1 + v + n_1\tilde{\mu}_1 \tag{33b}$$
$$\dot{\lambda_0} = -2\lambda_2 x_0 + 2\lambda_2 x_1 + 2\lambda_2\bar{\mu}_1 + \lambda_0 + 2x_0\lambda_0 \tag{33c}$$
$$\dot{\lambda_1} = 2\lambda_2 x_0 - 2\lambda_2 x_1 - 2\lambda_2\bar{\mu}_1 + 2\lambda_1 + 2x_1\lambda_1 \tag{33d}$$
$$\dot{z_0} = (x_1 - x_0 + \bar{\mu}_1)^2 + \tilde{\mu}_0{}^2 \tag{33e}$$
$$\dot{z_1} = \bar{\mu}_1{}^2 + \tilde{\mu}_1{}^2 \tag{33f}$$
$$0 = 2\lambda_2\tilde{\mu}_0 + n_0\lambda_0 \tag{33g}$$
$$0 = 2\lambda_2\bar{\mu}_1 + 2\lambda_3\bar{\mu}_1 - 2\lambda_2 x_0 + 2\lambda_2 x_1 \tag{33h}$$
$$0 = 2\lambda_3\tilde{\mu}_1 + n_1\lambda_1 \tag{33i}$$

with boundary conditions

$$z_i(0) - P_i x_i^2(0) = 0, \quad i = 0, 1 \tag{34a}$$
$$\lambda_i(0) + 2P_i x_i(0)\lambda_{i+2} = 0, \quad i = 0, 1 \tag{34b}$$
$$p z_i(T)^{p-1} - \lambda_{i+2} = 0, \quad i = 0, 1 \tag{34c}$$
$$\lambda_i(T) = 0, \quad i = 0, 1 \tag{34d}$$
$$z_0^p(T) + z_1^p(T) - 2 \geq 0 \tag{34e}$$

λ_2 and λ_3 are constants. The 2 is inserted into (34e) so that the approximation to the max is from the outside. This guarantees that our optimal v will be proper for the original problem.

For comparison purposes, we also used the standard $\hat{\beta}$ method on the linearized model: minimize $\int_0^T |v|^2 \, dt$ over the proper v given

$$\dot{x_0} = -x_0 + v + n_0\tilde{\mu}_0 \tag{35a}$$
$$\dot{x_1} = -2x_1 + v + n_1\tilde{\mu}_1 \tag{35b}$$
$$0 = x_1 - x_0 + \bar{\mu}_1 - \bar{\mu}_0 \tag{35c}$$

to compute the linearized test signal v_L.

For the original nonlinear problem, we found the auxiliary signal v using p-norm for parameter sets of $T = 1$, $n_i = 0.1$, $P_i = 10$ and $T = 5$, $n_i = 0.1$, $P_i = 10$. Comparison of the auxiliary signal found using the p-norm and v_L is in Figures 3 and 4.

The black and red curves in Figure 3 show that the linearized system was producing a much larger test signal than the true nonlinear system signal. In addition, the green
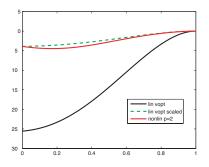

Fig. 3. v optimal comparison for $T = 1$, $n = 0.1$, and $P = 10$.

and red lines show that the shapes were also somewhat different. This difference in the shape of the nonlinear and linear test signals is even more pronounced on the longer test interval of Figure 4.
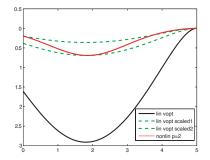

Fig. 4. v optimal comparison for $T = 5$, $n = 0.1$, and $P = 10$.

2) Computational observations: The results are obtained for $p = 2, 3$ and 4. For this particular example, there was no visible difference in results for different values of p.

Given a computed test signal we can evaluate the true noise bound by minimizing the noise such that there is a common output. This is done with SOCS. Using SOCS we can also examine whether a smaller multiple of a test signal would be proper for the nonlinear problem. On $[0, 5]$ we found that $0.21v_L$ was proper. For this same case the noise bound for the nonlinear test signal v_{NL} was 51. So there is room for improvement of the auxiliary signal v. In fact, $0.29v_{NL}$ was proper. Whether this was due to a local minimum for the nonlinear problem or to the difficulty of the optimization problem (or the difficulty in finding the bound), we are not certain at this time. The key point is that the computed nonlinear test signal was much better than the linear one.

In using SOCS, switching from the default discretization-integration method (trapezoidal rule) to the higher order Hermite-Simpson was important in obtaining auxiliary signals. The SOCS default of starting with trapezoidal method and then switching to Hermite-Simpson after a few iterations, often resulted in a feasible point not being found. By starting with Hermite-Simpson, that was avoided. Also, starting Hermite-Simpson helped reduce the running time. The solution times were still sensitive to starting values.

SOCS traces its total time in seconds, and those total times

are given in Table I. Running time for p-norm depended on the initial guesses for states and parameters. As an example, for one set of initial guesses total time was 231.9 s while for another total time was 1.97 s. The best time is given in the table. Code was run on a PC computer with Intel Pentium M 1.6GHz processor with 760MB of RAM.

TABLE I
CPU TIMES FOR EXAMPLE 1 WITH $n_i = 0.1, P_i = 10$.

T	Total SOCS time in seconds		
	linearized	2-norm	3-norm
1	1.03 s	11.64 s	1.3 s
5	1.64 s	1.97 s	2.08 s

The criteria for comparison was the cost function $\int_0^T |v|^2 \, dt$ of the computed auxiliary signal v. Allowed noise bound was set to 1. The actual noise bound for a given v was computed by SOCS with another optimization. Auxiliary signals v had to have calculated noise bound greater than 1 to be proper. Cost function and calculated noise bounds for different methods are in the Table II.

TABLE II
SIZE OF v AND COMPUTED NOISE BOUND WITH $n_i = 0.1, P_i = 10$.

T	Cost function			Noise bound		
	linear	2-norm	3-norm	linear	2-norm	3-norm
1	261.64	9.01	8.97	5579.2	1.038	1.025
5	22.98	0.954	0.951	1293.5	1.035	1.022

B. Example 2

We seek to minimize $\int_0^T |v|^2 \, dt$ over proper v given

$$\dot{x_0} = -x_0^2 + v + \tilde{\mu}_0 \tag{36a}$$
$$\dot{x_1} = -2x_1^2 + v + \tilde{\mu}_1 \tag{36b}$$
$$0 = x_1 - x_0 + \bar{\mu}_1 - \bar{\mu}_0 \tag{36c}$$

Here $P_i = 1$. This example is interesting in that the faulty and nonfaulty system have the same set point ($x = 0, v = 0$) and the same linearizations about this set point. Thus the linearization approach cannot be used at all on this example. For $p = 2$, the cost was 7.56, and the computed noise bound was 1.18. This is quite close to the desired value of 1. For $p = 3$ the cost was 7.37 and for $p = 4$ the cost was 7.26.

V. CONCLUSIONS

The construction of auxiliary test signal for fault detection in nonlinear systems is considered. Previously reported experiments had shown that signals computed from linearizations could sometimes be useful. Two theoretical results are presented. One shows that provided the linearized problems can be told apart, that a properly scaled version of the linear test signal can be used on the nonlinear system if the noise bounds are small enough. A second result showed that the original linear test signal can be used if the nonlinearities satisfy appropriate bounds. In addition,
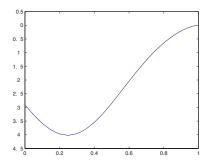

Fig. 5. Computed optimal test signal for Example 2.

we have given a direct optimization formulation suitable for highly nonlinear systems. Computational examples show that this can sometimes produce much smaller test signals then when using linearizations and also can sometimes find test signals for problems for which the linearized approach fails.

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