2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008



# Network-Based $H_\infty$ Control of Systems with Packet Dropout and Time-Varying Sampling Period

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***Abstract*— This paper is concerned with the problem of $H_\infty$ controller design for networked control systems (NCSs) with time delay, packet dropout and time-varying sampling period. The considered NCSs may receive more than one control input during a sampling period. A multi-objective optimization methodology in terms of linear matrix inequalities (LMIs) and the discrete Jensen inequality are adopted to deal with the problem of $H_\infty$ controller design for NCSs with time-varying sampling period. The proposed $H_\infty$ controller design is also applicable for NCSs with constant sampling period. The simulation results illustrate the effectiveness of the proposed $H_\infty$ controller design.**

## I. INTRODUCTION

Networked control systems (NCSs) have received increasing attentions in recent years. The flexibility and ease of maintenance of a system using network to transfer information is a very appealing goal. However, computer loads, network, sporadic faults, etc. may cause time delay, packet dropout and sampling period jitter, etc., which might be potential sources to poor performance of NCSs.

Many researchers have studied stability/stabilization for NCSs in the presence of network-induced delay [1], [2], [3], and [4], [5], [6] studied the problems of stability/stabilization for NCSs with packet dropout. [7] and [8] was concerned with the problem of optimal control for systems with time delay/packet dropout. For other methods dealing with time delay specifically, see also [9], [10], etc. There have also been considerable research efforts on $H_\infty$ control for systems with time-delay [11]-[15].

Just as we can see, the systems considered in [3], [5], [8], [13] are sampled with constant sampling period. In computer control applications, the sensor supposedly samples at a fixed nominal period, but computer loads, network, sporadic faults, etc. may cause time delay, packet dropout and sampling period jitter, etc. Recently, there are also considerable research efforts on time-varying sampling period [16]-[19]. [16], [17] studied the problems of stability analysis/controller design for systems with time-varying sampling period and time delay, and [16], [17] assumed that the control input $u$ was constant between sampling instants. [18] considered digital feedback control systems with time-varying sampling period consisting of an interconnection of a continuous-time nonlinear plant. [19] was concerned with the problem of $H_\infty$ controller design for NCSs with time-varying sampling period, long time delay and packet dropout, and [19] assumed that the actuator may receive zero or one control input during a sampling period.

To the best of our knowledge, for NCSs with time-varying sampling period, $H_\infty$ controller design and packet dropout have not received enough attention except in [19], and [16], [17], [19] assumed that the actuator may receive zero or one control input during a sampling period.

The purpose of this paper is to prove robustness of NCSs with respect to small variations of the sampling period. In this paper, we will consider the case that the actuator receives more than one control input during a sampling period, time delay and packet dropout are also taken into consideration, and $H_\infty$ controller design for NCSs with time-varying sampling period is presented, which are different from the existing results for systems with time-varying sampling period. If the actuator receives zero or one control input during a sampling period or constant sampling period is adopted, the proposed $H_\infty$ controller design is also applicable. The discrete Jensen inequality is adopted for controller design and no any redundant matrices are introduced, so the computational complexity of the obtained results may be reduced compared with the ones having redundant matrices.

This paper is organized as follows. In Section 2, the model of NCSs which may receive more than one control input during a sampling period is presented. By formulating a feasibility problem into a multi-objective optimization problem subject to LMIs constraints, Section 3 is dedicated to $H_\infty$ controller design for NCSs with time-varying sampling period, time delay and packet dropout. The results of numerical simulation are presented in Section 4. Conclusions are stated in Section 5.

## II. PRELIMINARIES AND PROBLEM STATEMENT

Consider a linear time-invariant system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 u(t) + B_2 \omega(t) \\ z(t) &= C_1 x(t) + D_1 u(t) \end{aligned} \tag{1}$$

This work was supported in part by Program for New Century Excellent Talents in University (NCET-04-0283), the Funds for Creative Research Groups of China (No. 60521003), Program for Changjiang Scholars and Innovative Research Team in University (No. IRT0421), the State Key Program of National Natural Science of China (Grant No. 60534010), the Funds of National Science of China (Grant No. 60674021) and the Funds of PhD program of MOE, China (Grant No. 20060145019), the 111 Project (B08015).

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where $x(t)$, $u(t)$, $z(t)$, $\omega(t)$ are the state vector, control input vector, controlled output, and disturbance input, respectively, and $\omega(t)$ is piecewise constant. $A$, $B_1$, $B_2$, $C_1$, $D_1$ are known constant matrices of appropriate dimensions. Throughout this paper, matrices, if not explicitly stated, are assumed to have appropriate dimensions.

For NCSs with time-varying sampling period, define $t_k$ as the $k$th sampling instant, $t_{k+1}$ as the $(k+1)$th sampling instant, $h_k$ as the length of the $k$th sampling period and $h$ as the ideal sampling period, then $h_k = t_{k+1} - t_k$. Suppose $\sigma$ is a scalar and $-h \le \sigma \le h$, $l$ is a positive integer and $l > 1$, define $\vartheta_1 = \{h,\ h \pm \sigma/l,\ h \pm 2\sigma/l,\ \cdots,\ h \pm \sigma\}$. In this paper, we suppose the sampling period $h_k \in \vartheta_1$, that is $h_k$ switches in the finite set $\vartheta_1$.

In the following, we will present the model of NCSs which may receive two control inputs during a sampling period, the case that the actuator receives more than two control inputs during a sampling period is discussed in Remark 2.

Suppose $u_{k-l_k}$ is the available control input at the instant $t_k$, during the sampling period $[t_k,\ t_{k+1}]$, two control inputs $u_{k-r_k}$ and $u_{k-\rho_k}$ reach the actuator at the instants $t_k + \varepsilon_{k1}$ and $t_k + \varepsilon_{k2}$, respectively, where $\varepsilon_{k1} \in [0,\ h_k]$, $\varepsilon_{k2} \in [0,\ h_k]$, and $\varepsilon_{k1} \le \varepsilon_{k2}$. In this paper, we suppose $l_m \le l_k \le l_M$, $r_m \le r_k \le r_M$, $\rho_m \le \rho_k \le \rho_M$, $\varepsilon_{k1}$ and $\varepsilon_{k2}$ switch in the finite set $\vartheta_2$, where $\vartheta_2 = \{\beta | \beta \in [0,\ h_k]\}$. Define $n$ as the maximum of consecutive packet dropout, then $l_M = r_M + n + 1$, on the other hand, since the time delay of the control inputs $u_{k-r_k}$ and $u_{k-\rho_k}$ may be shorter than a sampling period, then $r_m = \rho_m = 0$.

For NCSs with long time delay, packet dropout and time-varying sampling period, if the actuator receives two control inputs $u_{k-r_k}$ and $u_{k-\rho_k}$ during the sampling period $[t_k,\ t_{k+1}]$, the discrete time representation of the system (1) is as follows

$$\begin{aligned} x_{k+1} &= \Phi_k x_k + \Gamma_{l_k}^k u_{k-l_k} + \Gamma_{r_k}^k u_{k-r_k} + \Gamma_{\rho_k}^k u_{k-\rho_k} + \widetilde{\Gamma}_k \omega_k \\ z_k &= C_1 x_k + D_1 u_k \end{aligned} \tag{2}$$

where $\Phi_k = e^{Ah_k}$, $\Gamma_{l_k}^k = \int_0^{\varepsilon_{k1}} e^{A(h_k - s)} ds B_1$, $\Gamma_{r_k}^k = \int_{\varepsilon_{k1}}^{\varepsilon_{k2}} e^{A(h_k - s)} ds B_1$, $\Gamma_{\rho_k}^k = \int_{\varepsilon_{k2}}^{h_k} e^{A(h_k - s)} ds B_1$, $\widetilde{\Gamma}_k = \int_0^{h_k} e^{As} ds B_2$, $u_k = -K x_k$.

Define $\Phi_k$, $-\Gamma_{l_k}^k K$, $-\Gamma_{r_k}^k K$, $-\Gamma_{\rho_k}^k K$ and $\widetilde{\Gamma}_k$ as $\Psi_1$, $\Psi_2$, $\Psi_3$, $\Psi_4$ and $\Psi_5$, respectively. At the instant $t_k$, the available control input at the actuator is $-K x_{k-l_k}$, then (2) can be written as follows

$$\begin{aligned} x_{k+1} &= \Psi_1 x_k + \Psi_2 x_{k-l_k} + \Psi_3 x_{k-r_k} + \Psi_4 x_{k-\rho_k} + \Psi_5 \omega_k \\ z_k &= C_1 x_k - D_1 K x_{k-l_k} \end{aligned} \tag{3}$$

then the problem of $H_\infty$ controller design for (1) can be reduced to the corresponding problem for the system (3).

The following discrete Jensen inequality will be used in the sequel.

**Lemma 1 [20].** For any constant positive semi-definite symmetric matrix $M \in R^{m*m}$, two positive integers $\beta_1$ and $\beta_2$ satisfying $\beta_2 \ge \beta_1 \ge 1$, the following inequality holds

$$\begin{aligned} &-(\beta_2 - \beta_1 + 1) \textstyle\sum_{i=\beta_1}^{\beta_2} \psi^T(i) M \psi(i) \\ &\le -\left(\textstyle\sum_{i=\beta_1}^{\beta_2} \psi(i)\right)^T M \left(\textstyle\sum_{i=\beta_1}^{\beta_2} \psi(i)\right) \end{aligned} \tag{4}$$

## III. $H_\infty$ CONTROLLER DESIGN FOR NCSs

Based on the model presented in (3), we are now in a position to design the feedback gain $K$, which can make the system (3) asymptotically stable with the $H_\infty$ norm bound $\gamma_k$ (If the sampling period $h_k$ is adopted, the corresponding $H_\infty$ norm bound is denoted as $\gamma_k$).

**Theorem 1.** For given positive scalars $l_M$, $l_m$, $r_M$, $r_m$, $\rho_M$, $\rho_m$, if there exist symmetric positive definite matrices $N$, $\widetilde{Q}_i$ $(i = 1,\ \cdots,\ 7)$, $\widetilde{R}_j$ $(j = 1,\ \cdots,\ 5)$, and matrix $V$, scalars $\gamma_k > 0$, such that the following LMIs hold for every feasible value of $h_k$, $\varepsilon_{k1}$ and $\varepsilon_{k2}$ $(h_k \in \vartheta_1,\ \varepsilon_{k1} \in \vartheta_2,\ \varepsilon_{k2} \in \vartheta_2)$

$$\left[\begin{array}{cccccccccccccccc}
\widetilde{\Lambda}_{11} & 0 & \widetilde{R}_3 & \widetilde{R}_1 & 0 & \widetilde{R}_4 & 0 & \widetilde{R}_5 & 0 & N\Phi_k^T & \widetilde{\mathscr{I}_1} & \widetilde{\mathscr{I}_1} & \widetilde{\mathscr{I}_1} & \widetilde{\mathscr{I}_1} & \widetilde{\mathscr{I}_1} & NC_1^T \\
* & \widetilde{\Lambda}_{22} & \widetilde{R}_2 & \widetilde{R}_2 & 0 & 0 & 0 & 0 & 0 & \widetilde{\mathscr{I}_2} & \widetilde{\mathscr{I}_2} & \widetilde{\mathscr{I}_2} & \widetilde{\mathscr{I}_2} & \widetilde{\mathscr{I}_2} & \widetilde{\mathscr{I}_2} & -VD_1^T \\
* & * & \widetilde{\Lambda}_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \widetilde{\Lambda}_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\widetilde{Q}_4 & 0 & 0 & 0 & 0 & \widetilde{\mathscr{I}_3} & \widetilde{\mathscr{I}_3} & \widetilde{\mathscr{I}_3} & \widetilde{\mathscr{I}_3} & \widetilde{\mathscr{I}_3} & \widetilde{\mathscr{I}_3} & 0 \\
* & * & * & * & * & \widetilde{\Lambda}_{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\widetilde{Q}_6 & 0 & 0 & \widetilde{\mathscr{I}_4} & \widetilde{\mathscr{I}_4} & \widetilde{\mathscr{I}_4} & \widetilde{\mathscr{I}_4} & \widetilde{\mathscr{I}_4} & \widetilde{\mathscr{I}_4} & 0 \\
* & * & * & * & * & * & * & \widetilde{\Lambda}_{88} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\gamma_k I & \widetilde{\Gamma}_k^T & \widetilde{\Gamma}_k^T & \widetilde{\Gamma}_k^T & \widetilde{\Gamma}_k^T & \widetilde{\Gamma}_k^T & \widetilde{\Gamma}_k^T & 0 \\
* & * & * & * & * & * & * & * & * & -N & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & \widetilde{\mathscr{L}_1} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & \widetilde{\mathscr{L}_2} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & \widetilde{\mathscr{L}_3} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & \widetilde{\mathscr{L}_4} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & \widetilde{\mathscr{L}_5} & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & -\gamma_k I
\end{array}\right] < 0 \tag{5}$$

where

$$\begin{aligned} \widetilde{\Lambda}_{11} &= -N + (l_M - l_m + 1)\widetilde{Q}_1 + \widetilde{Q}_2 + \widetilde{Q}_3 + (r_M - r_m + 1)\widetilde{Q}_4 \\ &\quad + \widetilde{Q}_5 + (\rho_M - \rho_m + 1)\widetilde{Q}_6 + \widetilde{Q}_7 - \widetilde{R}_1 - \widetilde{R}_3 - \widetilde{R}_4 - \widetilde{R}_5 \\ \widetilde{\Lambda}_{22} &= -\widetilde{Q}_1 - 2\widetilde{R}_2 \\ \widetilde{\Lambda}_{33} &= -\widetilde{Q}_3 - \widetilde{R}_2 - \widetilde{R}_3 \\ \widetilde{\Lambda}_{44} &= -\widetilde{Q}_2 - \widetilde{R}_1 - \widetilde{R}_2 \end{aligned}$$

$$
\begin{aligned}
&\widetilde{\Lambda}_{66} = -\widetilde{Q}_5 - \widetilde{R}_4, && \widetilde{\Lambda}_{88} = -\widetilde{Q}_7 - \widetilde{R}_5\\
&\widetilde{\mathscr{J}_1} = N\Phi_k^T - N, && \widetilde{\mathscr{J}_2} = -V{\Gamma_{l_k}^k}^T\\
&\widetilde{\mathscr{J}_3} = -V{\Gamma_{r_k}^k}^T, && \widetilde{\mathscr{J}_4} = -V{\Gamma_{\rho_k}^k}^T\\
&\widetilde{\mathscr{L}_1} = l_M^{-2}(\widetilde{R}_1 - 2N), && \widetilde{\mathscr{L}_2} = (l_M - l_m)^{-2}(\widetilde{R}_2 - 2N)\\
&\widetilde{\mathscr{L}_3} = l_m^{-2}(\widetilde{R}_3 - 2N), && \widetilde{\mathscr{L}_4} = r_M^{-2}(\widetilde{R}_4 - 2N)\\
&\widetilde{\mathscr{L}_5} = \rho_M^{-2}(\widetilde{R}_5 - 2N)
\end{aligned}
$$

then with the control law

$$u_k = -Kx_k, \quad K = V^T N^{-1}$$

the system described by (3) is asymptotically stable with $H_\infty$ norm bound $\gamma_k$.

*Proof:* Let us consider the following Lyapunov function

$$
\begin{aligned}
V_k = {} & V_{1k} + V_{2k} + V_{3k} + V_{4k} + V_{5k} + V_{6k} + V_{7k} + V_{8k} + V_{9k}\\
& + V_{10k} + V_{11k} + V_{12k} + V_{13k} + V_{14k} + V_{15k} + V_{16k}
\end{aligned} \tag{6}
$$

where

$$
\begin{aligned}
V_{1k} &= x_k^T P x_k\\
V_{2k} &= \sum_{i=k-l_k}^{k-1} x_i^T Q_1 x_i\\
V_{3k} &= \sum_{i=-l_M+1}^{-l_m} \sum_{j=k+i}^{k-1} x_j^T Q_1 x_j\\
V_{4k} &= \sum_{i=k-l_M}^{k-1} x_i^T Q_2 x_i\\
V_{5k} &= \sum_{i=k-l_m}^{k-1} x_i^T Q_3 x_i\\
V_{6k} &= l_M \sum_{i=-l_M}^{-1} \sum_{j=k+i}^{k-1} \eta_j^T R_1 \eta_j\\
V_{7k} &= (l_M - l_m) \sum_{i=-l_M}^{-l_m-1} \sum_{j=k+i}^{k-1} \eta_j^T R_2 \eta_j\\
V_{8k} &= l_m \sum_{i=-l_m}^{-1} \sum_{j=k+i}^{k-1} \eta_j^T R_3 \eta_j\\
V_{9k} &= \sum_{i=k-r_k}^{k-1} x_i^T Q_4 x_i\\
V_{10k} &= \sum_{i=-r_M+1}^{-r_m} \sum_{j=k+i}^{k-1} x_j^T Q_4 x_j\\
V_{11k} &= \sum_{i=k-r_M}^{k-1} x_i^T Q_5 x_i\\
V_{12k} &= r_M \sum_{i=-r_M}^{-1} \sum_{j=k+i}^{k-1} \eta_j^T R_4 \eta_j\\
V_{13k} &= \sum_{i=k-\rho_k}^{k-1} x_i^T Q_6 x_i\\
V_{14k} &= \sum_{i=-\rho_M+1}^{-\rho_m} \sum_{j=k+i}^{k-1} x_j^T Q_6 x_j\\
V_{15k} &= \sum_{i=k-\rho_M}^{k-1} x_i^T Q_7 x_i\\
V_{16k} &= \rho_M \sum_{i=-\rho_M}^{-1} \sum_{j=k+i}^{k-1} \eta_j^T R_5 \eta_j
\end{aligned} \tag{7}
$$

$P$, $Q_1$, $\cdots$, $Q_7$, $R_1$, $\cdots$, $R_5$ are symmetric positive definite matrices and $\eta_j = x_{j+1} - x_j$.

From Lemma 1, we can see that

$$
\begin{aligned}
-l_M \sum_{i=k-l_M}^{k-1} \eta_i^T R_1 \eta_i &\le -\Big(\sum_{i=k-l_M}^{k-1} \eta_i\Big)^T R_1 \Big(\sum_{i=k-l_M}^{k-1} \eta_i\Big)\\
&= -(x_k - x_{k-l_M})^T R_1 (x_k - x_{k-l_M})
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
&-(l_M - l_m) \sum_{i=k-l_M}^{k-l_m-1} \eta_i^T R_2 \eta_i\\
&\le -(x_{k-l_m} - x_{k-l_k})^T R_2 (x_{k-l_m} - x_{k-l_k})\\
&\quad - (x_{k-l_k} - x_{k-l_M})^T R_2 (x_{k-l_k} - x_{k-l_M})
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
-l_m \sum_{i=k-l_m}^{k-1} \eta_i^T R_3 \eta_i &\le -\Big(\sum_{i=k-l_m}^{k-1} \eta_i\Big)^T R_3 \Big(\sum_{i=k-l_m}^{k-1} \eta_i\Big)\\
&= -(x_k - x_{k-l_m})^T R_3 (x_k - x_{k-l_m})
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
-r_M \sum_{i=k-r_M}^{k-1} \eta_i^T R_4 \eta_i &\le -\Big(\sum_{i=k-r_M}^{k-1} \eta_i\Big)^T R_4 \Big(\sum_{i=k-r_M}^{k-1} \eta_i\Big)\\
&= -(x_k - x_{k-r_M})^T R_4 (x_k - x_{k-r_M})
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
-\rho_M \sum_{i=k-\rho_M}^{k-1} \eta_i^T R_5 \eta_i &\le -\Big(\sum_{i=k-\rho_M}^{k-1} \eta_i\Big)^T R_5 \Big(\sum_{i=k-\rho_M}^{k-1} \eta_i\Big)\\
&= -(x_k - x_{k-\rho_M})^T R_5 (x_k - x_{k-\rho_M})
\end{aligned} \tag{12}
$$

Define $\triangle V_k = V_{k+1} - V_k$, then

$$\triangle V_{1k} = x_{k+1}^T P x_{k+1} - x_k^T P x_k \tag{13}$$

$$\triangle V_{2k} \le x_k^T Q_1 x_k + \sum_{i=k-l_M+1}^{k-l_m} x_i^T Q_1 x_i - x_{k-l_k}^T Q_1 x_{k-l_k} \tag{14}$$

$$
\begin{aligned}
\triangle V_{3k} &= \sum_{i=-l_M+1}^{-l_m} (x_k^T Q_1 x_k - x_{k+i}^T Q_1 x_{k+i})\\
&= (l_M - l_m) x_k^T Q_1 x_k - \sum_{i=k-l_M+1}^{k-l_m} x_i^T Q_1 x_i
\end{aligned} \tag{15}
$$

$$\triangle V_{4k} = x_k^T Q_2 x_k - x_{k-l_M}^T Q_2 x_{k-l_M} \tag{16}$$

$$\triangle V_{5k} = x_k^T Q_3 x_k - x_{k-l_m}^T Q_3 x_{k-l_m} \tag{17}$$

$$
\begin{aligned}
\triangle V_{6k} &= l_M \sum_{i=-l_M}^{-1} [\eta_k^T R_1 \eta_k - \eta_{k+i}^T R_1 \eta_{k+i}]\\
&= l_M^2 (x_{k+1} - x_k)^T R_1 (x_{k+1} - x_k) - l_M \sum_{i=k-l_M}^{k-1} \eta_i^T R_1 \eta_i
\end{aligned} \tag{18}
$$

$$\triangle V_{7k} = (l_M - l_m)\sum_{i=-l_M}^{-l_m-1}[\eta_k^T R_2 \eta_k - \eta_{k+i}^T R_2 \eta_{k+i}]$$
$$= (l_M - l_m)^2 (x_{k+1} - x_k)^T R_2 (x_{k+1} - x_k)$$
$$- (l_M - l_m)\sum_{i=k-l_M}^{k-l_m-1} \eta_i^T R_2 \eta_i \qquad (19)$$

$$\triangle V_{8k} = l_m \sum_{i=-l_m}^{-1} [\eta_k^T R_3 \eta_k - \eta_{k+i}^T R_3 \eta_{k+i}]$$
$$= l_m^2 (x_{k+1} - x_k)^T R_3 (x_{k+1} - x_k) - l_m \sum_{i=k-l_m}^{k-1} \eta_i^T R_3 \eta_i \qquad (20)$$

Similarly, we have

$$\triangle V_{9k} \le x_k^T Q_4 x_k + \sum_{i=k-r_M+1}^{k-r_m} x_i^T Q_4 x_i - x_{k-r_k}^T Q_4 x_{k-r_k} \qquad (21)$$

$$\triangle V_{10k} = (r_M - r_m) x_k^T Q_4 x_k - \sum_{i=k-r_M+1}^{k-r_m} x_i^T Q_4 x_i \qquad (22)$$

$$\triangle V_{11k} = x_k^T Q_5 x_k - x_{k-r_M}^T Q_5 x_{k-r_M} \qquad (23)$$

$$\triangle V_{12k} = r_M^2 (x_{k+1} - x_k)^T R_4 (x_{k+1} - x_k) - r_M \sum_{i=k-r_M}^{k-1} \eta_i^T R_4 \eta_i \qquad (24)$$

$$\triangle V_{13k} \le x_k^T Q_6 x_k + \sum_{i=k-\rho_M+1}^{k-\rho_m} x_i^T Q_6 x_i - x_{k-\rho_k}^T Q_6 x_{k-\rho_k} \qquad (25)$$

$$\triangle V_{14k} = (\rho_M - \rho_m) x_k^T Q_6 x_k - \sum_{i=k-\rho_M+1}^{k-\rho_m} x_i^T Q_6 x_i \qquad (26)$$

$$\triangle V_{15k} = x_k^T Q_7 x_k - x_{k-\rho_M}^T Q_7 x_{k-\rho_M} \qquad (27)$$

$$\triangle V_{16k} = \rho_M^2 (x_{k+1} - x_k)^T R_5 (x_{k+1} - x_k) - \rho_M \sum_{i=k-\rho_M}^{k-1} \eta_i^T R_5 \eta_i \qquad (28)$$

Combining (8)-(28) together, we have

$$\Delta V_k \le \xi_k{}^T (\Lambda + \Omega) \xi_k \qquad (29)$$

where $\Lambda$ is omitted here for briefness, and

$$\xi_k^T = \begin{bmatrix} x_k^T & x_{k-l_k}^T & x_{k-l_m}^T & x_{k-l_M}^T & x_{k-r_k}^T \\ x_{k-r_M}^T & x_{k-\rho_k}^T & x_{k-\rho_M}^T & \omega_k^T \end{bmatrix}$$

$$\begin{aligned} \Pi_1 &= \begin{bmatrix} \Psi_1 & \Psi_2 & 0 & 0 & \Psi_3 & 0 & \Psi_4 & 0 & \Psi_5 \end{bmatrix} \\ \Pi_2 &= \begin{bmatrix} \Psi_1 - I & \Psi_2 & 0 & 0 & \Psi_3 & 0 & \Psi_4 & 0 & \Psi_5 \end{bmatrix} \\ \Omega &= \Pi_1^T P \Pi_1 + l_M^2 \Pi_2^T R_1 \Pi_2 + (l_M - l_m)^2 \Pi_2^T R_2 \Pi_2 \\ &\quad + l_m^2 \Pi_2^T R_3 \Pi_2 + r_M^2 \Pi_2^T R_4 \Pi_2 + \rho_M^2 \Pi_2^T R_5 \Pi_2 \end{aligned}$$

From (3), we can see that $z_k = C_1 x_k - D_1 K x_{k-l_k}$, and $z_k$ can be written as $z_k = \Theta_1 \xi_k$, where $\Theta_1 = \begin{bmatrix} C_1 & -D_1 K & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, similarly, $\omega_k = \Theta_2 \xi_k$, where $\Theta_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$, for any nonzero $\xi_k$, we have

$$\gamma_k^{-1} z_k^T z_k - \gamma_k \omega_k^T \omega_k = \xi_k^T \Xi \xi_k$$

where $\Xi = \gamma_k^{-1} \Theta_1^T \Theta_1 - \gamma_k \Theta_2^T \Theta_2$, so

$$\gamma_k^{-1} z_k^T z_k - \gamma_k \omega_k^T \omega_k + \Delta V_k \le \xi_k^T \Upsilon \xi_k$$

where $\Upsilon = \Lambda + \Omega + \Xi$.

Using the Schur complement, it is easy to prove that if (5) is satisfied, we have $\Upsilon < 0$, then for any nonzero $\xi_k$, we have $\gamma_k^{-1} z_k^T z_k - \gamma_k \omega_k^T \omega_k + \Delta V_k < 0$.

Since $\gamma_k^{-1} z_k^T z_k - \gamma_k \omega_k^T \omega_k + \Delta V_k < 0$, then

$$\gamma_k^{-1} z_k^T z_k - \gamma_k \omega_k^T \omega_k < -\Delta V_k$$

Summing up $z_k^T z_k$, $\omega_k^T \omega_k$ and $\Delta V_k$ in the above inequality for $k = 0$ to $k = n$, using the zero initial condition, we have

$$\sum_{k=0}^{n} ||z_k||^2 < \gamma_k^2 \sum_{k=0}^{n} ||\omega_k||^2 - \gamma_k V_{n+1}$$

the above inequality holds for all $n$, let n $\to \infty$, we have

$$||z||_2^2 < \gamma_k^2 ||\omega||_2^2$$

If the disturbance input $\omega_k = 0$, (5) can ensure the asymptotic stability of the system described by (3), and if $\omega_k \neq 0$, we have $||z||_2^2 < \gamma_k^2 ||\omega||_2^2$, so if (5) is satisfied, the system described by (3) with $K = V^T N^{-1}$ is asymptotically stable with $H_\infty$ norm bound $\gamma_k$, this completes the proof. ■

**Remark 1.** Just as shown in Theorem 1, it is difficult to optimize all the $\gamma_k$ simultaneously, the linear weighted sum $\gamma_{sum}$ of $\gamma_k$ may be introduced to optimize $\gamma_k$. Suppose

$$\alpha_1 \gamma_1 + \alpha_2 \gamma_2 + \cdots + \alpha_{2l+1} \gamma_{2l+1} < \gamma_{sum}$$

where $\gamma_j$ $(j = 1,\ 2,\ \cdots,\ 2l+1)$ are the feasible values of $H_\infty$ norm bound $\gamma_k$, $\alpha_j$ are the weighting coefficients and $\alpha_j > 0$, the optimal $\gamma_k$ can be obtained by optimizing $\gamma_{sum}$. Generally speaking, for specific weighting coefficients $\alpha_p$ (where $p = 1,\ 2,\ \cdots,\ 2l+1$ and $p \neq j$), the larger the weighting coefficient $\alpha_j$ $(j = 1,\ 2,\ \cdots,\ 2l+1)$, the better the $H_\infty$ norm bound $\gamma_j$, one may choose appropriate weighting coefficients to get the desired $H_\infty$ norm bounds.

**Remark 2.** The $H_\infty$ controller design proposed in Theorem 1 can be extended easily to the case that the actuator receives more than two control inputs during a sampling period, it is omitted here.

**Remark 3.** Since constant sampling period is a special case of time-varying sampling period, the $H_\infty$ controller design method proposed in Theorem 1 is also applicable for NCSs with constant sampling period.

If constant sampling period $h$ is adopted, suppose $\varepsilon_{k1} \in [0,\ h]$, $\varepsilon_{k2} \in [0,\ h]$, $\varepsilon_{k1} \le \varepsilon_{k2}$, $\varepsilon_{k1}$ and $\varepsilon_{k2}$ switch in the finite set $\vartheta_3$, where $\vartheta_3 = \{\beta | \beta \in [0,\ h]\}$. If the actuator receives two control inputs during a sampling period, the discrete time representation of (1) is as follows

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma_{l_k} u_{k-l_k} + \Gamma_{r_k} u_{k-r_k} + \Gamma_{\rho_k} u_{k-\rho_k} + \widetilde{\Gamma} \omega_k \\ z_k &= C_1 x_k - D_1 K x_{k-l_k} \end{aligned} \qquad (30)$$

where $\Phi = e^{Ah}$, $\Gamma_{l_k} = \int_0^{\varepsilon_{k1}} e^{A(h-s)} ds B_1$, $\Gamma_{r_k} = \int_{\varepsilon_{k1}}^{\varepsilon_{k2}} e^{A(h-s)} ds B_1$, $\Gamma_{\rho_k} = \int_{\varepsilon_{k2}}^{h} e^{A(h-s)} ds B_1$, $\widetilde{\Gamma} = \int_0^h e^{As} ds B_2$, $u_k = -K x_k$.

Similar to Theorem 1, the following corollary presents the $H_\infty$ controller design for NCSs with constant sampling period.

**Corollary 1.** For given positive scalars $l_M$, $l_m$, $r_M$, $r_m$, $\rho_M$, $\rho_m$, if there exist symmetric positive definite matrices $N$, $\widetilde{Q}_i$ $(i=1,\ \cdots,\ 7)$, $\widetilde{R}_j$ $(j=1,\ \cdots,\ 5)$, and matrix $V$, scalar $\gamma>0$, such that the following LMIs hold for every feasible value of $\varepsilon_{k1}$ and $\varepsilon_{k2}$ $(\varepsilon_{k1}\in\vartheta_3,\ \varepsilon_{k2}\in\vartheta_3)$

$$
\left[\begin{array}{cccccccccccccccc}
\widetilde{\Lambda}_{11} & 0 & \widetilde{R}_3 & \widetilde{R}_1 & 0 & \widetilde{R}_4 & 0 & \widetilde{R}_5 & 0 & N\Phi^T & \mathscr{J}_1 & \mathscr{J}_1 & \mathscr{J}_1 & \mathscr{J}_1 & \mathscr{J}_1 & NC_1^T \\
\ast & \widetilde{\Lambda}_{22} & \widetilde{R}_2 & \widetilde{R}_2 & 0 & 0 & 0 & 0 & 0 & \mathscr{J}_2 & \mathscr{J}_2 & \mathscr{J}_2 & \mathscr{J}_2 & \mathscr{J}_2 & \mathscr{J}_2 & -VD_1^T \\
\ast & \ast & \widetilde{\Lambda}_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \widetilde{\Lambda}_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & -\widetilde{Q}_4 & 0 & 0 & 0 & 0 & \mathscr{J}_3 & \mathscr{J}_3 & \mathscr{J}_3 & \mathscr{J}_3 & \mathscr{J}_3 & \mathscr{J}_3 & 0 \\
\ast & \ast & \ast & \ast & \ast & \widetilde{\Lambda}_{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & -\widetilde{Q}_6 & 0 & 0 & \mathscr{J}_4 & \mathscr{J}_4 & \mathscr{J}_4 & \mathscr{J}_4 & \mathscr{J}_4 & \mathscr{J}_4 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \widetilde{\Lambda}_{88} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & -\gamma I & \widetilde{\Gamma}^T & \widetilde{\Gamma}^T & \widetilde{\Gamma}^T & \widetilde{\Gamma}^T & \widetilde{\Gamma}^T & \widetilde{\Gamma}^T & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & -N & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \mathscr{L}_1 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \mathscr{L}_2 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \mathscr{L}_3 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \mathscr{L}_4 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \mathscr{L}_5 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & -\gamma I
\end{array}\right] < 0 \qquad (31)
$$

where

$$
\begin{aligned}
\widetilde{\Lambda}_{11} &= -N+(l_M-l_m+1)\widetilde{Q}_1+\widetilde{Q}_2+\widetilde{Q}_3+(r_M-r_m+1)\widetilde{Q}_4 \\
&\quad +\widetilde{Q}_5+(\rho_M-\rho_m+1)\widetilde{Q}_6+\widetilde{Q}_7-\widetilde{R}_1-\widetilde{R}_3-\widetilde{R}_4-\widetilde{R}_5 \\
\widetilde{\Lambda}_{22} &= -\widetilde{Q}_1-2\widetilde{R}_2, \qquad \widetilde{\Lambda}_{33} = -\widetilde{Q}_3-\widetilde{R}_2-\widetilde{R}_3 \\
\widetilde{\Lambda}_{44} &= -\widetilde{Q}_2-\widetilde{R}_1-\widetilde{R}_2, \qquad \widetilde{\Lambda}_{66} = -\widetilde{Q}_5-\widetilde{R}_4 \\
\widetilde{\Lambda}_{88} &= -\widetilde{Q}_7-\widetilde{R}_5, \qquad \mathscr{J}_1 = N\Phi^T-N \\
\mathscr{J}_2 &= -V\Gamma_{l_k}{}^T, \qquad \mathscr{J}_3 = -V\Gamma_{r_k}{}^T \\
\mathscr{J}_4 &= -V\Gamma_{\rho_k}{}^T, \qquad \mathscr{L}_1 = l_M^{-2}(\widetilde{R}_1-2N) \\
\mathscr{L}_2 &= (l_M-l_m)^{-2}(\widetilde{R}_2-2N), \qquad \mathscr{L}_3 = l_m^{-2}(\widetilde{R}_3-2N) \\
\mathscr{L}_4 &= r_M^{-2}(\widetilde{R}_4-2N), \qquad \mathscr{L}_5 = \rho_M^{-2}(\widetilde{R}_5-2N)
\end{aligned}
$$

then with the control law

$$u_k=-Kx_k, \quad K=V^TN^{-1}$$

the system described by (30) is asymptotically stable with $H_\infty$ norm bound $\gamma$.

*Proof:* The proof is similar to the proof of Theorem 1, here it is omitted. ∎

In the following, we will illustrate the effectiveness of the proposed design method by an example.

TABLE I

THE $H_\infty$ NORM BOUNDS $(\alpha_1=1.8,\ \alpha_2=0.6)$

| $l_M$ | 4 | 5 | 6 |
|---|---|---|---|
| $\gamma_1$ | 4.7436 | 51.2995 | - |
| $\gamma_2$ | 5.2160 | 68.5423 | - |

TABLE II

THE $H_\infty$ NORM BOUNDS $(l_M=4)$

| | *Case* 1 | *Case* 2 | *Case* 3 |
|---|---|---|---|
| $\gamma_1$ | 5.1429 | 4.7400 | 5.3112 |
| $\gamma_2$ | 4.3308 | 5.2205 | 4.0759 |

## IV. SIMULATION RESULTS AND DISCUSSION

**Example 1.** To illustrate the effectiveness of the proposed $H_\infty$ controller design for NCSs with time-varying sampling period and packet dropout, we present an open loop unstable system as follows

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -0.0994 & 0.6708 \\ 0.4595 & -0.1881 \end{bmatrix} x(t) + \begin{bmatrix} 0.0372 \\ -0.2908 \end{bmatrix} u(t) \\
&\quad + \begin{bmatrix} 0.2450 \\ -0.8513 \end{bmatrix} \omega(t) \\
z(t) &= \begin{bmatrix} 0.3564 & 0.0788 \end{bmatrix} x(t) + 0.0942u(t)
\end{aligned} \qquad (32)
$$

Suppose the sampling period $h_k$ may switch among $h_1=0.08s$ and $h_2=0.1s$, $r_m=0$, $\rho_m=0$, $l_m=1$, $\rho_M=1$, $r_M=2$. Denote the $H_\infty$ norm bounds corresponding to sampling periods $0.08s$ and $0.1s$ as $\gamma_1$ and $\gamma_2$, respectively, suppose the weighting coefficients $\alpha_1=1.8$, $\alpha_2=0.6$, for simplicity of simulation, suppose $\varepsilon_{k1}=\varepsilon_{k2}$ and they may switch between $0.8h_1$ and $h_2$. Solving the LMIs presented in Theorem 1, we can get the $H_\infty$ norm bounds corresponding to different $l_M$ (see Table 1, '-' denotes that the LMIs are infeasible), from what we can see that the $H_\infty$ performance of system will degrade with the increase of $l_M$, similarly, the $H_\infty$ performance of system will degrade with the increase of $r_M$ and $\rho_M$, here it is omitted for space limit.

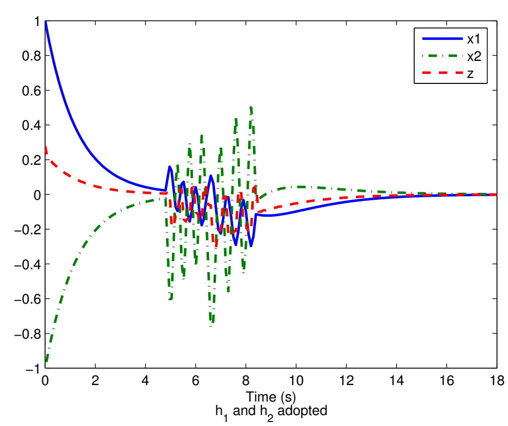


Fig. 1. State response and controlled output

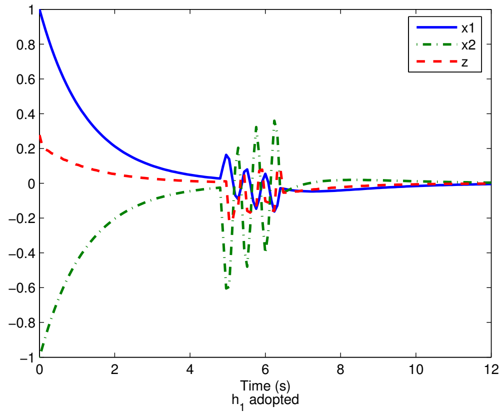


Fig. 2. State response and controlled output

Suppose $l_M = 4$, then the $H_\infty$ norm bounds corresponding to different weighting coefficients are shown in Table 2 (Case 1 is corresponding to $\alpha_1 = 1$, $\alpha_2 = 0.6$, Case 2 is corresponding to $\alpha_1 = 3$, $\alpha_2 = 1$, Case 3 is corresponding to $\alpha_1 = 2.2$, $\alpha_2 = 1.6$, respectively), from what we can see that different weighting coefficients $\alpha_1$ and $\alpha_2$ may lead to different $H_\infty$ norm bounds, one may choose appropriate weighting coefficients to get the desired $H_\infty$ performance.

Suppose the initial state of the system is $x_0 = [1 - 1]^T$ and the control inputs based on plant states $x_0,\ x_2,\ x_4,\ \cdots$ are transferred to the actuator successfully, while the control inputs based on plant states $x_1,\ x_3,\ x_5,\ \cdots$ are dropped. Suppose during the time interval $[0s,\ 6.4s)$, $[6.4s,\ 18s)$, the sampling periods are $0.08s$ and $0.1s$, respectively, if $h_1$ is adopted, $\varepsilon_{k1} = \varepsilon_{k2} = 0.8h_1$, and if $h_2$ is adopted, $\varepsilon_{k1} = \varepsilon_{k2} = h_2$. During the time interval $[4.8s,\ 6.4s)$, the disturbance inputs $5sin(j)$ $(j = 1,\ 2,\ \cdots,\ 20)$ are added into the system, during $[6.4s,\ 8.4s)$, another disturbance inputs $5sin(j)$ $(j = 1,\ 2,\ \cdots,\ 20)$ are added into the system. Suppose $\alpha_1 = 2.2$, $\alpha_2$=1.6, by solving the multi-objective optimization problem in Remark 1, we can get the controller gain $K = [-3.6719\ \ -4.1193]$, the plant state response and controlled output are pictured in Fig. 1.

If $l_M = 4$ and constant sampling period $h_1$ is adopted, $\varepsilon_{k1} = \varepsilon_{k2} = 0.8h_1$, by solving the LMIs in Corollary 1, we can get the $H_\infty$ norm bound $\gamma = 2.6890$, and the controller gain $K = [-4.4842\ \ -5.0145]$. During the time interval $[4.8s,\ 6.4s)$, the disturbance inputs $5sin(j)$ $(j = 1,\ 2,\ \cdots,\ 20)$ are added into the system, the plant state response and controlled output are pictured in Fig. 2.

Table 1, Table 2, Fig. 1 and Fig. 2 illustrate the effectiveness of the proposed $H_\infty$ controller design for NCSs with time-varying sampling period and constant sampling period.

## V. CONCLUSIONS

This paper studies the problem of $H_\infty$ controller design for NCSs with time-varying sampling period. The considered NCSs may receive more than one control input during a sampling period, time delay and packet dropout are also taken into consideration. The problem of $H_\infty$ controller design for NCSs with time-varying sampling period is converted into a multi-objective optimization problem in terms of LMIs, and the discrete Jensen inequality is adopted for controller design. The proposed $H_\infty$ controller design is also applicable for NCSs with constant sampling period. The simulation results have illustrated the effectiveness of the proposed $H_\infty$ controller design.

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