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Consensus Tracking under Directed Interaction Topologies: Algorithms and Experiments

Wei Ren

Abstract— **Consensus tracking problems with, respectively, bounded control effort and directed switching interaction topologies are considered when a time-varying consensus reference state is available to only a subgroup of a team. A consensus tracking algorithm explicitly accounting for bounded control effort is proposed and analyzed under a fixed directed interaction topology. Furthermore, convergence analysis for a consensus tracking algorithm is provided when the time-varying consensus reference state is available to a dynamically changing subgroup of the team under directed switching inter-vehicle interaction topologies. Experimental results of a formation control application are demonstrated on a multi-robot platform to validate one of the proposed consensus tracking algorithms.**

I. INTRODUCTION

Cooperative control often requires that individual vehicles share a consistent view of the objectives and the world. Consensus algorithms guarantee that vehicles sharing information have a consistent view of information that is critical to the coordination task. The instantaneous value of that information is the information state. By necessity, consensus algorithms are designed to be distributed, assuming only neighbor-to-neighbor interaction between vehicles. A vehicle updates the value of its information state based on the information states of its neighbors. The goal is to design an update law so that the information states of all vehicles in the network converge to a common value (see [1] and references therein). Consensus-type techniques have been used to solve formation control problems in [2]–[7], to name a few.

Current research in consensus algorithms primarily assumes that the consensus equilibrium is a weighted average or a weighted power mean of the initial information states and therefore constant. This assumption might not be appropriate when each vehicle's information state evolves over time, as occurs in formation maneuvering problems. In addition, many consensus algorithms ensure only that the information states converge to a common value but does not allow specification of a particular value. While this paradigm is useful for applications such as cooperative rendezvous where there is not a single correct value, there are many applications where there is a desired, or reference, information state. In this case the convergence issues include both convergence to a common value, as well as convergence of the common state to its reference value.

Consensus with a constant leader state under undirected switching inter-vehicle interaction topologies is addressed in the leader following case of [8]. Consensus with a constant leader state is further considered in [9], [10] in the context of a directed fixed interaction topology. Dynamic consensus is considered in [11]. In [12], consensus tracking algorithms are proposed and analyzed under a directed fixed interaction topology, where a time-varying *consensus reference state* is available to only a subset of the team members, called the *group leaders*. The consensus tracking algorithms are also applied in [7] to a formation control problem under a directed fixed interaction topology.

While [7], [12] have addressed the consensus tracking problem with a time-varying consensus reference state, the algorithms in [7], [12] do not explicitly account for actuator saturation. Furthermore, the convergence analysis in [12] is restricted to the case of a directed fixed interaction topology. In practice, both the group leaders and the directed inter-vehicle interaction topologies may be dynamically switching.

In this paper, we first propose a consensus tracking algorithm to account for actuator actuation and provide convergence analysis in the case of fixed group leaders and a directed fixed inter-vehicle interaction topology. We then provide convergence analysis for a consensus tracking algorithm in the case of dynamically changing group leaders and directed switching inter-vehicle interaction topologies. Finally, we experimentally implement and validate a consensus tracking algorithm for a formation control problem on our multi-robot platform in the case of dynamically changing group leaders and directed switching inter-vehicle interaction topologies. These results extend the consensus tracking results in [7], [12]. All results in this paper are based on directed interaction topologies. It is worthwhile to mention that an undirected interaction topology is a special case of a directed interaction topology.

II. PROBLEM STATEMENT

Suppose that there are n vehicles in a team and the information states of all vehicles satisfy single-integrator kinematics given by

$$\dot{\xi}_i = u_i, \quad i = 1, \ldots, n, \tag{1}$$

where $\xi_i \in \mathbb{R}^m$ is the information state of the i^{th} vehicle and $u_i \in \mathbb{R}^m$ is the control input.

The objective of this paper is to design distributed control laws for u_i, $i = 1, \ldots, n$, such that the information states of all vehicles converge to a time-varying consensus reference state with, respectively, bounded control effort and directed switching interaction topologies when the time-varying consensus reference state is available to only a subgroup of the team.

Suppose that the consensus reference state, denoted by ξ^r, satisfies

$$\dot{\xi}^r = f(t, \xi^r), \tag{2}$$

W. Ren is with the Department of Electrical and Computer Engineering, Utah State University, Logan, UT 84322, USA wren@engineering.usu.edu

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where $f(\cdot,\cdot)$ is piecewise continuous in t and locally Lipschitz in ξ^r.

We introduce a virtual vehicle with the states ξ^r and $\dot{\xi}^r$, named vehicle $n+1$ without loss of generality. Let $A_{n+1} = [a_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$ be the weighted adjacency matrix (see Appendix for graph theory notations) for the $n+1$ vehicles, denoting information flow among vehicles and whether a vehicle has access to the consensus reference state. In particular, $a_{ij} > 0$, $i,j = 1,\ldots,n$, if vehicle i receives information from vehicle j, $a_{i(n+1)} > 0$ if ξ^r and $\dot{\xi}^r$ are available to vehicle i, and $a_{(n+1)j} = 0$, $j = 1,\ldots,n+1$. Note that $a_{ii} = 0$, $\forall i$.

Before moving on, we need the following lemma:

Lemma 2.1: Suppose that $L_p \in \mathbb{R}^{p\times p}$ satisfies the property (11). Then the following conditions are equivalent: (i) L_p has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}_p = [1,\ldots,1]^T$ and all other eigenvalues have positive real parts; (ii) The directed graph of L_p has a directed spanning tree; (iii) The rank of L_p is $p-1$.

Proof: The equivalence of (i) and (ii) directly follows [13]. Note that the argument (i) implies that $\text{Nullity}(L_p) = 1$, which in turn implies $\text{Rank}(L_p) = p-1$. That is, (i) implies (iii). Next, we show that (iii) also implies (i). Given a system $\dot{x} = -L_p x$, where $x \in \mathbb{R}^p$, the solution is $x(t) = e^{-L_p t}x(0)$. Note that $e^{-L_p t}$ is a stochastic matrix with positive diagonal entries [13]. Then it follows that $\left\|e^{-L_p t}\right\|_\infty = 1$, $\forall t$, which in turn implies that the system $\dot{x} = -L_p x$ is marginally stable. Note that an LTI system $\dot{x} = Fx$, where $F \in \mathbb{R}^{p\times p}$, is marginally stable if and only if all eigenvalues of F have non-positive real parts and every eigenvalue with zero real parts should have its geometric multiplicity equal to its algebraic multiplicity. Also note that $-L_p$ has at least one zero eigenvalue with an associated eigenvector $\mathbf{1}_p$ because $-L_p$ has zero row sums. In addition, it follows from Gershgorin's disc theorem [14, page 334] that all nonzero eigenvalues of $-L_p$ have negative real parts because $-L_p$ is diagonally dominant and has non-positive diagonal entries. Because the system $\dot{x} = -L_p x$ is marginally stable, the geometric multiplicity of the eigenvalue zero of L_p equals its algebraic multiplicity. Note that $\text{Rank}(L_p) = p-1$ implies that $\text{Nullity}(L_p) = 1$, that is, the geometric multiplicity of the eigenvalue zero is one. It follows that the algebraic multiplicity of the eigenvalue zero is also one, which in turn implies that L_p has a simple zero eigenvalue. That is, (iii) implies (i). We conclude that (i), (ii), and (iii) are equivalent. ∎

III. CONSENSUS TRACKING ALGORITHMS

A. Bounded Control Inputs

We first consider consensus tracking with bounded control inputs. We propose a consensus tracking algorithm that guarantees that the maximum control effort is independent of the initial conditions of the information states as

$$u_i = \frac{1}{\eta_i}[\sum_{j=1}^{n} a_{ij}\hat{\dot{\xi}}_j + a_{i(n+1)}\dot{\xi}^r] - \frac{1}{\eta_i}\gamma\tanh[\sum_{j=1}^{n} a_{ij}(\xi_i - \xi_j) + a_{i(n+1)}(\xi_i - \xi^r)], \quad (3)$$

where $i = 1,\ldots,n$, a_{ij}, $i = 1,\ldots,n$, $j = 1,\ldots,n+1$, is the (i,j)th entry of the weighted adjacency matrix A_{n+1}, $\eta_i \stackrel{\triangle}{=} \sum_{j=1}^{n+1} a_{ij}$, $\hat{\dot{\xi}}_j$ is the estimate of $\dot{\xi}_j$, γ is a positive scalar, and $\tanh(\cdot)$ is defined componentwise. Note that a_{ij}, $i,j = 1,\ldots,n$, specify whether an interaction link from vehicle j to vehicle i exists, and $a_{i(n+1)}$, $i = 1,\ldots,n$, specify whether vehicle i has access to ξ^r and $\dot{\xi}^r$.

Note that in (3) each vehicle's control input depends on its local neighbors' information states and their derivative estimates because of the tracking nature of (3). In practical implementation, the derivatives can be estimated by numerical differentiation of the local neighbors' information states. For example, in the simplest case, we can let $\hat{\dot{\xi}}_j = \frac{\xi_j[k]-\xi_j[k-1]}{T}$, where k is the discrete-time index and T is the sampling period. Next, we will show conditions under which each control input is bounded and independent of the initial information states, and consensus tracking is achieved under a directed fixed interaction topology.

Theorem 3.1: Let $\epsilon_j \stackrel{\triangle}{=} \hat{\dot{\xi}}_j - \dot{\xi}_j$ and $\delta \stackrel{\triangle}{=} [\delta_1^T,\ldots,\delta_n^T]^T$ with $\delta_i \stackrel{\triangle}{=} \sum_{j=1}^{n} a_{ij}\epsilon_j$. Suppose that $\|\delta\| < \sqrt{mn}\gamma\theta$, where $0 < \theta < 1$, $f(t,\xi^r)$ in (2) is bounded, and the directed graph of A_{n+1} has a directed spanning tree[1]. Using (3) for (1), u_i is bounded, $\|u_i\|_\infty$ is independent of $\xi_i(0)$, and the closed-loop system is input-to-state stable with $\xi_i - \xi^r$ being the state and δ_i being the input. In addition, when $\delta_i(t) \to 0$, $i = 1,\ldots,n$, as $t \to \infty$, $\xi_i(t) \to \xi^r(t)$, $i = 1,\ldots,n$, as $t \to \infty$.

Proof: Noting that all entries of the last row of A_{n+1} are zero, it follows that the virtual vehicle $n+1$ with the states ξ^r and $\dot{\xi}^r$ does not have any incoming interaction link. As a result, if the directed graph of A_{n+1} has a directed spanning tree, then each vehicle except the virtual vehicle $n+1$ has at least one incoming interaction link, which implies that $\eta_i \neq 0$, $i = 1,\ldots,n$. Thus (3) is well defined.

Let $e_i \stackrel{\triangle}{=} \sum_{j=1}^{n} a_{ij}(\xi_i - \xi_j) + a_{i(n+1)}(\xi_i - \xi^r)$. Note that (3) can be written as

$$(\sum_{j=1}^{n+1} a_{ij})u_i = \sum_{j=1}^{n} a_{ij}\dot{\xi}_j + \sum_{j=1}^{n} a_{ij}\epsilon_j + a_{i(n+1)}\dot{\xi}^r - \gamma\tanh(e_i),$$

which implies

$$\dot{e}_i = -\gamma\tanh(e_i) + \delta_i \quad (4)$$

by noting that $\dot{\xi}_i = u_i$. It thus follows that (4) can be written in matrix form as

$$(L_{n\times(n+1)} \otimes I_m)\underline{u} = -\gamma\tanh(e) + \delta, \quad (5)$$

where $\underline{u} \stackrel{\triangle}{=} [u_1^T,\ldots,u_n^T,\dot{\xi}^{r^T}]^T$, $e \stackrel{\triangle}{=} [e_1^T,\ldots,e_n^T]^T$, $\otimes$ denotes the Kronecker product (see [15] for its properties), and $L_{n\times(n+1)}$ is an $n\times(n+1)$ matrix with $\ell_{ii} = \sum_{j=1,j\neq i}^{n+1} a_{ij}$ and $\ell_{ij} = -a_{ij}$, $i,j = 1,\ldots,n$, and $\ell_{i(n+1)} = -a_{i(n+1)}$, $i = 1,\ldots,n$.

Let $L_{(n+1)\times(n+1)} = \begin{bmatrix} L_{n\times(n+1)} \\ 0_{1\times(n+1)} \end{bmatrix}$. Note that according to the definition of $L_{(n+1)\times(n+1)}$, the directed graph of

[1]Equivalently, the virtual vehicle $n+1$ has a directed path to all vehicles 1 to n.

$L_{(n+1)\times(n+1)}$ has a directed spanning tree if and only if the directed graph of A_{n+1} has a directed spanning tree. Also note that $L_{(n+1)\times(n+1)}$ satisfies the property (11) with $p = n+1$. From the arguments (ii) and (iii) of Lemma 2.1, it follows that rank$(L_{(n+1)\times(n+1)}) = n$ if and only if the directed graph of A_{n+1} has a directed spanning tree. This in turn implies that rank$(L_{n\times(n+1)}) = n$ if and only if the directed graph of A_{n+1} has a directed spanning tree because all entries in the last row of $L_{(n+1)\times(n+1)}$ are zero. Rewrite $L_{n\times(n+1)} = [W|b]$, where $W = [w_{ij}] \in \mathbb{R}^{n\times n}$ is given as $w_{ij} = -a_{ij}$, $i \neq j$, and $w_{ii} = \sum_{j=1,j\neq i}^{n+1} a_{ij}$, and $b \in \mathbb{R}^{n\times 1}$ is given as $b = [-a_{1(n+1)}, \ldots, -a_{n(n+1)}]^T$. Noting that $L_{n\times(n+1)}$ has $n+1$ columns and each of its row sum is zero, it follows that the last column of $L_{n\times(n+1)}$ is dependent on its first n columns, where $b = -W\mathbf{1}_n$. As a result, it follows that Rank(W) = Rank$([W|b]) = n$ if and only if the directed graph of A_{n+1} has a directed spanning tree.

Note that (5) can be written as

$$(W \otimes I_m)u + (b \otimes I_m)\dot{\xi}^r = -\Gamma \tanh(e) + \delta, \qquad (6)$$

where $u = [u_1^T, \ldots, u_n^T]^T$. Because W has full rank and therefore is invertible if and only if the directed graph of A_{n+1} has a directed spanning tree, the boundedness of u_i comes from the fact that b, $\dot{\xi}^r$, $\tanh(\cdot)$, W^{-1}, and δ are all bounded. In addition, it is straightforward to see that $\|u_i\|_\infty$ is independent of $\xi_i(0)$ because $\|\tanh(\cdot)\|_\infty \leq 1$.

Consider a positive-definite function $V = \mathbf{1}_{mn}^T \log(\cosh(e))$, where $\log(\cosh(\cdot))$ is defined componentwise. It follows from (4) that the derivative of V is given by $\dot{V} = \dot{e}^T \tanh(e) = -\gamma \|\tanh(e)\|^2 + \delta^T \tanh(e) \leq -\gamma(1-\theta)\|\tanh(e)\|^2 - \gamma\theta\|\tanh(e)\|^2 + \|\delta\|\,\|\tanh(e)\|$. Noting that $|\tanh(x)| \leq 1$, $\forall x \in \mathbb{R}$, and $\|\delta\| < \sqrt{mn}\gamma\theta$, it follows that $\dot{V} \leq -\gamma(1-\theta)\|\tanh(e)\|^2$ if $\|\tanh(e)\| > \frac{\|\delta\|}{\gamma\theta}$. Therefore, it follows from Theorem 4.19 in [16] that the closed-loop system of (1) using (3) is input-to-state stable with e being the state and δ being the input.

Note that $e = (W \otimes I_m)\xi + (b \otimes I_m)\xi^r$, where $\xi \triangleq [\xi_1^T, \ldots, \xi_n^T]^T$. Also note from the previous argument that $W^{-1}b = -\mathbf{1}_n$ if and only if the directed graph of A_{n+1} has a directed spanning tree. It follows that $\xi - \mathbf{1}_n \otimes \xi^r = (W^{-1} \otimes I_m)e$. Similarly, the closed-loop system of (1) using (3) is input-to-state stable with $\xi - \mathbf{1}_n \otimes \xi^r$ being the state and δ being the input, which in turn implies the first statement of the theorem. When $\delta_i(t) \to 0$, $i = 1, \ldots, n$, as $t \to \infty$, it follows from the property of input-to-state stability that $\xi_i(t) \to \xi^r(t)$, $i = 1, \ldots, n$, as $t \to \infty$. ∎

B. Directed Switching Interaction Topologies

We then consider a consensus tracking algorithm given as

$$u_i = \frac{1}{\eta_i(t)} \sum_{j=1}^{n} a_{ij}(t)[\hat{\dot{\xi}}_j - \gamma(\xi_i - \xi_j)] + \frac{1}{\eta_i(t)} a_{i(n+1)}(t)[\dot{\xi}^r - \gamma(\xi_i - \xi^r)], \qquad (7)$$

where $i = 1, \ldots, n$, $a_{ij}(t)$, $i = 1, \ldots, n$, $j = 1, \ldots, n+1$, is the (i,j)th entry of the weighted adjacency matrix $A_{n+1}(t)$ at time t, γ is a positive scalar, $\eta_i(t) \triangleq \sum_{j=1}^{n+1} a_{ij}(t)$, and $\hat{\dot{\xi}}_j$ is the estimate of $\dot{\xi}_j$. Note that (7) generalizes the algorithm proposed in [12] to allow time-varying or dynamically switching weights $a_{ij}(t)$, $i = 1, \ldots, n$, $j = 1, \ldots, n+1$.

Similar to (3), each vehicle's control input in (7) depends on its local neighbors' information states and their derivative estimates because of the tracking nature of (7). Next, we will show conditions under which consensus tracking is achieved in the case of dynamically changing group leaders under directed switching inter-vehicle interaction topologies.

Theorem 3.2: Let δ_i be defined as in Theorem 3.1. Let t_0 be the initial time. Also let $t_1, t_2, \ldots$ be the switching times for the directed graph of $A_{n+1}(t) = [a_{ij}(t)] \in \mathbb{R}^{(n+1)\times(n+1)}$ defined in (7). Suppose that $A_{n+1}(t)$ is piecewise continuous and each nonzero (and therefore positive) entry of $A_{n+1}(t)$ is chosen from a compact set $[\underline{a}, \bar{a}]$, where $\underline{a}$ an $\bar{a}$ are positive constants. Also suppose that the directed switching graph of $A_{n+1}(t)$ has a directed spanning tree across each interval $[t_j, t_{j+1})$, $j = 0, 1, \ldots$. Using (7) for (1), the closed-loop system is input-to-state stable with $\xi_i - \xi^r$ being the state and δ_i being the input. In addition, when $\delta_i(t) \to 0$, $i = 1, \ldots, n$, as $t \to \infty$, $\xi_i(t) \to \xi^r$, $i = 1, \ldots, n$, as $t \to \infty$.

Proof: The same argument as that in the first paragraph of the proof of Theorem 3.1 can be used to show that $\eta_i(t) \neq 0$, $i = 1, \ldots, n$, across each interval $[t_j, t_{j+1})$, $j = 0, 1, \ldots$ if the directed switching graph of $A_{n+1}(t)$ has a directed spanning tree across that interval. Thus (7) is well defined across each interval $[t_j, t_{j+1})$, $j = 0, 1, \ldots$.

Note that (7) can be written as

$$\begin{aligned}&[W(t) \otimes I_m]u + [b(t) \otimes I_m]\dot{\xi}^r \\ &\quad = -\gamma\left([W(t) \otimes I_m]\xi + [b(t) \otimes I_m]\xi^r\right) + \delta,\end{aligned} \qquad (8)$$

where u, ξ, δ, $W(t)$, and $b(t)$ are defined as in the proof of Theorem 3.1 except that $W(t)$ and $b(t)$ are switching in (8). Following the proof of Theorem 3.1, we can show that $W(t)$ is invertible across each interval $[t_j, t_{j+1})$, $j = 0, 1, \ldots$ if and only if the directed graph of $A_{n+1}(t)$ has a directed spanning tree across that interval.

We next show that $W^{-1}(t)$ is bounded under the assumption of the theorem. Noting that each nonzero entry of $A_{n+1}(t)$ is chosen from a compact set, it follows that each nonzero entry of $W(t)$ is also within a compact set. Two matrices are said to have the same structure if their positive, zero, and negative entries are in the same places. Under the assumption that the directed graph of $A_{n+1}(t)$ is switching but has a directed spanning tree across each interval $[t_j, t_{j+1})$, the number of possible directed graphs of $A_{n+1}(t)$ is finite. It then follows that there are a finite number of possible structures for $A_{n+1}(t)$, which implies that there are a finite number of possible structures for $W(t)$. For each possible structure of $W(t)$, $W^{-1}(t)$ exists across each interval $[t_j, t_{j+1})$, which implies that det$(W(t)) \neq 0$ across each interval $[t_j, t_{j+1})$ if the directed graph of $A_{n+1}(t)$ has a directed spanning tree across that interval. Thus for each possible structure of $W(t)$, det$(W(t))$ is within a compact set that does not include zero. Also note that for each possible structure of $W(t)$, all entries of its adjoint are within a compact set. It thus follows that each entry of $W^{-1}(t)$ is within a compact set.

Noting that $b(t) = -W(t)\mathbf{1}_n$ and $W(t)$ is invertible across each interval $[t_j, t_{j+1})$ under the assumption of the theorem, it follows that $-W^{-1}(t)b(t) = \mathbf{1}_n$. Note that $\dot{\xi} = u$, we rewrite (8) as $\dot{\xi} - \mathbf{1}_n \otimes \dot{\xi}^r = -\gamma(\xi - \mathbf{1}_n \otimes \xi^r) + [W^{-1}(t) \otimes I_m]\delta$. The statements of the theorem then follow directly. ■

Note that the tracking nature of the algorithm (7) requires more stringent conditions for convergence under directed switching interaction topologies than those for the standard consensus algorithm of the form $u_i = -\sum_{j=1}^{n} a_{ij}(\xi_i - \xi_j)$ as studied in [8], [13], [17], where the final consensus equilibrium is a constant.

IV. EXPERIMENTAL VALIDATION ON A MULTI-ROBOT PLATFORM

In this section, we apply a consensus tracking algorithm to a formation control problem and experimentally validate the application on a multi-robot platform under directed switching interaction topologies. In particular, four wheeled mobile robots are required to maintain a square formation while their geometric center (GC) is required to follow a circle clockwise.

Let (r_{xi}, r_{yi}), θ_i, and (v_i, ω_i) denote, respectively, the Cartesian position, orientation, and linear and angular speed of the i^{th} robot. The kinematic equations for the i^{th} robot are

$$\dot{r}_{xi} = v_i \cos(\theta_i), \quad \dot{r}_{yi} = v_i \sin(\theta_i), \quad \dot{\theta}_i = \omega_i. \tag{9}$$

To focus on the main issue, we feedback linearize (9) for a fixed point off the center of the wheel axis denoted as (x_i, y_i), where $x_i = r_{xi} + d_i \cos(\theta_i)$ and $y_i = r_{yi} + d_i \sin(\theta_i)$ with $d_i = 0.15$ m. Letting $\begin{bmatrix} v_i \\ \omega_i \end{bmatrix} = \begin{bmatrix} \cos(\theta_i) & \sin(\theta_i) \\ -\frac{1}{d_i}\sin(\theta_i) & \frac{1}{d_i}\cos(\theta_i) \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \end{bmatrix}$, gives $\begin{bmatrix} \dot{x}_i \\ \dot{y}_i \end{bmatrix} = \begin{bmatrix} u_{xi} \\ u_{yi} \end{bmatrix}$, which is a simplified kinematic equation but is sufficient for the purpose of this paper.

In our implementation, a consensus tracking algorithm is applied on the group level, where the robots share their understandings of the geometric center of the team and reach consensus on the time-varying reference trajectory for the geometric center of the team in a distributed manner. Based on the group-level consensus tracking algorithm, a local control law is then applied for robot level control, where the robots are driven to follow their desired positions. As a result, information sharing and consensus occur on the group level rather than on the robot level.

On the group level, let $\xi^r \triangleq [x_c^r, y_c^r, \theta_c^r]$ denote the consensus reference state for the team of robots, where (x_c^r, y_c^r) and θ_c^r represent, respectively, the reference position and orientation for the geometric center of the team. We consider the case where the group leaders that have access to ξ^r and $\dot{\xi}^r$ are dynamically changing and the directed inter-robot interaction topologies are also dynamically changing. Because ξ^r and $\dot{\xi}^r$ are only available to a dynamically changing subgroup of the team, each robot maintains a local variable, $\xi_i = [x_{ci}, y_{ci}, \theta_{ci}]^T$, which denotes the i^{th} robot's understanding of ξ^r. Here ξ_i is the information state for the i^{th} robot. We apply the consensus tracking algorithm (7) to guarantee that $\xi_i(t)$ tracks $\xi^r(t)$, $i = 1, \ldots, n$, for all $\xi_i(0)$. In our experiments, the algorithm (7) is implemented in discrete time and $\widehat{\dot{\xi}}_j$ is set to be $\frac{\xi_j[k] - \xi_j[k-1]}{T}$.

On the robot level, each robot determines its desired position (x_i^d, y_i^d) as $\begin{bmatrix} x_i^d \\ y_i^d \end{bmatrix} = \begin{bmatrix} x_{ci} \\ y_{ci} \end{bmatrix} + \begin{bmatrix} \cos(\theta_{ci}) & -\sin(\theta_{ci}) \\ \sin(\theta_{ci}) & \cos(\theta_{ci}) \end{bmatrix} \begin{bmatrix} \tilde{x}_{iF} \\ \tilde{y}_{iF} \end{bmatrix}$, where $[\tilde{x}_{iF}, \tilde{y}_{iF}]^T$ represents the desired deviation vector of the i^{th} robot relative to the geometric center of the team. A simple local control law of the form

$$\begin{aligned} u_{xi} &= \dot{x}_i^d - k_{xi}(x_i - x_i^d) \\ u_{yi} &= \dot{y}_i^d - k_{yi}(y_i - y_i^d), \end{aligned} \tag{10}$$

where $k_{xi} > 0$ and $k_{yi} > 0$, is applied to guarantee that $x_i(t) \to x_i^d(t)$ and $y_i(t) \to y_i^d(t)$ as $t \to \infty$.

With both the group level control law (7) and the vehicle level control law (10), if $\xi_i(t)$ tracks $\xi^r(t)$, $x_i(t)$ tracks $x_i^d(t)$, and $y_i(t)$ tracks $y_i^d(t)$, $i = 1, \ldots, n$, then the desired formation shape is maintained and the geometric center of the formation follows its reference trajectory.

In our experiment, the consensus reference state $\xi^r = [x_c^r, y_c^r, \theta_c^r]$ satisfies

$$\dot{x}_c^r = v_c^r \cos(\theta_c^r), \quad \dot{y}_c^r = v_c^r \sin(\theta_c^r), \quad \dot{\theta}_c^r = \omega_c^r,$$

where $v_c^r = \frac{9\pi}{500}$ m/sec, $\omega_c^r = \frac{\pi}{50}$ rad/sec, $(x_c^r(0), y_c^r(0)) = (0, 0)$ m, and $\theta_c^r(0) = 0$ degree. In addition, we let $\tilde{x}_{iF} = \ell_i \cos(\phi_i)$ and $\tilde{y}_{iF} = \ell_i \sin(\phi_i)$, where $\ell_i = 0.6$ m and $\phi_i = \pi - \frac{\pi}{4} i$ rad, $i = 1, \ldots, 4$. Thus the lateral length of the square formation is $0.6\sqrt{2}$ m.

Fig. 1 shows a team of four AmigoBots at Utah State University. The robots can communicate with each other through ethernet with TCP/IP protocols. The robots rely on encoder and sonar data for their position and orientation measurements.

Fig. 1. Multi-robot experimental platform at Utah State University.

We assume that the directed graphs of $A_{n+1}(t)$, where $A_{n+1}(t)$ is defined in (7), switch randomly from the set $\bar{\mathcal{G}} = \{\mathcal{G}_1, \ldots, \mathcal{G}_6\}$ as shown in Fig. 2 with a switching time around 10 seconds. In Fig. 2, a subscript ℓ denotes a group leader that has access to ξ^r and $\dot{\xi}^r$, a subscript f denotes a follower, and a link from node j to node i denotes that $a_{ij} > 0$, $i, j = 1, \ldots, 4$, in (7). Note that all $\mathcal{G}_i$, $i = 1, \ldots, 6$, have a directed spanning tree with node ξ^r being the root.

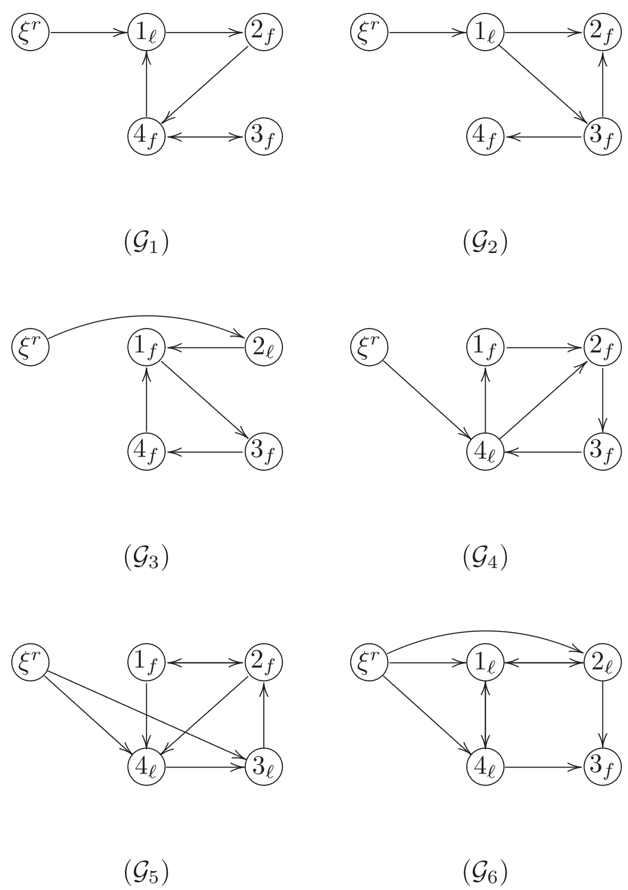

Fig. 2. Directed switching graphs of $A_{n+1}(t)$.

Fig. 3(a) shows the trajectories of the four robots at $t \in [0, t_f]$ sec and snapshots at $t = 0, \frac{t_f}{3}, \frac{2t_f}{3}$ sec, where t_f is the ending time of the experiment. Note that the four robots maintain a tight formation in our experiment even when both the group leaders and the directed inter-robot interaction topologies are dynamically changing. Fig. 3(b) shows the formation maintenance errors, defined as $e_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} - \delta_{ij}$, where δ_{ij} is the desired separation distance between robots i and j. That is, e_{ij} represents the difference between the desired and actual separation distance between the robots. Note that the formation maintenance error are between -6 cm and 5 cm in our experiment. Fig. 3(c) shows the consensus tracking errors for the geometric center position, defined as $\sqrt{(x_c^r - x_{ci})^2 + (y_c^r - y_{ci})^2}$. The consensus tracking errors for the geometric center are below 1.6 cm. Fig. 3(d) shows the consensus tracking errors for the geometric center orientation, defined as $\theta_c^r - \theta_{ci}$. The consensus tracking errors for the geometric center orientation are between -0.1 degrees and 0.15 degrees.

V. CONCLUSION AND FUTURE WORK

We have proposed and analyzed a consensus tracking algorithm accounting for bounded control effort and have shown a convergence condition for a consensus tracking algorithm when the group leaders are dynamically changing and the directed inter-vehicle interaction topologies are switching. We have also applied a consensus tracking algorithm to a formation control problem on our multi-robot experimental platform. Experimental results have shown the effectiveness of the algorithm even in the presence of dynamically changing group leaders and directed switching inter-robot interaction topologies. The experimental movies of the current paper can be found at `http://www.neng.usu.edu/ece/faculty/wren/videos/Amigobots/multi-leader-experiments.wmv`.

APPENDIX

A weighted graph consists of a node set $\mathcal{V} = \{1, \ldots, p\}$, an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $A_p = [a_{ij}] \in \mathbb{R}^{p \times p}$. An edge (i, j) in a weighted directed graph denotes that vehicle j can obtain information from vehicle i, but not necessarily vice versa. In contrast, the pairs of nodes in a weighted undirected graph are unordered, where an edge (i, j) denotes that vehicles i and j can obtain information from one another. The weighted adjacency matrix A_p of a weighted directed graph is defined such that a_{ij} is a positive weight if $(j, i) \in \mathcal{E}$, while $a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. The weighted adjacency matrix A_p of a weighted undirected graph is defined analogously except that $a_{ij} = a_{ji}$, $\forall i \neq j$, since $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$. If the weights are not relevant, then a_{ij} is set equal to 1 for all $(j, i) \in \mathcal{E}$. In this paper, self edges are not allowed, i.e. $a_{ii} = 0$.

A directed path is a sequence of edges in a directed graph of the form $(i_1, i_2), (i_2, i_3), \ldots$, where $i_j \in \mathcal{V}$. An undirected path in an undirected graph is defined analogously. A directed graph has a directed spanning tree if there exists at least one node having a directed path to all other nodes. An undirected graph is connected if there is an undirected path between every pair of distinct nodes.

Let the matrix $L_p = [\ell_{ij}] \in \mathbb{R}^{p \times p}$ be defined as $\ell_{ii} = \sum_{j=1, j \neq i}^{p} a_{ij}$ and $\ell_{ij} = -a_{ij}$, $i \neq j$. The matrix L_p satisfies the conditions

$$\ell_{ij} \leq 0, \quad i \neq j, \qquad \sum_{j=1}^{p} \ell_{ij} = 0, \quad i = 1, \ldots, p. \tag{11}$$

For an undirected graph, the Laplacian matrix L_p is symmetric positive semi-definite. However, L_p for a directed graph does not have this property. In both the undirected and directed cases, 0 is an eigenvalue of L_p with the associated eigenvector $\mathbf{1}_p$, where $\mathbf{1}_p$ is a $p \times 1$ column vector of all ones. In the case of undirected graphs, 0 is a simple eigenvalue of L_p and all other eigenvalues are positive if and only if the undirected graph is connected [18]. In the case of directed graphs, 0 is a simple eigenvalue of L_p and all other eigenvalues have positive real parts if and only if the directed graph has a directed spanning tree [13].

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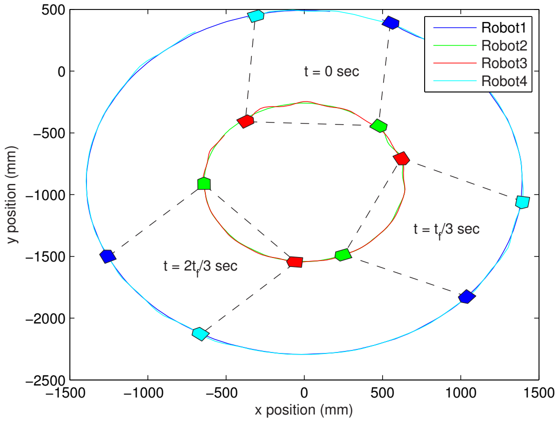
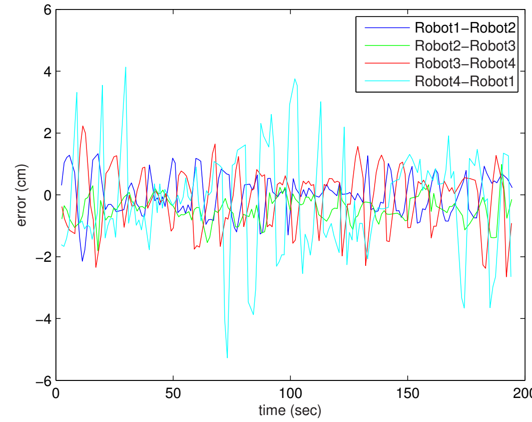
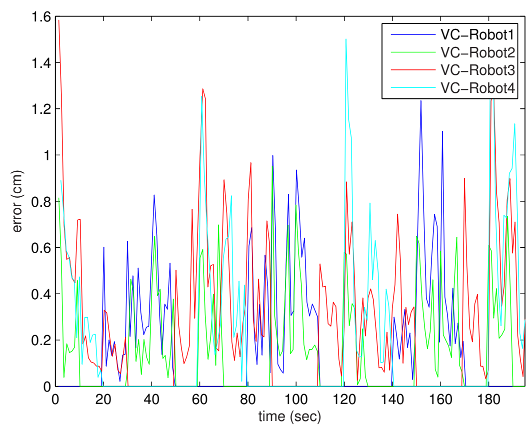


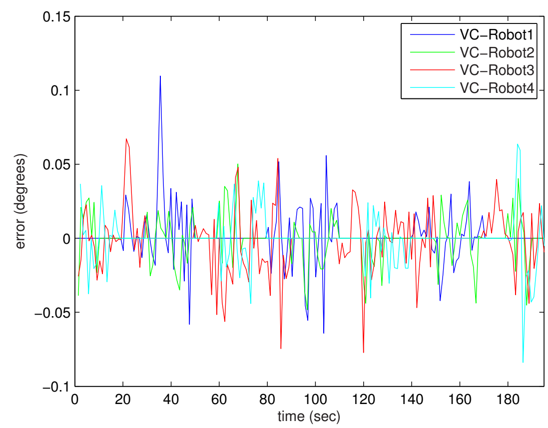
(a) Trajectories of the four robots



(b) Formation maintenance errors



(c) Consensus tracking errors for GC position



(d) Consensus tracking errors for GC orientation

Fig. 3. Consensus tracking experimental results with four mobile robots.

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