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A new approach to explicit MPC using self-optimizing control

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***Abstract*— Model predictive control (MPC) is a favored method for handling constrained linear control problems. Normally, the MPC optimization problem is solved on-line, but in 'explicit MPC' an explicit piecewise affine feedback law is computed and implemented [1]. This approach is similar to 'self-optimizing control,' where the idea is to find simple pre-computed policies for implementing optimal operation, for example, by keeping selected controlled variable combinations c constant. The 'nullspace' method [2] generates optimal variable combinations, which turn out to be equivalent to the explicit MPC feedback laws, that is, $c = u-Kx$, where K is the optimal state feedback matrix in a given region. More importantly, this link gives new insights and also some new results. One is that regions changes may be identified by tracking the variables c for neighboring regions.**

I. Introduction

Consider the general static optimization problem [2]:

$$\begin{aligned} \min_{u_0,x} \quad & J_0(x,u_0,d) \\ \text{s.t. } & f_i(x,u_0,d) = 0, \quad i \in \mathcal{E} \\ & h_i(x,u_0,d) \geq 0, \quad i \in \mathcal{I}, \end{aligned} \tag{P1}$$

where $x \in \mathbb{R}^{n_x}$ are the states, $u_0 \in \mathbb{R}^{n_{u_0}}$ are the inputs, and $d \in \mathcal{D} \subset \mathbb{R}^{n_d}$ are disturbances. By discretization and reformulation this may also represent some dynamic optimization problems. Usually f is a model of the physical system, whilst h is a set of inequality constraints that limits the operation (e.g., physical limits on temperature measurements or flow constraints). In addition to (P1) we have measurements on the form

$$y_0 = f^y(x,u_0,d). \tag{1}$$

In this work the emphasis is on *implementation of the solution to* (P1). This means that the optimization problem (P1) is solved off-line to generate a 'control policy' which is suitable for on-line implementation, with particular emphasis on remaining close to optimal solution when there are unknown disturbances. That is, we search for 'control policies' such that the cost J_0 remains optimal or close to optimal when disturbances occur without the need to reoptimize.

A. Self-optimizing control

In our previous work on 'self-optimizing control' we have looked for simple control policies to implement optimal operation, and in particular 'what should we control' (choice of controlled variables (CV's)). Using off-line optimization we may determine regions where different sets of active constraints are active, and implementation of optimal operation is then in each region to:

1) Control the active constraints.
2) For the remaining unconstrained degrees of freedom: Control 'self-optimizing' variables $c = Hy$ which have the property that keeping them constant ($c = c_s$) indirectly achieves close-to optimal operation (with a small loss), in spite of disturbances d. We here allow for linear measurement combinations, $c = Hy$. There are here two factors that should be considered:
   a) Disturbances d. Ideally, we want the optimal value of c (c_{opt}) to be independent of d.
   b) Measurements errors n^y. The loss should be insensitive to these.

B. Relationship to explicit MPC

Consider a simple static optimization problem $\min_u J(u,d)$, where u are the unconstrained degrees of freedom and the states x and the active constraints have been eliminated by substitution. For the quadratic case

$$\begin{aligned} J(u,d) &= [u \quad d]^T S [u \quad d] \\ \text{where } S &= \begin{bmatrix} J_{uu} & J_{ud} \\ J_{ud}^{\mathrm{T}} & J_{dd} \end{bmatrix}. \end{aligned} \tag{2}$$

In addition we have available 'measurements' $y = G^y u + G^d d$. A key result, which is the basis for this paper, is

For a quadratic optimization problem there exists (infinitely many) linear measurement combinations $c = Hy$ that are optimally invariant to disturbances d.

One sees immediately that there may be some link to explicit MPC, because the discrete form MPC problem can be written as a static quadratic problem. The link is: If we let y contain the inputs u and the states x, then the 'self-optimizing' variable combination $c = Hy$ is the same as the explicit MPC feedback law (control policy), i.e. $c = u-Kx$. (This is shown in section III.)

Based on this, we provide in this contribution some *new* ideas on explicit MPC:

1) We propose that tracking the variables c (deviation from optimal feedback law) for all regions, may be used as a local method to detect when to switch between regions.
2) We may use our results to include measurement error in y (e.g. in x and u) when deriving the optimal explicit MPC.
3) We may extend the results to output feedback ($c = u - Ky$) by including in y present and past outputs (and not present states x).

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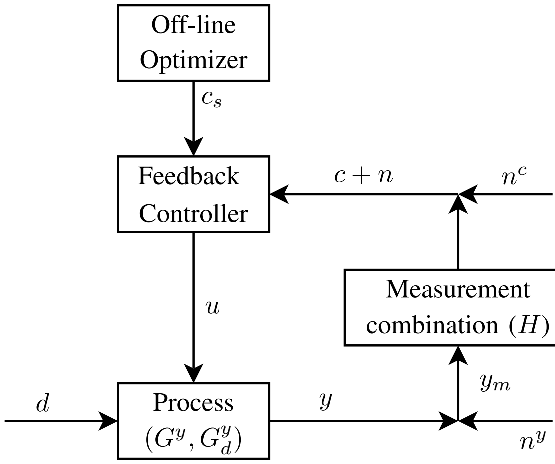

Fig. 1. Block diagram of a feedback control structure including an optimization layer [4].

4) We can also extend the results to the case where only a subset of the states are measured (but in this case there will be a loss, which we can quantify). This may be of interest even in the unconstrained LQ case.

In this paper the basic framework and issue (1) are discussed. In [3] it is shown how the results can be extended to handle items (2)-(4), both with theorems and examples.

II. RESULTS FROM SELF-OPTIMIZING CONTROL

A. Steady state conditions

Once the set of active constraints in (P1) is known we can form the reduced problem and the unconstrained degrees of freedom u can be determined. The unconstrained measurements are

$$y = G^y u + G^y_d d, \tag{3}$$

and y contain information about the present state and disturbances (y may include u_0 and d, but not the active constraints.) The (measured) value of y_m available for implementation is

$$y_m = y + n^y, \tag{4}$$

where n^y represents uncertainty in the measurement of y including uncertainty of implementation in u.

The following theorem describes a method to find linear invariants that yields zero loss from optimality when the invariants are controlled at constant setpoint. The theorem is based on the 'nullspace method' presented in [2]. Figure 1 illustrates how the H matrix is used to linearly combine measurements (and square down the plant).

Theorem 1: (Linear invariants for quadratic optimization problem [4]) Consider an unconstrained quadratic optimization problem in the variables u (input vector of length n_u) and d (disturbance vector of length n_d)

$$\min_u J(u,d) = \begin{bmatrix} u & d \end{bmatrix} \begin{bmatrix} J_{uu} & J_{ud} \\ {J_{ud}}^{\mathrm{T}} & J_{dd} \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix} \tag{5}$$

In addition, there are 'measurement variables' $y = G^y u + G^y_d d$.

If there exists $n_y \geq n_u + n_d$ independent measurements (where 'independent' means that the matrix $\tilde{G}^y = \begin{bmatrix} G^y & G^y_d \end{bmatrix}$ has full rank), then the optimal solution to (5) has the property that there exists $n_c = n_u$ linear variable combinations (constraints) $c = Hy$ that are invariant to the disturbances d. The optimal measurement combination matrix H is found by either: (1): Let $F = \frac{\partial y^{\text{opt}}}{\partial d^{\mathrm{T}}}$ be the optimal sensitivity matrix evaluated with constant active constraints. Under the assumptions stated above possible to select the matrix H in the left nullspace of F, $H \in \mathcal{N}(F^{\mathrm{T}})$, such that

$$HF = 0 \tag{6}$$

(2): If $n_y = n_u + n_d$:

$$H = M_n^{-1} \tilde{J} (\tilde{G}^y)^{-1}, \tag{7}$$

where $\tilde{J} = \begin{bmatrix} J_{uu}^{1/2} & J_{uu}^{-1/2} J_{ud} \end{bmatrix}$ and $\tilde{G}^y = \begin{bmatrix} G^y & G^y_d \end{bmatrix}$ is the augmented plant. M_n^{-1} may be seen as a free parameter. (Note that $M_n = J_{cc}$ is the Hessian of the cost with respect to the c-variables; in most cases we select $M_n = I$ for convenience.)

Remark 1: The sensitivity F matrix can be obtained from

$$F = -\left(G^y J_{uu}^{-1} J_{ud} - G^y_d\right). \tag{8}$$

Remark 2: An equivalent formulation is: Assume that there exists a set of independent measurements y and that the (operational) constraint $c \triangleq Hy = c_s$ (where c_s is a constant) is added to the problem. Then there exists an H that does not change the solution to (5). In terms of operation, this means that zero loss (optimal operation) is obtained by controlling $n_c = n_{u_0}$ variables $c = Hy$ with a constant set-point policy $c = c_s$, where H is selected according to theorem 1.

Theorem 1 may be extended:

Lemma 1: (Linear invariants for constrained quadratic optimization methods) Consider an optimization problem of the form

$$\begin{aligned} \min_{u_0, x} J_0 = \begin{bmatrix} x & u_0 & d \end{bmatrix} S \begin{bmatrix} x \\ u_0 \\ d \end{bmatrix} \\ \text{s.t. } Ax + Bu + Cd = 0 \\ \tilde{A}x + \tilde{B}u + \tilde{C}d \leq 0, \end{aligned} \tag{9}$$

with $\det(A) \neq 0$ and $[\tilde{A}\ \tilde{B}]$ full row rank.

Assume that the disturbance space has been partitioned into n_a critical regions. In each region there are $n^i_u = n_{u_0} - n^i_A \geq 0$ unconstrained degrees of freedom, where $n^i_A \leq n_m$ is the number of optimally active constraints in region i.

If there exists a set of independent unconstrained measurements $y^i = (G^y)^i u^i + (G^y_d)^i d$ in each region i, such that $n_{y^i} \geq n_{u^i} + n_d$, the optimal solution to (9) has the property that there exists variable combinations $c^i = H^i y^i$ that are invariant to the disturbances d in the critical region i. The corresponding optimal H^i may be obtained from Theorem 1.

Within each region, optimality requires that $c^i - c^i_s = 0$ (where c^i_s is a constant). From continuity of the solution, we have that c^i is continuous across the boundary of region i. This implies that the elements in the variable vector $c^i - c^i_s$ will change sign or remain zero when crossing into or from a neighboring region.

Proof: See internal report [5]. ■

B. Implementation of optimal solution

For the case of no measurement error, $n^y = 0$, Theorem 1 shows that for the solution to quadratic optimization problems, variable combinations $c = Hy$ that are invariant to the disturbances can be found. In section III this insight will be used as a new approach to the explicit MPC problem.

III. Application to explicit MPC

We will now look at the model predictive control problem (MPC) with constraints on inputs and outputs. For a discussion on MPC in a unified theoretical framework see [6].

The following discrete MPC formulation is based on [7]. Consider the state-space representation of a given process model:

$$x(t+1) = Ax(t) + Bu(t) \tag{10}$$

$$y_0(t) = Cx(t), \tag{11}$$

subject to the following constraints:

$$y_{\min} \le y_0(t) \le y_{\max} \tag{12}$$

$$u_{\min} \le u(t) \le u_{\max}, \tag{13}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively, subscripts min and max denote the lower and upper bounds, respectively, and (A, B) is stabilizable. MPC problems for regulating to the origin can then be posed as the following optimization problem:

$$\begin{aligned}
\min_U J(U, x(t)) &= {x_{t+N_y|y}}^{\mathrm{T}} P x_{t+N_y|t} + \\
&+ \sum_{k=0}^{N_y-1} \left[{x_{t+k|t}}^{\mathrm{T}} Q x_{t+k|t} + {u_{t+k}}^{\mathrm{T}} R u_{t+k} \right] \\
\text{s.t. } & y_{\min} \le y_{t+k|t} \le y_{\max}, \quad k = 1, \dots, N_c \\
& u_{\min} \le u_{t+k} \le u_{\max}, \quad k = 0, 1, \dots, N_c \\
& x_{t|t} = x(t) \\
& x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k}, \quad k \ge 0 \\
& y_{t+k|t} = Cx_{t+k|t}, \quad k \ge 0 \\
& u_{t+k} = Kx_{t+k|t}, \quad N_u \le k \le N_y
\end{aligned}$$

where $U \triangleq \{u_t, \dots, u_{t+N_u-1}\}$, $Q = Q^{\mathrm{T}} \ge 0$, $R = R^{\mathrm{T}} > 0$, $P \ge 0$, $N_y \ge N_u$, and K is some feedback gain. [7] show that by substitution of the model equations, the problem can be rewritten on the form

$$\begin{aligned}
&\min_U \frac{1}{2} U^{\mathrm{T}} H U + x(t)^{\mathrm{T}} F U + \frac{1}{2} x(t)^{\mathrm{T}} Y x(t) \\
&\text{s.t. } GU \le W + Ex(t)
\end{aligned} \tag{14}$$

The MPC control law is based on the following idea: At time t, compute the optimal solution $U^*(t) = \{u^*_t, \dots, u^*_{t+N_u-1}\}$ and apply $u(t) = u^*_t$ [1].

Remark 3: The trade-off between robustness and performance is included in the weights in the MPC cost function and in the constraints.

If we let the initial state $x(t)$ be treated as a disturbance, (14) can be written as:

$$\begin{aligned}
&\min_U \frac{1}{2} \begin{bmatrix} U^{\mathrm{T}} & d^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} H & F \\ F & Y \end{bmatrix} \begin{bmatrix} U \\ d \end{bmatrix} \\
&\text{s.t. } GU \le W + Ed,
\end{aligned} \tag{15}$$

and we observe that (15) is on the same form as (9), where the model equations $f(x, u_0, d) = 0$ have already been substituted into the objective function.

A property of the solution to (15) is that the disturbance space (initial state space) is divided into critical regions. In the i'th critical region there are $n^i_u = n_U - n^i_A$ unconstrained degrees of freedom, where n^i_A is the number of active constraints in region i.

As we discuss in section III-A, a possible set of measurements y is the current state and the inputs, $y^{\mathrm{T}} = \begin{bmatrix} x^{\mathrm{T}} & u^{\mathrm{T}} \end{bmatrix}$. We further note that causality is not an issue here, as we have the information at the current time.

A. Exact measurements of all states (state feedback)

The following theorem is well known, but we will here prove the theorem using the nullspace method.

Theorem 2: (Optimal state feedback [1]) The control law $u(t) = f(x(t))$, $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, defined by the MPC problem, is continuous and piecewise affine

$$f(x) = K^i x + g^i \quad \text{if } H^i x \le k^i, \quad i = 1, \dots, N_{\text{mpc}} \tag{16}$$

where the polyhedral sets $\{H^i x \le k^i\}$, $i = 1, \dots, N_{\text{mpc}} \le N_r$ are a partition of the given set of states X.

In this case causality is not a problem and from Theorem 1 the optimal solution is simply $u = Kx + g$ (i.e. $c = u - (Kx - g)$). Note that $n_d = n_x$ in this case.

Proof:

We consider the explicit MPC formulation as in (15). First we consider the unconstrained case. Let $y = (U, x)$ be the set of candidate measurements. With this choice of measurements and disturbances on the present state, we form the process model:

$$\Delta y = G^y \Delta U + G^y_d \Delta d \tag{17}$$

$$G^y = \begin{bmatrix} 0_{n_x \times (n_u N_u)} \\ I_{(n_u N_u) \times (n_u N_u)} \end{bmatrix} \in \mathbb{R}^{(n_x + n_u N_u) \times (n_u N_u)} \tag{18}$$

$$G^y_d = \begin{bmatrix} I_{n_x \times n_x} \\ 0_{(n_u N_u) \times n_x} \end{bmatrix} \in \mathbb{R}^{(n_x + n_u N_u) \times n_x}. \tag{19}$$

We then get the optimal sensitivity as

$$F = \frac{\partial y^{\text{opt}}}{\partial d^{\text{T}}} = -\left(G^y J_{uu}^{-1} J_{ud} - G_d^y\right) = \tag{20}$$

$$-\left(\begin{bmatrix} 0_{n_x \times (n_u N_u)} \\ (J_{uu}^{-1} J_{ud})_{(n_u N_u) \times n_x} \end{bmatrix} - \begin{bmatrix} I_{n_x \times n_x} \\ 0_{(n_u N_u) \times n_x} \end{bmatrix}\right) \tag{21}$$

$$= \begin{bmatrix} I_{n_x \times n_x} \\ -J_{uu}^{-1} J_{ud} \end{bmatrix} \tag{22}$$

We now search for a matrix H that gives a non-trivial solution to $HF = 0$:

$$\begin{bmatrix}(H_1)_{(n_u N_u) \times n_x} & (H_2)_{(n_u N_u) \times (n_u N_u)}\end{bmatrix} \begin{bmatrix} I_{n_x \times n_x} \\ J_{uu}^{-1} J_{ud} \end{bmatrix} = \tag{23}$$

$$= H_1 - H_2 \left(J_{uu}^{-1} J_{ud}\right) = 0 \tag{24}$$

To ensure a non-trivial solution we can for example choose $H_2 = I_{(n_u N_u) \times n_u N_u}$. Then we must have $H_1 = J_{uu}^{-1} J_{ud}$, and hence the *optimal combination c of x and U* becomes

$$c = Hy = J_{uu}^{-1} J_{ud} x + U = 0 \in \mathbb{R}^{(n_u N_u)} \tag{25}$$

In the internal report by [5] it is shown how the affine term in (16) enters as a function of the active constraints. ■

Remark 4: (Comparison with previous results on unconstrained MPC) In (25) the state feedback gain matrix is given as $J_{uu}^{-1} J_{ud}$. This is gives the same result as conventional MPC, see equation (3) in [8].

Remark 5: These are not new results but the alternative proof leads to some new insights. The most important is probably that the "self-optimizing" variables $c^i = u - (K^i x + g^i)$ which are optimally zero in region i, may be used for identifying when to switch between regions (Theorem 3) rather than using a "centralized" approach, for example based on a state tree structure search. This seems to be new. Another insight is to understand why a simple feedback solution must exist in the first place. A third is to allow for new extensions.

Theorem 3: (Optimal region for explicit MPC detection using feedback law) The variables $c = u_k - (Kx_k + g)$ can be used to identify region changes.

Proof: See report by [5]. ■

Remark 6: Neighboring regions with the same feedback law (including regions where the feedback law is to keep the input saturated) can be merged (provided that the regions remain convex or if the "crossings" inside a non-convex region due to the optimal direction of the process in closed loop only occurs in the convex part of the region). This may greatly reduce the number of regions compared to presently used enumeration schemes. Note that the number of c-variables that need to be tracked to detect region changes is only equal to the number of inputs n_{u_0} times the number of distinct merged regions. Because of the merging of regions, this may be a small number even with a large input or control horizon and with output (state) constraints.

We present a simple example from [1] that confirms that our switching policy based on tracking the sign of the c-variables works in practice.

Algorithm 1 Detect current region and calculate u_k

Require: CR_{k-1}, i.e. the region of the last sample time, and x_k

$u_k = K(CR_{k-1}) + g(CR_{k-1})$
[Regions, α] = Neighbors(CR_{k-1})
for $i = 1$ to length(Regions) **do**
  $c_k(i) = \alpha_i \left(u_k - (K(\text{Regions}(i)) + g(\text{Regions}(i)))\right)$
end for
if sign($c_k(i) \neq -1$) **then**
  CR_k = Regions(i)
else
  $CR_k = CR_{k-1}$
end if
return $u_k = K(CR_k)x_k + g(CR_k)$, CR_k

Example 3.1 (Optimal switching): This example is taken from [1] (with correction), and is included here to demonstrate optimal switching using the sign change of $c = u - Kx$ as the criterion. The system is:

$$y(t) = \frac{2}{s^2 + 3s + 2} u(t).$$

With a sampling time $T = 0.1$ seconds the following state-space representation is obtained:

$$x(t+1) = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix} x(t) + \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1.4142 \end{bmatrix} x(t)$$

One observes that only the last state is measured, but it will be assumed that both states are known (measured) in the remainder of this example.

The task is to regulate the system to the origin while fulfilling the input constraint

$$-2 \le u(t) \le 2. \tag{26}$$

The objective function to be minimized is

$$\min x_{t+2|t}{}^{\text{T}} P x_{t+2|t} + \sum_{k=0}^{1} \left[x_{t+k|t}{}^{\text{T}} x_{t+k|t} + 0.01 u_{t+k}^2\right] \tag{27}$$

subject to the constraints and $x_{t|t} = x(t)$.

P solves the Lyapunov equation $P = A^{\text{T}} P A + Q$, where $Q = I$ in this case. The optimal control problem can be solved for example using the MPT toolbox [9]. The P-matrix is numerically:

$$P = \begin{bmatrix} 5.5461 & 4.9873 \\ 4.9873 & 10.4940 \end{bmatrix}$$

To illustrate ideas a simulation from $x_0 = (1, 1)$ was done. State space trajectories and inputs are shown in figures 2 and 3. As long as the state is in the input-constrained region where $u^{\text{opt}} = -2$, the linear combination $c = u_k - Kx_k$ remains positive. One chooses to leave the input-constrained region when c_k becomes zero. As one observes, this happens at time instant 8, where the process indeed is on the boundary between the input-saturated region and the center region.

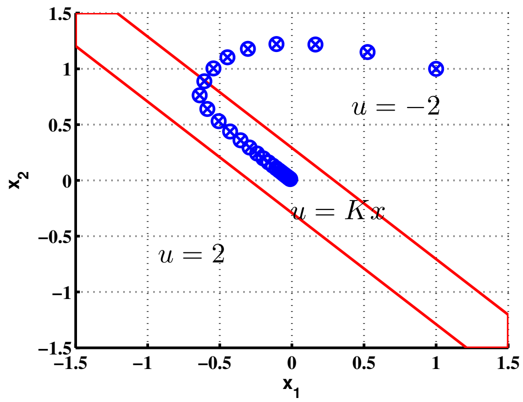

Fig. 2. Partition of state space for first input. (Example 3.1.)

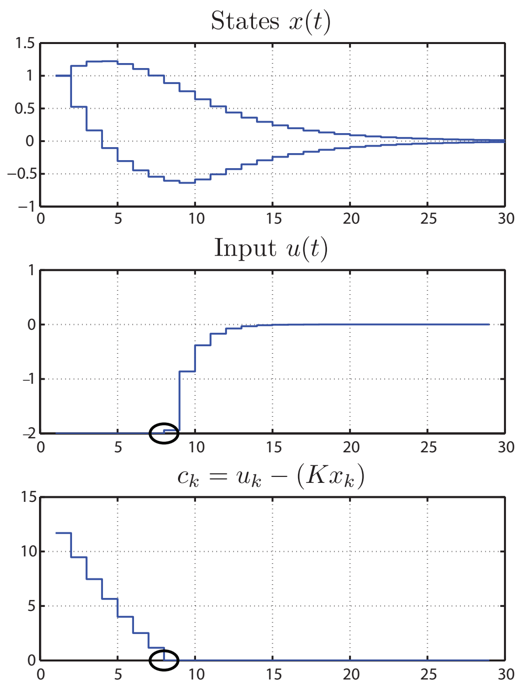

Fig. 3. Closed loop MPC with region detection using $u_k - (Kx_k)$. (Example 3.1.)

After the switching the controller for the center region is implemented. The state trajectory is the same as in [1].

The reason for why c never becomes negative is because both states are assumed measured at the present time and hence optimal switching is achieved. This can be understood from the algorithm 1, where we show how the current critical region (CR_k) is tracked and how the current input u_k is calculated.

Example 3.2 (Double integrator): Consider the double integrator disussed by [1], $y(t) = 1/s^2 u(t)$, and its equivalent discrete-time state-space representation,

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k, \qquad y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k,$$

which is obtained by setting $\ddot{y}(t) = (\dot{y}(t+T_s) - \dot{y}(t))/T_s$,

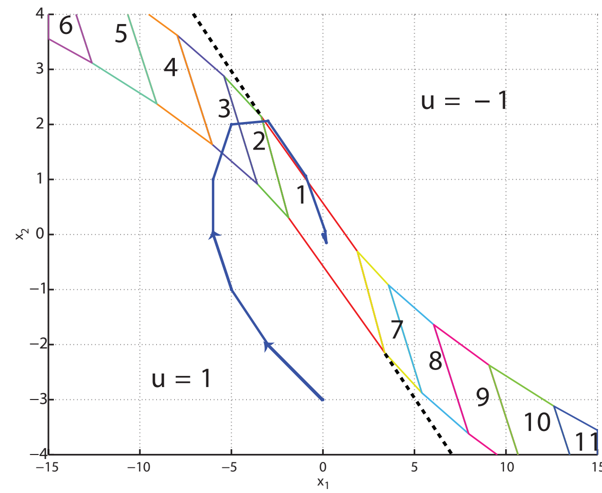

Fig. 4. Regions for double integrator example (Example 3.2).

$\dot{y}(t) = (y(t+T_s) - y(t))/T_s$, $T_s = 1$. The control objective is to regulate the system to the origin while minimizing the quadratic cost funcion $J = \sum_{t=0}^{\infty} y(t)^{\mathrm{T}} y(t) + \frac{1}{10} u^2$ subject to the input constraint $-1 \leq u(t) \leq 1$. The infinite horizion control problem can be converted to a finite horizion problem by solving [1], [10]:

$$K_{\mathrm{LQ}} = -(R + B^{\mathrm{T}} P B)^{-1} B^{\mathrm{T}} P A,$$
$$P = (A + B K_{\mathrm{LQ}})^{\mathrm{T}} P (A + B K_{\mathrm{LQ}}) + K_{\mathrm{LQ}}{}^{\mathrm{T}} R K_{\mathrm{LQ}} + Q$$

to obtain the unconstrained feedback gain K_{LQ} and the final state weight matrix P (see example 3.1). In this case we get $K_{\mathrm{LQ}} = \begin{bmatrix} 0.8166 & 1.7499 \end{bmatrix}$ and $P = \begin{bmatrix} 2.1429 & 1.2246 \\ 1.2246 & 1.3996 \end{bmatrix}$. For demonstration purposes we choose $N_u = 6$, and by solving the paramteric program we get 73 regions initially. In this case there are 11 regions of unsaturated control actions, which agrees with the general result of $(2N_u - 1)$ regions given in [1]. Merging all regions where the first optimal input is the same, leaves us with the 11 unsaturated regions, and two regions for which the optimal input is either at the high or low constraint. The final partitioning with 13 regions is shown in figure 4. We note that [1] find 57 regions after their merging scheme.

Considering figure 4 one observes that the input-saturated regions are non-convex. However, optimally, this process moves clockwise in the state space, and we observe that the "non-convex" crossings will not occur in practise. The remaing boundaries then form convex regions (indicated by the dashed lines in the figure.)

Figure 5 shows the evolution of the invariants c^i in each region when we start the simulation at $x_0 = (0, -3)$ and close the loop by using the optimal inputs. We start in the input-saturated region $u = 1$, and need to track the invariants for regions 1,2,3,4,5, and 6 to determine optimal switching. We should switch to unsaturated control when one the variables c^1 to c^6 becomes zero or changes sign. As one sees, this happens for c^3 at $t = 6$, so we change to this region. After using the feedback law for region 3 for one

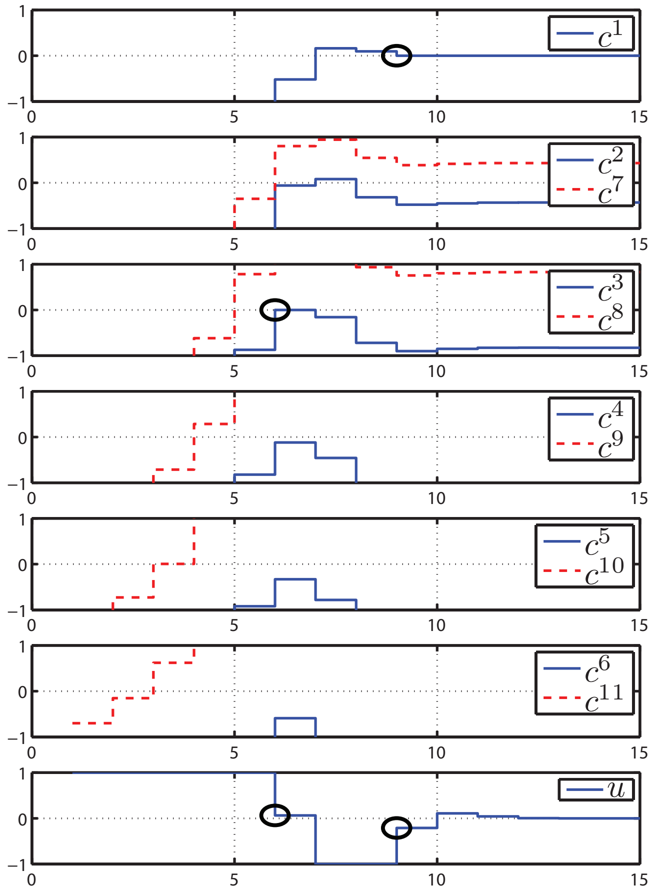

Fig. 5. Invariants for double integrator example.

sample time, we reach the other input constraint $u = -1$ at $t = 8$. Now, to decide when to leave this constrained region we track the invariants for regions 1,7,8,9,10,11, and we observe that at $t = 9$ the invariant for the center region becomes zero, hence we switch control to this region.

Note that in this example, where we have only input constrained regions, the challenge is to decide when to leave the input constraints. Note that the converse crossing can also be tracked using their invariants on the form $c_k = u_k - g$.

The idea of using directionality (clockwise movement in this case) to reduce the number of reigons in explicit MPC can be generalized by using the directional derivative of the process under optimal control, $(A - BK^i)$, together with the normal vectors to the boundaries of the regions, and by some normalization scheme we remove all boundaries for which crossings under optimal control will not occur. Here K^i is the optimal feedback gain for region i.

IV. Discussion

In this paper we have described the link between self-optimizing control and explicit MPC. This link has been used to propose a new method for detecting region changes. This new method lets us reduce the number of regions by merging all regions for which the first input is the same. In its simple form presented in this paper, it does not handle non-convex regions, but we noted that for some processes directionality of the process in closed loops implies that the non-convex crossings may be ignored.

In a forthcoming contribution [3] we show how the results can be extended to output feedback and how to find invariants that give minimal loss when controlled at constant set points also when we have noisy measurements. We further show how we one choose the order of the controller and we show by examples that the resulting controller will have performance in the order of magnitude of LQG controllers.

The most important problem of using results from steady state self-optimizing control is causality, in steady state optimization all measurements are available at the current time (i.e. $t \to \infty$), but in dynamic optimization we may need to find invariants between measurements at current and future times and then switch the invariants back to get a casual controller, but this controller will be non-optimal by construction. Also this is discussed in more detail in [3].

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