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Filtering of Interaction Rules in Cooperation

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Abstract— **Many cooperative control strategies focus on stability concerns to the deficit of global structure in a distributed cooperation task. We present a method of ensuring provable convergence of decentralized switching systems using ad-hoc definitions of proximity graphs, where convergence is measured by a potential function defined on the graph. The method depends on the proper filtering of the time-varying proximity graph structure so as to maintain convergence characteristics. We demonstrate the approach with an underactuated system.**

I. Introduction

We present a technique that uses a locally-defined energy function that leads to a guarantee that a potential globally defined for the system converges for the coordinated control algorithm in the face of arbitrarily changing proximity graph structures. The adaptive control system weighs the changing network with internal models of the system in order to provide converging responses with acceptable performance. This is accomplished through an internal estimate of the stability margin along with the use of consensus for purposes of performance improvement. Depending on the controller chosen, this can be achieved using linear controllers with quadratic Lyapunov functions.

This work is in contrast to other approaches that focus on nonsmooth analytical approaches to showing stability and convergence. The advantage of the approach we show here is that it does not depend on any particular proximity graph structure, and correspondingly does not depend on any particular potential function for potential-based methods. Proofs of stability have been produced for such systems (e.g., [1], [2]), but typically these proofs impose constraints on the dynamics of the system and the proximity graph. For example, the results in [1] apply only to a specific potential function on the unit-disk graph, and the results in [2] apply to another particular potential function on a Voronoi graph. The difficulty associated with these prior works is that the stability results leave little room for task specification; tasks must be framed in terms of what can be achieved in a stable manner and may therefore be limited to stable area coverage or "flocking" through a series of obstacles. Moreover, the task specification will likely change over time, thus introducing discrete changes into the equations of motion. Finally, heuristics that are not easily combined with these approaches are often helpful for various tasks, such as collision avoidance and other safety-critical elements of the task specification. The key point is that the control mechanism should dictate task specification to the minimum extent possible.

To this end, we have developed a coordinated control algorithm that operates in the following context. Assume that a graph G can be computed given the state of every agent and that the goal is to minimize this potential. For instance, such a potential could be a measure of deviation from a desired distance between two agents, in which case minimizing the potential corresponds to successfully creating a formation. However, allowing arbitrary switching in the graph G due state changes–and corresponding changes in the control–can potentially lead to instability. Filtering the changes in the graph in such a way that convergence of the state-dependent potential is ensured avoids this issue. This is the topic of the present paper.

II. Motivating Example: Connectivity Using the Gabriel Graph

Let R be the set of agents. Let $\mathcal{G}$ be the set of graphs over the vertices R. Let the *sensor graph* G_S be a graph where R is the vertex set, and there is an edge (or "link") between two vertices r_1 and $r_2 \in R$ iff agents r_1 and r_2 can both sense each other. Let the *control graph* (also referred to as the *neighbor graph*) G_N be a graph where R is the vertex set, and there is an edge between two vertices r_1 and $r_2 \in R$ iff agents r_1 and r_2 are interacting for control purposes. To simplify notation, we will understand S to be the edge set of G_S, N to be the edge set of G_N, and S_i and N_i to the corresponding set for a given agent i. The graph G_N will be defined by a time-varying switching function σ, which we will describe in terms of a graph construction algorithm. Note that N (the set of neighbors used in the control law) is necessarily a subset of S (the set of sensed neighbors).

A Gabriel graph [3], [4], [5] is a graph $G_N(\mathbf{x}(t))$ that dictates which data is incorporated into the control laws. There is a link between agents A and B if and only if for all other agents Z, the interior angle $\angle AZB$ is acute. Equivalently, there is a link between agents A and B iff there are no other agents within the circle with diameter $\overline{AB}$.

The Gabriel graph switching function provides many advantages; chief among these is provable connectivity of the graph [4]. The Gabriel graph is also well-suited to providing uniform coverage of an area, as it creates a mesh of acute triangles. The Gabriel graph is a planar graph [4], so it does not suffer from high edge density when the agents are close together. However, the Gabriel graph depends on links being created with *non-zero virtual potential*; that is, the potential function defined on the graph will generally jump up when an edge is generated between two vertices i and j. This complicates any proof of stability, as virtual energy may be added to the system as the topology (and therefore the control law) changes. Even for linear, stable systems, arbitrary switching can lead to instability. This is what we wish to avoid in a completely generic manner.

In our approach to showing convergence, instead of computing a limit on the switching frequency explicitly, we use a notion of a global "energy reserve" (first introduced in [6], [7]) to create a convergence-guaranteeing limiting effect on the switching rate. (The idea behind this name is that if a switch will increase the value of the Lyapunov function, there must be enough energy reserve to compensate for this increase.) We find this approach intuitive and moreover straightforward to implement in our distributed scenarios, in which switching events are detected locally. Although any global quantity can be problematic, we will demonstrate that a local estimate of this quantity based upon a zero sum consensus algorithm is sufficient to establish convergence.

III. Problem Description

The general problem we wish to address is how one takes a control law calculated using a proximity graph definition that depends on the state and implements it on an underlying dynamic, cooperative system. The translation into physical actions should take into account stability, stability margin, and correctness (in terms of successfully completing a command or returning a failure result). In this paper we focus on stability (at least in the classical linear systems sense of bounded input bounded output stability). We address the issue of margin as a natural byproduct of how we solve the stability problem. We do not address correctness, though it is an important problem. However, the method presented here allows one to specify arbitrary proximity graph rules, hence potentially moving the correctness question into the graph design domain.

The primary difficulty is that a control $u(G, \mathbf{x})$ (where $G \in \mathcal{G}$ is a graph and $\mathbf{x}$ is the state) that has no information about low-level convergence characteristics may have to be *modified* to preserve convergence. Hence, we will require a mapping $\begin{array}{rcl}\chi : \mathcal{G} & \rightarrow & \mathcal{G} \\ G & \mapsto & G_E\end{array}$ that maps a *desired* proximity graph G to a stably implementable proximity graph G_E. An example of such a motivating scenario is discussed in the next section.

We would like to have a system that has provable high-level properties (e.g., connectivity of the network topology) while maintaining low-level characteristics such as stability (of the physical system), stability margin, and performance metrics. The basic approach is to translate the proximity graph G to an alternative G_E by using a mapping χ that is essentially a dynamically updated guard condition that protects the stability of the system. Hence, χ may be thought of as a means of "filtering" the effects of G based on a stability condition.

We show here that if one has a stable cooperative system for each possible network state G_N, then one may use an adaptive control strategy to guarantee stability in a decentralized manner. In particular, the idea is to associate with each agent i a value E_i which is defined as the solution to the following differential equation, where E_i has an arbitrarily chosen nonnegative initial value.

$$\begin{aligned} \dot{E}_i(t) &= -k_e d_i(t) \text{ if no switch in graph occurs} \\ E_i(t) &= \lim_{\tilde{t} \to t^-} E_i(\tilde{t}) - \Delta V \text{ otherwise} \end{aligned}$$

where $k_e > 0$ is the same constant for all agents, $0 < k_e < 1$ (this will be formalized shortly in Eqs. (5) and (6)). The value d_i is a local conservative estimate of the stability margin of the system, and is critical to maintaining stability. The value E_i is initialized to a nonnegative value and then evolves according to Equation 5 as long as the network topology is not changing. Whenever there is a switch, E_i is re-initialized to the value given in Equation 6 by subtracting ΔV. We call E_i the *local energy reserve*, and it should be thought of as a local estimate of stability margin relative to the *hybrid* system. (Moreover, we will see that replacing E_i with an estimate of E_i can be shown to provide global stability so long as the estimate is conservative.)

This brings us to the simple change necessary to stabilize the system. The modified control $u(\chi(G_N), \mathbf{x}(t))$ is identical to $u(G_N, \mathbf{x}(t))$, except for the added condition that any switch in the control graph that would cause $\hat{E}_i < 0$ is prohibited. This result provides guaranteed convergence, and one is guaranteed to eventually be able to implement any graph G. It is worth noting that the evolution of E is only used in the calculation of χ–it doesn't affect which controls u are admissible for the system. Additionally, this computation is decentralized: agents only need access to local values E_i, d_i, and local estimates of changes in the Lyapunov functions as the network topology changes.

The key idea is that we are using the evolution of the energy reserve E_i to systematically *block* changes in proximity graph if they will lead to instability (that is, χ blocks new graphs until stability can be ensured). However, *typically* $u(\chi(G), x) = u(G, x)$ in systems that do not have aggressive controller gains [6]. Hence, χ, though a conservative approach to preserving stability, often does not come into play.

IV. General Result

Consider a set of agents R and a time-varying switching signal $\begin{array}{rcl}\sigma : \mathbb{R} & \rightarrow & \mathcal{G} \\ t & \mapsto & G_N\end{array}$ that determines the proximity graph and is constant except for discrete changes at times $t_1 .... t_n$ on the interval $[t_0, t_f]$. Assume that the state for each agent i is $\mathbf{x} \in M$, the governing equations are $\dot{\mathbf{x}} = f(\mathbf{x})$, and that the switching function changes f over time, $\sigma : (\mathbf{x}, t) \longrightarrow f$. The equations of motion of interest are as follows:

$$\begin{aligned}\ddot{\mathbf{x}}_i &= u_i \quad (1)\\ u_i &= \begin{cases} -\dot{\mathbf{x}} & \tau(\sigma) < T \\ \overline{u}_i(\chi(G), \mathbf{x}(t)) & \tau(\sigma) \geq T \end{cases} \quad (2)\end{aligned}$$

where $\overline{u}_i$ stabilize $\mathbf{x}$ for each choice of G, $\tau(\sigma)$ is the length of time since the last change in the network topology N_i, and T is a time-delay before u can decrease the Lyapunov function. The filter χ will be defined shortly. We assume that for each time interval $(t_j...t_{j+1})$ (we will call this interval τ_j), there exists a global potential function $\mathbf{V}_{\sigma(\tau_j)}$ such that $\mathbf{V}_{\sigma(\tau_j)}$ is positive-definite, $\dot{\mathbf{V}}_{\sigma(\tau_j)}$ is negative semi-definite, and $\ddot{\mathbf{V}}_{\sigma(\tau_j)}$ is bounded. (This is satisfied, for instance, under the conditions on the graph Laplacian discussed in [1].) We define the overall potential function $\mathbf{V}_{\sigma(t)}$ to be equal to $\mathbf{V}_{\sigma(\tau_j)}$ on the interval $(t_j...t_{j+1})$, for all j.

Define the quantity s_i such that:

$$s_i(t) = \frac{1}{2}\left(\sum_{j\in N_i}\left(\lim_{t\to\tilde{t}^+} P(\mathbf{x}_i, \mathbf{x}_j) - \lim_{t\to\tilde{t}^-} P(\mathbf{x}_i, \mathbf{x}_j)\right)\right), \quad (3)$$

where $P(\mathbf{x}_i, \mathbf{x}_j)$ is the potential between agent i and j. Moreover, each agent can determine an estimate $\hat{s}_i$ such that $\sum_{i\in R}\hat{s}_i \geq \sum_{i\in R} s_i$ (often for our purposes $\hat{s}_i = s_i$). This quantity captures the instantaneous change in potential due to the link switching. The factor of 1/2 is present because each link connects to two agents, and thus will be counted twice. It is thus easy to show that the following holds:

$$\sum_{i\in R} s_i = \lim_{t\to\tilde{t}^+}(\mathbf{V}_{\sigma(t)}) - \lim_{t\to\tilde{t}^-}(\mathbf{V}_{\sigma(t)}) \quad (4)$$

Associate with each agent i a value E_i which is called the *local energy reserve*, and is defined as the solution to a differential equation. E_i has an arbitrarily chosen nonnegative initial value and evolves according to the following:

$$\dot{E}_i(t) = \begin{cases} 0 & \text{if } s_i(t) = 0 \text{ and } \tau(\sigma) < T \\ -k_e d_i(t) + w_i & \text{if } s_i(t) = 0 \text{ and } \tau(\sigma) \geq T \end{cases} \quad (5)$$

$$E_i(t) = \lim_{\tilde{t}\to t^-} E_i(\tilde{t}) - s_i(t) \text{ otherwise} \quad (6)$$

where k_e is a global constant, $0 < k_e < 1$ and $\sum_i w_i = 0$ (which will show up as a zero-sum consensus [8] term later). Notice that E_i is initialized to a nonnegative value and then evolves according to Equation 5 as long as s_i is zero (that is, on intervals with no switches). Whenever $s_i \neq 0$ (there is a switch), E_i is re-initialized to the value given in Equation 6.

Each agent maintains a local estimate $\hat{E}_i$, which is initially greater than zero and evolves as follows:

$$\dot{\hat{E}}_i(t) = \begin{cases} 0 & \text{if } \hat{s}_i(t) = 0 \text{ and } \tau(\sigma) < T \\ -k_e d_i(t) + w_i & \text{if } \hat{s}_i(t) = 0 \text{ and } \tau(\sigma) \geq T \end{cases} \quad (7)$$

$$\hat{E}_i(t) = \lim_{\tilde{t}\to t^-} \hat{E}(\tilde{t}) - \hat{s}_i(t) \text{ otherwise} \quad (8)$$

Let the global values E and $\hat{E}$ be defined such that

$$E = \sum_{i\in R} E_i \quad (9)$$

$$\hat{E} = \sum_{i\in R} \hat{E}_i \quad (10)$$

We will call E the *global energy reserve*.

This brings us to the graph filter definition that provides convergence, defined by

$$\chi(G(t), \mathbf{x}(t), t) = \begin{cases} G & \text{if } \hat{E}_i > 0 \\ \lim_{\tilde{t}\to t^-} \chi(G(t), \mathbf{x}(t), \tilde{t}) & \text{otherwise.} \end{cases}$$

The filter χ is an identity on $\mathcal{G}$, except for the added condition that any switch that would cause $\hat{E}_i < 0$ for any agent i is prohibited. Note that the value of $\hat{E}_i$ cannot decrease in the absence of switching if $d_i \leq 0$ for all i (this can be thought of as a conservative estimate of the stability margin of the system for a graph at time t). Also, this computation is decentralized; the agents only need access to the local values E_i, d_i, and s_i.

The immediate consequence of modifying σ in this way is that $\hat{E} \geq 0$, since it is the sum of all nonnegative terms. It follows from Equations 9 and 10 and the definitions of s_i and $\hat{s}_i$ that $E \geq \hat{E}$. Thus if $\hat{E} \geq 0$, then $E \geq 0$ as well. This allows us to prove the following statement.

Theorem 4.1: The states in the system in Eq (1) and (2) all converge to a state of of unchanging potential for any sequence of graphs $G(t)$.

Proof: For purposes of notational simplicity, we will take $\mathbf{V}$ to denote $\mathbf{V}_{\sigma(t)}$ for the remainder of this section unless otherwise specified.

We start with the case $T = 0$ and then adapt accordingly for the case $T \neq 0$. On a time interval going from t_0 to t_f, let the proximity graph $G(t)$ change at times t_i. We will show stability of the system using the function $\mathbf{W}$, defined as:

$$\mathbf{W} = \mathbf{V} + E$$

The function $\mathbf{V}$ is positive-definite on any interval (t_i, t_{i+1}) by the assumption that for each static choice of G the system is stable with negative semi-definite derivative. Moreover, since $E > 0$ by definition, it is clear that $\mathbf{W} \geq 0$. Differentiating, we see that on any interval (t_i, t_{i+1}) on which there are no switches:

$$\dot{\mathbf{W}} = \dot{\mathbf{V}} + \dot{E}.$$

Note that

$$\dot{E} = \sum_{i\in R}\dot{E}_i = \sum_{i\in R} -k_e d_i + w_i = \sum_{i\in R} -k_e d_i \quad (11)$$

because of the zero-sum property. Substituting for $\dot{E}$ (with $T = 0$) gives:

$$\dot{\mathbf{W}} = \dot{\mathbf{V}} + \sum_{i\in R} -k_e d_i \leq 0, \quad (12)$$

so $\dot{\mathbf{W}}$ is negative definite on the interval (t_i, t_{i+1}) $\forall i$.

To address the times t_i, we must look back to the definition of s_i:

$$\lim_{\tilde{t}\to t^+} \mathbf{V}(\tilde{t}) = \lim_{\tilde{t}\to t^-} \mathbf{V}(\tilde{t}) + \sum_{i\in R} s_i(t).$$

Thus, at any instant t when a switch occurs (that is, when any $s_i \neq 0$),

$$\lim_{\tilde{t}\to t^+} \mathbf{W}(\tilde{t}) = \lim_{\tilde{t}\to t^-} \mathbf{V}(\tilde{t}) + \sum_{i\in R} s_i(t) + E(t)$$

Substituting for E from Equation 6,

$$\lim_{\tilde{t}\to t^+} \mathbf{W}(\tilde{t}) = \lim_{\tilde{t}\to t^-} \mathbf{V}(t) + \sum_{i\in R} s_i(t) + \lim_{\tilde{t}\to t^-} E(\tilde{t}) - \sum_{i\in R} s_i(t) \tag{13}$$

which simplifies in the following way:

$$\begin{aligned} \lim_{\tilde{t}\to t^+} \mathbf{W}(\tilde{t}) &= \lim_{\tilde{t}\to t^-} \mathbf{V}(t) + \lim_{\tilde{t}\to t^-} E(\tilde{t}) \\ &= \lim_{\tilde{t}\to t^-} \mathbf{W}(\tilde{t}) \end{aligned} \tag{14}$$

Thus, the discontinuity in $\mathbf{W}$ has been removed, as the limits from both sides are the same. Further, since $\dot{\mathbf{V}}$ is negative semi-definite by assumption, $0 < k_e < 1$, and $\dot{\mathbf{V}} < \sum_{i\in R} k_e d_i \leq 0$, it must be the case that $\dot{\mathbf{W}}$ is negative semi-definite.

We now follow the proof of Barbalat's lemma [9], which states that if $f(t)$ is lower bounded, $\dot{f}(t)$ is negative semi-definite, and $\dot{f}(t)$ is uniformly continuous (or equivalently, $\ddot{f}(t)$ is finite), then $\dot{f}(t)$ approaches zero as t approaches infinity. Unfortunately, Barbalat's Lemma as stated does not apply to our system because at the times t_i, the function $\dot{\mathbf{W}}(t_i)$ discontinuous. However, these discontinuities are separable, allowing the basic result to still hold. The true generalization of Barbalat's lemma requires the technical use of meagre functions [10], which are heavy machinery for what (for our purposes) is a reasonably straight forward result. We will show that $\dot{\mathbf{W}} \to 0$ as $t \to \infty$. To see this, suppose $\dot{\mathbf{W}}$ did not go to zero as as $t \to \infty$. Then there exists a sequence of times $t_n \to \infty$ such that $|\dot{\mathbf{W}}| > \epsilon\ \forall\ n \in \mathbb{N}$. On all intervals $[t_i, t_{i+1})$ $\ddot{\mathbf{W}}$ is bounded, so on these intervals $\dot{\mathbf{W}}$ is uniformly continuous. Because of this, there exists a δ such that $|t_n - t| < \delta \Rightarrow |\dot{\mathbf{W}}(t_n) - \dot{\mathbf{W}}(t)| \leq \epsilon/2$ on any interval that does not include t_i. We know that $\dot{\mathbf{W}}$ is integrable by the existence of $\mathbf{W}$ (which is bounded below by 0 and above by $W(0)$), which means that the quantity $|\int_0^{t_n+\delta} \dot{\mathbf{W}} dt - \int_0^{t_n+\delta} \dot{\mathbf{W}} dt| \to 0$ as $n \to \infty$. Hence, $|\int_{t_n}^{t_n+\delta} \dot{\mathbf{W}} dt| \to 0$ as $n \to \infty$, which implies that $\int_{t_n}^{t_n+\delta} |\dot{\mathbf{W}}| dt \to 0$ as $n \to \infty$ (since $\dot{\mathbf{W}} \leq 0$). Now, if $t_i \in [t_n, t_n+\delta]$, the value of $\dot{\mathbf{W}}(t_i)$ does not affect the integral since $\dot{\mathbf{W}}(t_i) \in co\{\lim_{\tilde{t}\to t_i^+} \dot{\mathbf{W}}(\tilde{t}), \lim_{\tilde{t}\to t_i^-} \dot{\mathbf{W}}(\tilde{t})\}$. Hence, $\int_{t_n}^{t_n+\delta} |\dot{\mathbf{W}}| dt = \int_{t_n}^{t_n+\delta_{t_i}} |\dot{\mathbf{W}}| dt + \int_{t_n+\delta_{t_i}}^{t_n+\delta} |\dot{\mathbf{W}}| dt \geq \frac{\epsilon\delta_{t_i}}{2} + \frac{\epsilon(\delta-\delta_{t_i})}{2} = \frac{\epsilon\delta}{2}$. This contradicts the convergence of the Riemann integral and therefore contradicts the integrability. Hence, $\dot{\mathbf{W}} \to 0$ as $t \to \infty$. It follows directly (since $\dot{E} \to 0$ as $t \to 0$) that $\dot{\mathbf{V}}_{\sigma(t)} \to 0$ as $t \to \infty$. That is, all agents reach a state of unchanging potential.

To address the case where $T \neq 0$, we simply need to confirm that $\dot{W} \leq 0$ when using $u_i(\chi(G_N(t)), \mathbf{x}(t))$. Again, we have

$$\mathbf{W} = \mathbf{V} + E > 0$$

and

$$\dot{\mathbf{W}} = \dot{\mathbf{V}} + \dot{E}.$$

Hence, we need to evaluate $\dot{E}$ and $\dot{\mathbf{V}}$.

Now consider a time interval $[t_j ... t_{j+1})$. First, let $T < t_{j+1} - t_j$. Then $\dot{E} = 0$ on $[t_j ... t_j + T)$ and $\dot{E} = \sum_{i\in R} -k_e d_i$ on $[t_j + T ... t_{j+1})$, hence $\dot{E} \leq 0$ on the complete interval. Under these same conditions, $\dot{\mathbf{V}}$ a new Lyapunov function has derivative $-\dot{\mathbf{x}}^T \dot{\mathbf{x}}$ when $t \in (t_j, t_j + T)$ and $\dot{\mathbf{W}} = \sum_{i\in R}(1 - k_e) d_i$ when $t \in (t_j + T, t_{j+1})$. Therefore, on this interval a different $V(t)$ (corresponding to the Euclidean norm on $\mathbb{R}^2$) is decreasing. In the case that $T \geq t_{j+1} - t_j$, $\dot{E} = 0$ on that interval. We simply consider the next time t_i such that $t_j + T \in (t_i, t_{i+1})$. Then, by the logic just given, we have $\dot{\mathbf{W}} \leq 0$ and are done. ■

Note that in the proof of Theorem 4.1 we are effectively changing both at what time changes in G are allowed to occur and potentially *if* they are allowed to occur if the delay T is too large. However, it is important to notice that network dropouts do not affect the analysis; a link can always be lost because that will only decrease potential energy associated with the control, but it may not be possible to add it back in. If a communication is re-established, the link still may not be added back into the control graph; thus, it is possible to control the switch in the positive direction. In general, it is necessary to define systems such that uncontrollable events cannot increase the overall potential.

Now we may state the algorithm for ensuring convergence in the face of arbitrary time-varying proximity graph topologies.

Algorithm for Filtering Proximity Graphs

Given a proximity graph $G(\mathbf{x}(t))$:

1) Choose a set of initial values $\hat{E}_i$ greater than zero;
2) Update $\hat{E}_i$ using Eq. (7) and (8);
3) Apply χ to $G(\mathbf{x}(t))$ using $\hat{E}_i$;
4) Calculate the control law $\overline{u}$ using $\chi(G(\mathbf{x}(t)))$.

Note that the algorithm is completely decentralized and only adds one state ($\hat{E}$) to each vehicle that needs to be maintained.

V. Example: Connected Target Tracking with Underactuated Dynamics Using Filtered Gabriel Graphs Interactions

We now introduce an example that takes advantage of Thm. 4.1. We assume we have each agent i with the normalized nonholonomic vehicle dynamics:

$$\begin{array}{rclrcl} \dot{x}^i &=& v_x^i & \dot{v}_x^i &=& \cos(\theta) u_1^i \\ \dot{y}^i &=& v_x^i & \dot{v}_y^i &=& \sin(\theta) u_1^i \\ \dot{\theta}^i &=& v_\theta^i & \dot{v}_\theta^i &=& u_2^i \end{array} \tag{15}$$

and control laws defined in the next sections. We will show in detail how the hybrid filtering works for this system, first generating the potentials for target tracking, collision avoidance, and the Gabriel graph itself. We will assume that the control has the following structure:

$$
\begin{aligned}
u_1^i &= \begin{cases} \overline{u}_i & cond < \epsilon \\ -\dot{x} - \dot{y} & \text{else} \end{cases} \\
u_2^i &= \begin{cases} \frac{d}{dt}\arctan(\dot{y}, \dot{x}) & cond < \epsilon \\ -K_\theta(\theta(t) - \angle(\sum_j \nabla P(\mathbf{x}_i, \mathbf{x}_j)) & \text{else} \end{cases}
\end{aligned}
$$

where $cond = |\theta(t) - \angle(\sum_j \nabla P(\mathbf{x}_i, \mathbf{x}_j)|$. This control ensures that the vehicle turns when it is not oriented properly and, when it is, it follows the Gabriel graph control laws. To generate $\overline{u}$, we create separate potentials for target tracking, collision avoidance, and our ad-hoc proximity graph of choice, the Gabriel graph itself.

Although the decision to prohibit a switch is made by each agent based on its local energy reserve, it may be desirable to allow switches to occur whenever the *global* energy reserve is sufficiently large. That is, we do not want to prevent a switch due to low energy reserves in one part of the system, when there are sufficient energy reserves unused somewhere else. Thus, we need some mechanism for sharing information about the energy reserve levels between agents.

We will take advantage of the *average-consensus* algorithm described by Olfati-Saber and Murray [8]. This algorithm allows a distributed set of agents to reach a consensus on a common global value, while sharing information only with their local neighbors. If an agent i has a set of neighbors S_i that it can sense,

$$\bar{u}_i = \sum_{l \in S_i} (E_l - E_i). \tag{16}$$

The system evolves somewhat differently, as the times when we must prohibit a switch have changed due to the differing *local* values of E, but the system meets all the conditions necessary for the proof in Section IV (in Eq.(11)) because the *global* behavior of E still has the required properties. However, as described in [8], all of the local energy reserves will now converge to a single value.

The consensus function [8] is just one example of a valid consensus function. In fact, any consensus algorithm with the zero-sum property is acceptable, as is clear from the proof of Thm. 4.1. The consensus on E is independent of the normal control of the system, although a faster consensus will improve performance in terms of convergence rate.

To generate a control law from the Gabriel Graph set of neighbors for a given agent i, we choose the following:

$$\overline{u}_i = \big[\sum_{j \in N_i} k_s(\|\mathbf{x}_i - \mathbf{x}_j\| - l_0)\hat{\mathbf{v}}_{ij}\big] - k_d\dot{\mathbf{x}}_i$$

where $\mathbf{x}_i$ represents the Cartesian coordinates describing the agent's position, $\ddot{\mathbf{x}}_i$ is the agent's acceleration, $\dot{\mathbf{x}}_i$ is the agent's velocity, N_i is the set of links connected to this agent, and $\hat{\mathbf{v}}_{ij}$ is the unit vector from agent i to agent j. Control constants are the natural length (l_0), the stiffness (k_s), and the damping coefficient (k_d). We require that the system be symmetric: if an agent a has a link connected to agent b, then agent b must have a link connected to agent a.

For each interval (t_i, t_{i+1}) between switches, the potential function is:

$$\mathbf{V}_{\sigma(\tau_j)} = \sum_{i \in R} \big[\sum_{j \in N_i} P(\mathbf{x}_i, \mathbf{x}_j) + \dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i\big].$$

Since P is conservative (in this case a quadratic function), it can be shown that:

$$\dot{\mathbf{V}}_{\sigma(\tau_j)} = \sum_{i \in R} -k_d \dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i$$

and hence we simply let:

$$d_i = -k_d \dot{\mathbf{x}}_\mathbf{i}^\mathbf{T} \dot{\mathbf{x}}_\mathbf{i}.$$

We define s_i such that:

$$s_i = \sum_{j \in N_i^+} P(\mathbf{x}_i, \mathbf{x}_j) - \sum_{j \in N_i^-} P(\mathbf{x}_i, \mathbf{x}_j)$$

where N_i^+ represents the limit of N_i from the right, and N_i^- represents the limit of N_i from the left. Lastly, we allow $\dot{E}_i$ to evolve as in Eqs. (5) and (6).

While our proof based on Barbalat's lemma is convenient for smooth potentials, it is not the only technique that is compatible with the energy reserve approach. For example, consider the work of Tanner et. al. in [11]. A control input u and Lyapunov function $\mathbf{V}$ are presented (we have changed the notation slightly to match the conventions used here):

$$u_i = -\sum_{j \in N_i} (\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j) - \sum_{j \in N_i} \nabla P(\mathbf{x}_i, \mathbf{x}_j)$$

where P is some potential function that approaches infinity as $\mathbf{x}_i$ approaches $\mathbf{x}_j$, and has a unique minimum when agents i and j are at a desired distance. N_i is the set of neighboring agents within some threshold distance of agent i. The Lyapunov function is then

$$\mathbf{V} = \frac{1}{2} \sum_{i \in R} \big[\sum_{j \in N_i} P(\mathbf{x}_i, \mathbf{x}_j) + \dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i)\big]$$

and

$$\dot{\mathbf{V}} = \dot{\mathbf{x}}^T L \dot{\mathbf{x}}$$

where L is the Laplacian of the neighbor graph G_N.

It is simple to add an energy reserve to $\mathbf{V}$, with $d_i = \dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j$. This modifies the Lyapunov function as shown:

$$\mathbf{V} = \frac{1}{2} \sum_{i \in R} \big[\sum_{j \in N_i} P(\mathbf{x}_i, \mathbf{x}_j) + \dot{\mathbf{x}}_i^T \dot{\mathbf{x}}_i)\big] + E$$

$$\dot{\mathbf{V}} = (1 - k_e)\dot{\mathbf{x}}^T L \dot{\mathbf{x}}$$

This change carries through the rest of the analysis. The results in [11] are preserved with the addition of an energy reserve, which allows for more flexibility in specifying a switching function.

For some systems, using the modified switching function may have implications for collision avoidance. If its energy reserve is depleted, an agent may not allow a switch that is necessary in order to prevent a collision. However, it is possible (and fairly straightforward) to design a system that

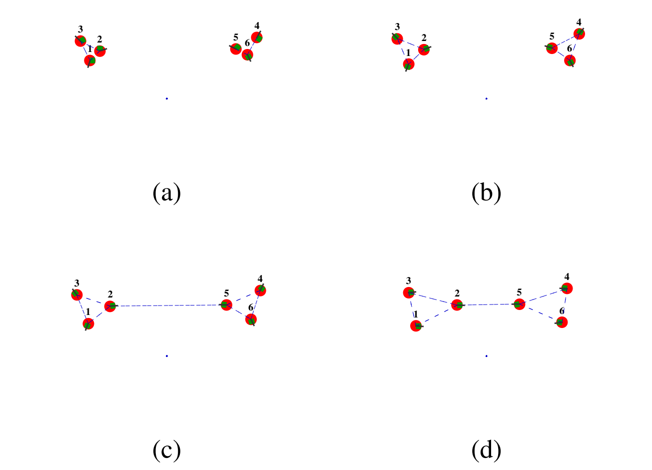


Fig. 1. Simulation of Gabriel graph maintaining connectivity. (Agent orientation is indicated by the small black line pointing in the direction the agent is facing.) Figures (a) and (b) shows links being created between nearby agents, but the long link connecting the two groups is being filtered by χ until Figure (c). After they have connected, they come closer together in (d). Movies of this simulation may be found at *http://robotics.colorado.edu/AMAI2007*.

does not depend on switching for collision avoidance. For example, consider the following control law:

$$\overline{u}_i = \Big[\sum_{j \in N_i} \nabla P_1(\mathbf{x}_i, \mathbf{x}_j)\Big] + \Big[\sum_{k \in R} \nabla P_2(\mathbf{x}_i, \mathbf{x}_k)\Big] - k_d \dot{\mathbf{x}}_i$$

where N_i is the set of neighbors according to some relation (such as a Gabriel graph), and R is the set of all agents. Suppose that P_1 and P_2 are both conservative functions, and that $P_2(\mathbf{x}_i, \mathbf{x}_k)$ approaches infinity as $\mathbf{x}_i$ approaches $\mathbf{x}_k$. It may be the case that P_2 is a "short-range" potential–it rapidly becomes small as the distance between the agents increases.

Similar to our previous examples, this system satisfies all of the requirements for Theorem 4.1. In addition, since P_2 affects all pairs of robots at all times, no collision can occur without overcoming an infinite potential. (We choose $P_2(\mathbf{x}_i, \mathbf{x}_j) = 1/\|r_{ij}\|$ for purposes of simulation, where r_{ij} is the distance from agent i to agent j.) A continuity argument such as that given in [11] is adequate for showing that collisions are avoided.

It should be noted that some care must be taken to ensure that a collision-avoidance term does not cause unintended consequences. For example, a poorly-chosen control law may avoid collisions but allow undesired local minima in the potential function. While terms such as P_2 do not affect our ability to cause convergence, they may alter system performance.

A. Simulation of Filtered Gabriel Graph interactions with Underactuated Dynamics

A group of six mobile agents with dynamics in Eq. (15) are given initial conditions in the region $(-600, 600) \times (-200, 600)$ cm such that two groups of three are substantially separated, as seen in Fig. 1(a). The Gabriel graph structure dictates that a link must be established between the two groups if they are within sensing range of each other. However, first the two groups separately create connections separately (seen in Fig. 1(b)) as the Gabriel graph will inject too much energy into the system to be able to guarantee convergence. Then, after the energy reserve condition is met, the connection between the two groups is made in Fig. 1(c), after which the two groups converge together, as seen in Fig. 1(d).

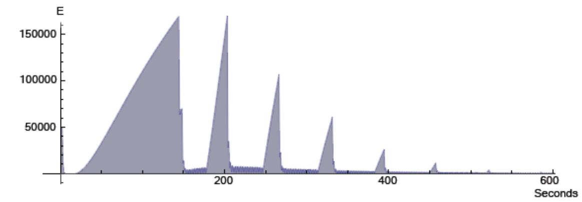


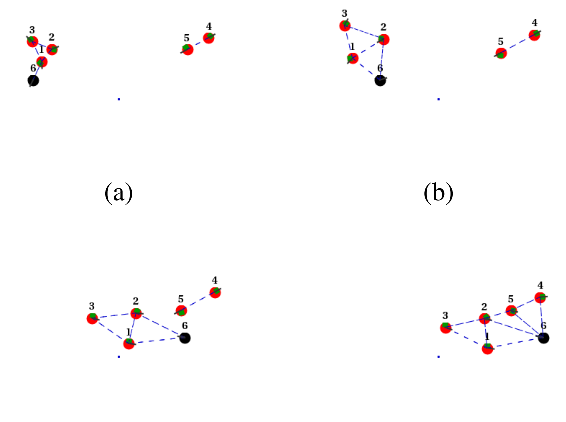
Fig. 2. The energy reserve E for agent 3 versus time. Drops in the value indicate that there is enough energy reserve to incorporate a new link with another agent.

Figure 2 shows the energy reserve for agent 3. Near time zero, connections are made to both agents 1 and 2, substantially increasing the potential energy. Then the energy reserve remains flat while the vehicle re-orients itself. Then, it must wait until the first drop is seen to create the link between the two groups. Without the consensus algorithm in Eq. (16), this could have taken longer to take place, but it would have nevertheless eventually created the link. This simulation was additionally performed on a hardware test-bed the Roomba robotic vacuum cleaner manufactured by iRobot as a base. This provides for an inexpensive hardware platform that can integrate sensors and distributed computing. The Roomba measures 32cm across and uses a simple differential drive system. (Hence, its dynamics are underactuated as in Eq.(15).) We are using a Linux-based board (the TS-7260 from Technologic Systems) as the deployed computer. We use wireless for broadcasting configuration data to neighbors. The experimental results are, perhaps not surprisingly, nearly identical to those seen in Fig.1.

B. Potentials and Simulations for Target Tracking

Consider a system in which there are potentials between the agents, as well as between the agents and targets in the environment. It may be the case that targets can appear, disappear, change position, and/or change characteristics in such a way as to inject large amounts of virtual energy into the system. Our technique can be applied to prevent destabilization of the system due to target behavior or false generation of targets.

For example, let R be a set of agents and T a set of targets, each of which may appear and disappear arbitrarily. Let the



(a) (b) (c) (d)

Fig. 3. Simulation of Gabriel graph tracking an intruder. This is similar to Fig.1, but here we see that agent 5 is unable to track the intruder (agent 6) immediately. Nevertheless, agents 1,2,4 and 5 are eventually tracking the intruder convergently. Movies of this simulation may be found at *http://robotics.colorado.edu/AMAI2007*.

control law for agent i be the following:

$$\overline{u}_i = \Big[\sum_{j\in N_i} \nabla P_1(\mathbf{x}_i, \mathbf{x}_j)\Big] + \Big[\sum_{k\in N} \nabla P_2(\mathbf{x}_i, \mathbf{x}_k)\Big] + \Big[\sum_{k\in T} \nabla P_T(\mathbf{x}_i)\Big] - k_d \dot{\mathbf{x}}_i$$

where $P_1 = \|r_{ij}\|$ is the potential function acting between the agents due to the Gabriel graph, $P_2 = 1/\|r_{ij}\|$ is the potential function acting between the agents for purposes of collision avoidance, and P_T is the potential function acting between agents and targets.

If there are no restrictions on the appearance of targets, then targets may inject an arbitrary amount of energy into the system. This is not desirable, as the continued appearance of targets, or the appearing and disappearing of a few targets in an unfortunate pattern, could destabilize the system and/or cause collisions between the agents. Modifying the switching function according to our technique will remove this problem.

In the simulation seen in Fig. 3, a group of five mobile agents with dynamics in Eq. (15) are given initial conditions in the region $(-600, 600) \times (-200, 600)$ cm such that two groups are substantially separated, as seen in Fig. 3(a) and another vehicle (agent 6) is passing through them that needs to be tracked. Initially, only agent 1 tracks agent 6, but soon agent 2's energy reserve is large enough for it to track agent 6 as well (seen in Fig. 3(b)). Because of the rapid switching that occurs for agent 5 between creating a link with agent 2 and agent 6, agent 5 must wait longer to make these connections (seen in Fig. 3(c)). At the end, agent 6 has been handed off to agents 2,4, and 5 successfully, as seen in Fig. 3(d).

VI. Conclusions

In this paper we have introduced an approach to cooperative control that focuses on monitoring and filtering the admissible changes in network graph topology used in a cooperative control law according to a stability criterion. This method can be distributed across a network of agents by additionally using consensus algorithms like those found in [8]. This approach leads to a flexible method of guaranteeing convergence for arbitrary network graphs, and explicitly avoids instabilities due to the graph topology switching.

The results presented here allow one to use arbitrarily chosen proximity graph definitions in the control law specification, which allows more flexibility in task specification. Moreover, the presented approach can be adapted to hierarchical heterogeneous systems almost without modification [12], where there are different types of agents with different priorities. However, this work only addresses convergence under arbitrarily switching graph structures, not configuration stability (which will be a focus of ongoing research).

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