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Robust Control and Time-Domain Specifications for Systems of Delay Differential Equations via Eigenvalue Assignment

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Abstract— **An approach for eigenvalue assignment for systems of delay differential equations (DDEs), based upon the Lambert W function, is applied to the problem of robust control design for perturbed systems of DDEs, and to the problem of time-domain specifications. The real stability radius, which measures the ability of a system to preserve its stability under a certain class of real perturbations, can be computed from known nominal coefficients of the DDE representing the system. In this paper, considering the stability radius, the real part of the eigenvalues is assigned. Also, time-domain specifications for the transient response of systems of DDEs are improved in a way similar to systems of ordinary differential equations using the eigenvalue assignment approach.**

I. Introduction

One of the main concerns for control engineers is to maintain robust stability and good performance to meet time-domain specifications [25]. One way to achieve such goals is to assign the eigenvalues of the original system to some prescribed values. Much has been done to develop such pole placement, or eigenvalue assignment, design methods, for systems of ordinary differential equations (ODEs). However, systems of delay differential equations (DDEs) have an infinite number of eigenvalues, and it is not practically feasible to assign all of them. Thus, the usual pole placement design techniques in ODEs cannot be applied without considerable modification to systems of DDEs [22].

An approach for eigenvalue assignment for systems of DDEs was developed by Yi *et al.* [27], based on the solution in terms of the Lambert W function [28]. Using that approach, summarized in Section II of this paper, one can design a linear feedback controller to place the rightmost eigenvalues at the desired positions in the complex plane and, thus, stabilize the system. In that study, instead of all eigenvalues of systems of DDEs, the critical subset, which are rightmost and, thus, determine stability of the system, of the infinite eigenspectrum is assigned. This is possible because the eigenvalues are expressed in terms of the parameters of the system and each one is distinguished by a branch of the Lambert W function. The advantages of the Lambert W function-based approach over other existing methods (e.g., Finite Spectrum Assignment [17], Continuous Pole Placement [20] etc.) have been discussed in [27]. In this paper, that approach is extended to the problems of robust stability of systems with uncertain parameters, and to time-domain specifications.

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When uncertainty exists in the coefficients of the system, a robust control law, which can guarantee stability, is required. To realize robust stabilization, calculating allowable uncertainty (i.e., stability radius by a method presented in [11]), the rightmost eigenvalues are placed at an appropriate distance from the imaginary axis in order to guarantee stability. Usually, the robust control problem for systems of DDEs has been handled using linear matrix inequalities (LMI) or the algebraic Riccati equation (ARE) (see, e.g., [14], [16], [18], [21] and the references therein). Even though such approaches can be applied to more general types of time delay systems (e.g., systems with multiple delays, time-varying delay), they provide only sufficient conditions and are substantially conservative because of dependence on the selection of involved cost functions and their coefficients.

The method developed in [27] also makes it possible to assign simultaneously the real and imaginary parts of a critical subset of the eigenspectrum for the first time. Therefore, similar guidelines to those for systems of ODEs to improve transient response, can be used for systems of DDEs via eigenvalue assignment.

II. Lambert W Function-Based Eigenvalue Assignment Method

This section summarizes the solution and eigenvalue assignment for DDEs based on the matrix Lambert W function.

A. Solution in terms of the Lambert W function

Consider a real linear time-invariant (LTI) system of DDEs with a single constant delay,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-h) + \mathbf{B}\mathbf{u}(t) & t > 0 \\ \mathbf{x}(t) &= \mathbf{g}(t) & t \in [-h, 0) \\ \mathbf{x}(t) &= \mathbf{x}_0 & t = 0 \end{aligned} \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is a state vector; $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{A_d} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times r}$ are system matrices; $\mathbf{u}(t) \in \mathbb{R}^{r\times 1}$ is a function representing the external excitation. A specified preshape function, $\mathbf{g}(t)$ and an initial point, $\mathbf{x}_0$ are defined in the Banach space of continuous mappings [9]. In [28], the solution, which is expressed in terms of the matrix Lambert W function, to (1) was developed and given by

$$\mathbf{x}(t) = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^I + \int_0^t \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-\xi)}\mathbf{C}_k^N \mathbf{B}\mathbf{u}(\xi)d\xi \tag{2}$$

where

$$\mathbf{S}_k = \frac{1}{h}\mathbf{W}_k(\mathbf{A_d} h \mathbf{Q}_k) + \mathbf{A} \tag{3}$$

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The coefficient $\mathbf{C}_k^I$ in (2) is a function of $\mathbf{A}$, $\mathbf{A_d}$, h and the preshape function, $\mathbf{g}(t)$ and the initial condition, $\mathbf{x}_0$, while $\mathbf{C}_k^N$ is a function of $\mathbf{A}$, $\mathbf{A_d}$, h and does not depend on $\mathbf{g}(t)$ or $\mathbf{x}_0$. The numerical and analytical methods for computing $\mathbf{C}_k^I$ and $\mathbf{C}_k^N$ were developed in [1], [29]. The following condition is used to solve for the unknown matrix $\mathbf{Q}_k$

$$\mathbf{W}_k(\mathbf{A_d}h\mathbf{Q}_k)e^{\mathbf{W}_k(\mathbf{A_d}h\mathbf{Q}_k)+\mathbf{A}h} = \mathbf{A_d}h \quad (4)$$

The matrix Lambert W function, $\mathbf{W}_k(\mathbf{H}_k)$, which satisfies the definition [7],

$$\mathbf{W}_k(\mathbf{H}_k)e^{\mathbf{W}_k(\mathbf{H}_k)} = \mathbf{H}_k \quad (5)$$

is complex valued, with a complex argument, $\mathbf{H}_k$, and has an infinite number of branches for $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$. The principal ($k = 0$) and other ($k \neq 0$) branches of the Lambert W function can be calculated analytically [7], or using commands already embedded in the various commercial software packages, such as Matlab, Maple, and Mathematica.

The solution, $\mathbf{Q}_k$, for each branch, k, to Eq. (4) is obtained numerically, for a variety of initial conditions, such as using the *fsolve* function in Matlab. Conditions for convergence of the infinite series in (2) have been studied in [2], [3], [9], [15]. For example, for a bounded external excitation, $\mathbf{u}(t)$, if the coefficient, $\mathbf{A_d}$, is nonsingular, the infinite series converges to the solution.

Note that, compared with results by other existing methods for the series expansion of solution to DDEs, where eigenvalues are obtained from exhaustive numerical computation, the solution in terms of the Lambert W function has an analytical form expressed with the parameters of the DDE in (1), i.e., $\mathbf{A}$, $\mathbf{A_d}$ and h. Hence, one can determine how those parameters are involved in the solution and, furthermore, how each parameter affects each eigenvalue and the solution. Also, each eigenvalue is distinguished in terms of k, which indicates the branch of the Lambert W function. For these reasons, the Lambert W function-based approaches have been applied to control problems and extended to other cases (see, e.g., [4], [5], [6], [10], [12], [24]).

B. Stability by the principal branch

The solution form in equation (2) reveals that the stability condition for the system of (1) depends on the eigenvalues of the matrix $\mathbf{S}_k$, and, thus, also on the matrix $e^{\mathbf{S}_k}$. A time delayed system characterized by (1) is asymptotically stable if and only if all the eigenvalues of $\mathbf{S}_k$, $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$, have negative real parts. Computing the matrices $\mathbf{S}_k$ for an infinite number of branches is not practically feasible. However, if coefficient matrix $\mathbf{A_d}$ does not have repeated zero eigenvalues, we have observed that the eigenvalues obtained using the principal branch ($k = 0$) are the rightmost ones and determine the stability of the system [31]. For the scalar case, it is proved that the root obtained using the principal branch always determines the stability of the system using monotinicity of the real part of the Lambert W function with respect to its branch k [24]. Such a proof can readily be extended to systems of DDEs where $\mathbf{A}$ and $\mathbf{A_d}$ commute and, thus, are simultaneously triangularizable. Although such a proof is not available in the case of general matrix-vector DDEs, we observe the same behavior in all the examples we have considered. We use that observation as the basis not only to determine stability of systems of DDEs, but also to place a subset of the eigenspectrum at desired locations.

C. Eigenvalue Assignment

The solutions (2) in terms of the matrix Lambert W function have also been used to develop an approach for eigenvalue assignment for systems of DDEs by Yi *et al.* [27]. In designing a linear feedback controller for a delayed system, represented by DDEs in (1), because there exists an infinite number of solution matrices, $\mathbf{S}_k$, and the number of control parameters is finite, it is not feasible to assign all of them at once. Just by using the classical pole placement method for ODEs, placement of a selected finite number of eigenvalues by may cause other uncontrolled eigenvalues to move to the right-half plane (RHP) [20]. However, the subsequent approach for control design using the matrix Lambert W function provides proper control laws without such loss of stability.

Consider the system in (1) and a generalized feedback containing current and delayed states

$$\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t) + \mathbf{K_d}\mathbf{x}(t-h) \quad (6)$$

Then, the closed-loop system becomes

$$\dot{\mathbf{x}}(t) = \{\mathbf{A} + \mathbf{BK}\}\mathbf{x}(t) + \{\mathbf{A_d} + \mathbf{BK_d}\}\mathbf{x}(t-h) \quad (7)$$

The controllability of such system, using the solution form of (2) was studied in [32]. The gains, $\mathbf{K}$ and $\mathbf{K_d}$ are determined as follows. First, select desired eigenvalues, $\lambda_{i,desired}$ for $i = 1, \cdots, n$, and set an equation so that the selected eigenvalues become those of the matrix $\mathbf{S}_0$ as

$$\lambda_i(\mathbf{S}_0) = \lambda_{i,desired} \quad (8)$$

for $i = 1, \cdots, n$, where, $\lambda_i(\mathbf{S}_0)$ is i^{th} eigenvalue of the matrix $\mathbf{S}_0$. Second apply the new two coefficient matrices $\mathbf{A}' \equiv \mathbf{A} + \mathbf{BK}$ and $\mathbf{A_d}' \equiv \mathbf{A_d} + \mathbf{BK_d}$, as Eq. (7) to Eq. (4) and solve numerically to obtain the matrix $\mathbf{Q}_0$ for the principal branch ($k = 0$). Note that $\mathbf{K}$ and $\mathbf{K_d}$ are unknown matrices with all unknown elements, and the matrix $\mathbf{Q}_0$ is a function of the unknown $\mathbf{K}$ and $\mathbf{K_d}$. For the third step, substitute the matrix $\mathbf{Q}_0$ from Eq. (4) into Eq. (3) to obtain $\mathbf{S}_0$ and its eigenvalues as the function of the unknown matrix $\mathbf{K}$ and $\mathbf{K_d}$. Finally, Eq. (8) with the matrix, $\mathbf{S}_0$, is solved for the unknown $\mathbf{K}$ and $\mathbf{K_d}$ using numerical methods, such as *fsolve* in Matlab. Depending on the structure or parameters of a given system, there exists limitations on the rightmost eigenvalues and some values are not permissible. In that case, the above approach does not yield any solution for $\mathbf{K}$ and $\mathbf{K_d}$. To resolve the problem, one may try again with fewer desired eigenvalues, or different values of the desired rightmost eigenvalues. Then, the solution, $\mathbf{K}$ and $\mathbf{K_d}$, is obtained numerically for a variety of initial conditions by an empirical trial and error procedure [27].

III. Stability Radius

The real stability radius, the norm of the minimum destabilizing perturbations, was obtained for linear systems of ODEs and a computable formula for the exact real stability radius was presented by Qiu *et al.* [26]. The real stability radius measures the ability of a system to preserve its stability under a certain class of real perturbations. The formula was extended to perturbed linear systems of DDEs in [11].

Assume that the perturbed system (1) can be written in the form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \{\mathbf{A}+\delta\mathbf{A}\}\mathbf{x}(t)+\{\mathbf{A_d}+\delta\mathbf{A_d}\}\mathbf{x}(t-h)\\ &= \{\mathbf{A}+\mathbf{E}\Delta_1\mathbf{F}_1\}\mathbf{x}(t)+\{\mathbf{A_d}+\mathbf{E}\Delta_2\mathbf{F}_2\}\mathbf{x}(t-h)\end{aligned} \tag{9}$$

where $\mathbf{E}\in\mathbb{R}^{n\times m}$, $\mathbf{F}_i\in\mathbb{R}^{l_i\times n}$, and $\Delta_i\in\mathbb{R}^{m\times l_i}$ denotes the perturbation matrix. Provided that the unperturbed system (1) is stable, the real structured stability radius of (9) is defined as [11]

$$r_{\mathbb{R}} = \inf\{\sigma_1(\Delta) : \text{system is unstable}\} \tag{10}$$

where $\Delta = [\Delta_1\ \Delta_2]$ and $\sigma_1(\Delta)$ denotes the largest singular value of Δ. The largest singular value, $\sigma_1(\Delta)$ is equal to the operator norm of Δ, which measures the size of Δ by how much it lengthens vectors in the worst case. Thus, the stability radius in (10) represents the size of the smallest perturbations in parameters, which can cause instability of a system. And the real stability radius problem concerns the computation of the real stability radius when the nominal system is known. The stability radius is computed from [11]

$$r_{\mathbb{R}} = \left\{\sup_{\omega}\inf_{\gamma\in(0,1]}\sigma_2\left(\begin{bmatrix}\Re(\Omega(j\omega)) & -\gamma\Im(\Omega(j\omega))\\ \gamma^{-1}\Im(\Omega(j\omega)) & \Re(\Omega(j\omega))\end{bmatrix}\right)\right\} \tag{11}$$

where

$$\Omega(s) = \begin{bmatrix}\mathbf{F}_1\\ \mathbf{F}_2e^{-hs}\end{bmatrix}(s\mathbf{I}-\mathbf{A}-\mathbf{A_d})^{-1}\mathbf{E} \tag{12}$$

In (11), it is not practically feasible to compute the supremum value for the whole range of $\omega\in(-\infty,\infty)$. However, for the value ω^*, which satisfies

$$\omega^* < \bar{\sigma}(\mathbf{A})+\bar{\sigma}(\mathbf{A_d})+\bar{\sigma}(\mathbf{E})\underline{\sigma}[\mathbf{W}(0)]\bar{\sigma}([\mathbf{F}_1\mathbf{F}_2]) \tag{13}$$

where

$$\mathbf{W}(0) = \begin{bmatrix}\mathbf{F}_1\\ \mathbf{F}_2\end{bmatrix}(-\mathbf{A}-\mathbf{A_d})^{-1}\mathbf{E} \tag{14}$$

Then,

$$\text{restab}(\omega^*) \le \text{restab}(\omega) \tag{15}$$

where

$$\text{restab}(\omega) = \left\{\inf_{\gamma\in(0,1]}\sigma_2\left(\begin{bmatrix}\Re(\Omega(j\omega)) & -\gamma\Im(\Omega(j\omega))\\ \gamma^{-1}\Im(\Omega(j\omega)) & \Re(\Omega(j\omega))\end{bmatrix}\right)\right\} \tag{16}$$

Therefore, one has only to check $\omega\in[0,\omega^*]$ to obtain the supremum value in (11).

The obtained stability radius from (11) provides a basis for assigning eigenvalues for robust stability of systems of DDEs with uncertain parameters.

IV. Design of Robust Feedback Controller

In this section, an algorithm is presented for the calculation of feedback gains to maintain stability for uncertain systems of DDEs. The approach to eigenvalue assignment using the Lambert W function is used to design robust linear feedback control laws, combined with the stability radius concept. The feedback controller can be designed to stabilize the nominal delayed system (1) using the method outlined in Section II-*C* [27]. However, if the system has uncertainties in the coefficients, which can be introduced by static perturbations of the parameters or can arise in estimating the parameters, the designed controller cannot guarantee stability. Therefore, a *robust* feedback controller is required when uncertainty exists in the parameters. Such a controller can be realized by providing sufficient margins in assigning the rightmost eigenvalues of the delayed system. However, conservative margins over those required can raise problems, such as cost of control. The stability radius, outlined in Section III, provides a reasonable measurement of how large the margin should be.

The basic idea of the proposed algorithm is to shift the rightmost eigenvalue to the left by computing the gains in the linear feedback controller [27] and increase the stability radius until it becomes larger than the uncertainty of the coefficients. Then, one can obtain a robust controller to guarantee stability of the system with uncertainty.

Algorithm 1: Designing a robust feedback controller for systems of DDEs with uncertainty.

- Step 1. Compute the radius, r_1, from actual uncertainties in parameters of given delayed system, (i.e., $r_1 = \sigma_1(\Delta)$).
- Step 2. Using the eigenvalue assignment method presented in Section II, compute $\mathbf{K}$ and $\mathbf{K_d}$, to stabilize the system.
- Step 3. Then, compute the theoretical stability radius of the stabilized system, r_2 from Eq. (11).
- Step 4. If $r_1 > r_2$, then, the system can be destabilized by the uncertainties. Therefore, go to Step 2 and increase the margin (compute $\mathbf{K}$ and $\mathbf{K_d}$ to move the rightmost eigenvalues more to the left).

Example 1: From [27], consider a system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}-1 & -1\\ 0 & -0.9\end{bmatrix}\mathbf{x}(t-0.1)+\begin{bmatrix}0\\ 1\end{bmatrix}\mathbf{u}(t) \tag{17}$$

Without feedback control, the system in (17) has one unstable eigenvalue 0.1098. Using feedback control as in (6), designed by the method presented in [27], if the desired rightmost eigenvalue is -1.0000, the computed gains are $\mathbf{K} = [-0.1391 - 1.8982]$ and $\mathbf{K_d} = [-0.1236 - 1.8128]$, and the stability radius in (10) is 0.6255. However, if the system (17) has uncertainties in the parameter

$$\dot{\mathbf{x}}(t) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \delta\mathbf{A} \right\} \mathbf{x}(t) + \left\{ \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix} + \delta\mathbf{A_d} \right\} \mathbf{x}(t-0.1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t) \tag{18}$$

and $\sigma(\delta\mathbf{A}+\delta\mathbf{A_d}) = 0.7$, the system can become unstable due to uncertainty. To ensure stability, set the desired rightmost eigenvalue to be -2.0000, then the computed gains are $\mathbf{K} = [-0.1687 - 3.6111]$ and $\mathbf{K_d} = [1.6231 - 0.9291]$, and the stability radius in (10) increases to 0.8832. Therefore, the system can remain stable despite the uncertainty ($\sigma(\delta\mathbf{A} + \delta\mathbf{A_d}) = 0.7$). Table I shows the gains, $\mathbf{K}$ and $\mathbf{K_d}$, corresponding to the several subsets of eigenvalues of $\mathbf{S}_0$.

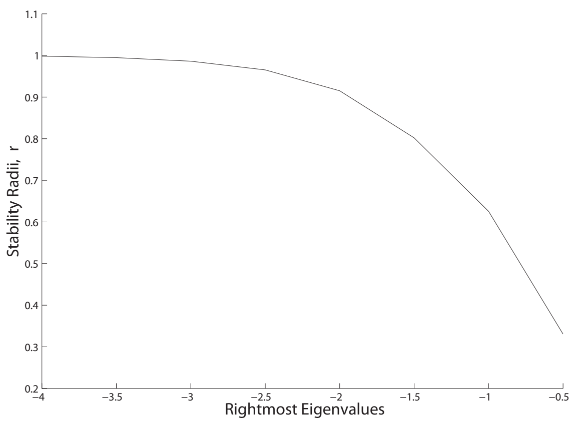

Fig. 1. As the eigenvalue moves left, then the stability radius increases consistently, which means, improved robustness.

The computed stability radii versus the rightmost eigenvalues, moving from -0.5 to -4 are shown in Figure 1. As seen in the figure, for the system (18), as the eigenvalue moves left, the stability radius increases monotonically. Note that, in general, an explicit relationship between the stability radius and the rightmost eigenvalues is not available, and moving the rightmost eigenvalues further to the left does not always lead to an increase of stability radius [20]. However, as shown above, by comparing the stability radius and uncertainty for a given system, Algorithm 1 can be used to achieve robust stability of TDS with uncertainty.

Michiels *et al.*, [19] developed an algorithm to maximize the stability radius by calculating its sensitivity with respect to the feedback gain of a type of TDS. However, in maximizing it, the rightmost eigenvalues can be moved to undesired positions and one can lose control of the system response. If the system has relatively small uncertainty, instead of maximizing the stability radius, one can focus more on the position of eigenvalues to improve the transient response of the system, which will be discussed in the subsequent section.

V. Time-Domain Specifications

To meet design specifications in the time-domain, PID-based controllers have been combined with a graphical approach [23], LQG method using ARE [25], or Smith predictors [13]. These methods are available for systems with control delays. For systems with state delays, linear matrix inequality approaches have been used (see, e.g., [18] and the references therein). In this section, the Lambert W function-based approach, presented in Section II, is applied to achieve time-domain specifications via eigenvalue assignment. Unlike other existing methods (e.g., Continuous Pole Placement in [20]), for the first time the Lambert W function-based approach can be used to assign the imaginary parts of system eigenvalues as well as their real parts for a critical subset of the infinite eigenspectrum. It is not practically feasible to assign the entire eigenspectrum; however, just by assigning some finite, but rightmost, eigenvalues the transient response of systems of DDEs can be improved to meet time-domain specifications for desired performance.

Example 2: Consider the system in (17). Table II shows the gains, $\mathbf{K}$ and $\mathbf{K_d}$, corresponding to the several subsets of eigenvalues of $\mathbf{S}_0$, which have a real part, -0.2, and different imaginary parts, $\pm 0.2i, \pm 0.5i$, and $\pm 1.0i$.

The eigenvalue is written as $\lambda = \sigma \pm j\omega_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$, the requirements for a step response are expressed in terms of the quantities, such as the rise time, t_r, the settling time, t_s, the overshoot, M_p, and the peak time, t_p. In case of ODEs, if the system is 2^{nd} order without zeros, the quantities have exact representations:

$$t_r = \frac{1.8}{\omega_n}, t_s = \frac{4.6}{\sigma}, M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}, t_p = \frac{\pi}{\omega_d} \tag{19}$$

For all other systems, however, these provide rough approximations, and can only provide a starting point for the design iteration [8]. Figure 2 shows the responses corresponding to the rightmost eigenvalues considered in Table II. Not surprisingly, the approximate values from (19) in Table III are not exactly same as the results from the responses from Figure 2. But, for this example, the guidelines for ODEs still work well in the case of DDEs.

Figure 3 shows two responses corresponding to the several subsets of eigenvalues of $\mathbf{S}_0$, which have different real parts (-0.2 and -1.0) with the same imaginary part (± 1.0). As seen in the figure, the settling time, the rise time, and overshoot decrease, but the peak time remain almost the same, and, for this example, the guidelines for ODEs still work well in case of DDEs. The approach presented in this section is straightforward for systems of ODEs. However, it represents the first approach to assign the real and imaginary parts of the eigenvalues simultaneously to meet time-domain specifications for time delay systems.

In this approach, we assign the real and imaginary parts of only the rightmost (i.e., for $k = 0$) eigenvalues. Even though DDEs have an infinite number of eigenvalues, as seen in the above examples, just by controlling the rightmost ones, one can achieve time-domain specifications with linear feedback controllers. The approach presented follows the

TABLE I

THE GAINS, $\mathbf{K}$ AND $\mathbf{K_D}$, OF THE LINEAR FEEDBACK CONTROLLER IN (6) CORRESPONDING TO EACH RIGHTMOST EIGENVALUES. COMPUTED BY USING THE APPROACH FOR EIGENVALUE ASSIGNMENT PRESENTED IN [27].

	K	**K_d**
-0.5 & -6	[-0.6971 -1.6893]	[-0.7098 -1.5381]
-1.0 & -6	[-0.1391 -1.8982]	[-0.1236 -1.8128]
-1.5 & -6	[-0.3799 -1.6949]	[1.0838 -2.3932]
-2.0 & -6	[0.8805 -2.1095]	[0.9136 -2.3932]
-2.5 & -6	[1.8716 -2.1103]	[0.8229 -2.5904]
-3.0 & -6	[2.5777 -1.7440]	[0.7022 -2.9078]
-3.5 & -6	[2.8765 -1.6818]	[0.9721 -3.1311]
-4.0 & -6	[3.1144 -1.5816]	[1.1724 -3.3304]

TABLE II

GAINS, $\mathbf{K}$ AND $\mathbf{K_D}$, AND PARAMETERS CORRESPONDING TO THE SEVERAL SUBSETS OF EIGENVALUES OF $\mathbf{S}_0$.

Rightmost Eigenvalues	$-0.2 \pm 0.2i$	$-0.2 \pm 0.5i$	$-0.2 \pm 1.0i$	$-0.5 \pm 1.0i$
σ	-0.2	-0.2	-0.2	-0.5
ω_d	0.2	0.5	1.0	1.0
ω_n	0.2828	0.5385	1.0198	1.1180
$\zeta = -\sigma/\omega_n$	0.7	0.3714	0.1961	0.4472
K	[0.0584 -1.7867]	[0.1405 -1.7998]	[0.4311 -1.8152]	[0.2380 -2.1656]
K_d	[0.6789 2.3413]	[0.7802 2.3204]	[1.1421 2.2124]	[0.9027 1.9451]

TABLE III

COMPARISON OF THE ACTUAL RESULTS FOR AND THE APPROXIMATIONS USING EQ. (19) OF TIME-DOMAIN SPECIFICATIONS FOR FIGURE 2.

	t_r		t_s		M_p		t_p	
Rightmost Eigenvalues	approximate (Eq. (19))	actual	approximate	actual	approximate	actual	approximate	actual
$-0.2 \pm 0.2i$	6.3640	6.9	23	23	4.60	6	15.7080	14.6
$-0.2 \pm 0.5i$	3.3425	2.5	23	23	28.46	31	6.2832	6.0
$-0.2 \pm 1.0i$	1.7650	0.8	23	27	53.35	75	3.1416	2.4

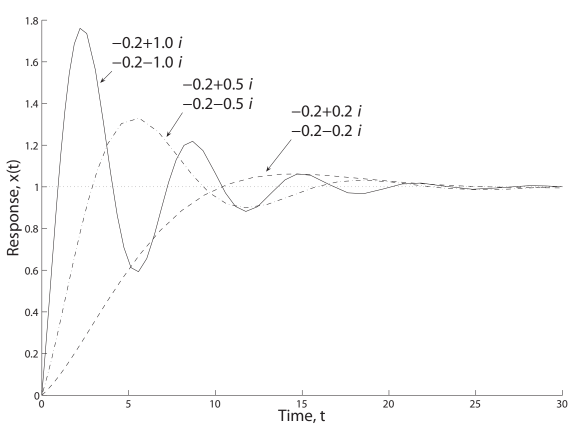

Fig. 2. Responses of the system in (17) with the feedback (6) corresponding to the rightmost eigenvalues in Table II with different imaginary parts of the rightmost eigenvalues.

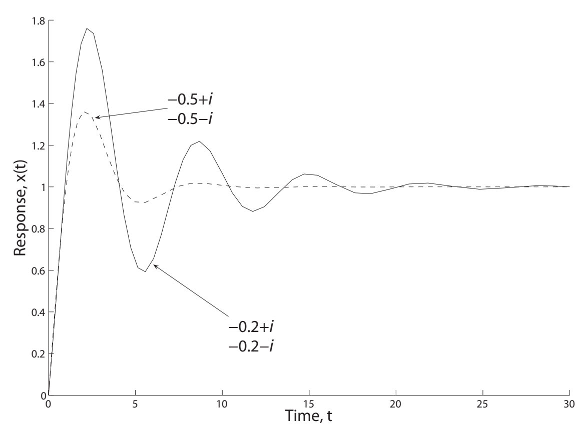

Fig. 3. Responses of the system in (17) with the feedback (6) corresponding to the rightmost eigenvalues in Table II with different real parts of the rightmost eigenvalues.

simple design guidelines for ODEs, and provides an effective rule of thumb to improve the transient response of systems of DDEs.

VI. Conclusions and Future Work

We have applied the eigenvalue assignment method based on the Lambert W function to design linear robust feedback controllers and to meet time-domain specifications of LTI systems of DDEs with a single delay. An algorithm for design of feedback controllers to maintain stability for uncertain systems of DDEs is presented. With the algorithm, considering the size of the uncertainty in the coefficients of systems of DDEs via the stability radius, one can find appropriate gains of the linear feedback controller by assigning the rightmost eigenvalues. The procedure presented in this paper can be applied to uncertain systems, where uncertainty in the system parameters cannot be ignored.

To improve the transient response of time delay systems, the design guideline for systems of ODEs has been used via the Lambert W function-based eigenvalue assignment. The presented approach is quite standard in case of ODEs. However, it has not been previously possible to use such methods for systems of DDEs. Because, unlike ODEs, DDEs have an infinite number of eigenvalues, controlling them has not been feasible due to the lack of analytical solution form. Using the approach based upon the solution form in terms of the matrix Lambert W function, the analysis for robustness and transient response can be extended from ODEs to DDEs as presented in this paper. The presented method, which is directly related to the position of the rightmost eigenvalues, provides an accurate and effective approach to analyze stability robustness and transient response of DDEs. Even though it is not feasible to assign all of the infinite eigenvalues of TDS, just by assigning the rightmost eigenvalues, one can control systems of DDEs in a way similar to systems of ODEs. This is the advantage of the Lambert W function-based approach over other existing methods.

VII. Acknowledgments

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References

[1] F. M. Asl and A. G. Ulsoy, "Analysis of a System of Linear Delay Differential Equations," *J. Dyn. Syst. Meas. Control*, vol. 125, 2003, pp. 215-223.

[2] H. T. Banks and A. Manitius, "Projection Series for Retarded Functional Differental Equations with Applications to Optimal Control Problems," *J. of Differential Equs*, vol. 18, 1975, pp. 296-332.

[3] R. E. Bellman and K. L. Cooke, *Differential-Difference Equations*, Academic Press, New York: 1963.

[4] Y. Chen and K. L. Moore, "Analytical Stability Bound for Delayed Second-Order Systems with Repeating Poles Using Lambert Function W," *Automatica*, vol. 38, 2002, pp. 891-895.

[5] Y. Chen and K. L. Moore, "Analytical Stability Bound for a Class of Delayed Fractional-Order Dynamic Systems," *Nonlinear Dyn*, vol. 29, 2002, pp. 191-200.

[6] Y. Cheng and C. Hwang, "Use of the Lambert W Function for Time-Domain Analysis of Feedback Fractional Delay Systems," *IEE Proc. Control Theory Appl.*, vol. 153, 2006, pp. 167-174.

[7] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth, "On the Lambert W Function," *Adv. Comput. Math.*, vol. 5, 1996, pp. 329-359.

[8] G. F. Franklin, J. D. Powell and A. Emami-Naeini, *Feedback Control of Dynamic Systems*. Upper Saddle River, NJ: Pearson Prentice Hall, Pearson Prentice Hall; 2005.

[9] J. K. Hale and Lunel, S. M. V., *Introduction to Functional Differential Equations*, Springer-Verlag, New York: 1993.

[10] P. Hövel and E. Schöll, "Control of Unstable Steady States by Time-Delayed Feedback Methods," *Physical Review E*, vol. 72, 2005, pp. 1-7.

[11] G. Hu and E. J. Davison, "Real Stability Radii of Linear Time-Invariant Time-Delay Systems", *Syst Control Lett*, vol. 50, 2003, pp. 209-219.

[12] C. Hwang and Y. Cheng, "A Note on The Use of the Lambert W Function in the Stability Analysis of Time-Delay Systems," *Automatica*, vol. 41, 2005, pp. 1979-1985.

[13] I. Kaya, "Autotuning of a New PI-PD Smith Predictor Based on Time Domain Specifications", *ISA Trans*, vol. 42, 2003, pp. 559-575.

[14] X. Li and C. E. deSouza, "Output Feedback Stabilization of Linear Time-Delay Systems", in *Stability and Control of Time-Delay Systems* L. Dugard and E. I. Verriest, Eds. New York: Springer, 1998, pp. 241–258.

[15] S. M. V. Lunel, *Exponential Type Calculus for Linear Delay Equations*, Centrum voor Wiskunde en Informatica, Amsterdam, The Netherlands: 1989.

[16] M. S. Mahmoud, *Robust Control and Filtering for Time-Delay Systems*, Marcel Dekker, New York: 2000.

[17] A. Manitius and A. W. Olbrot, "Finite Spectrum Assignment Problem for Systems with Delays," *IEEE Trans. Auto. Cont.*, vol. 24, 1979, pp. 541-553.

[18] W. J. Mao and J. Chu, "D-stability for Linear Continuous-Time Systems with Multiple Time Delays", *Automatica*, vol. 42, 2006, pp. 1589-1592.

[19] W. Michiels and D. Roose, "An eigenvalue Based Approach for the Robust Stabilization of Linear Time-Delay Systems," *Int J Control*, vol. 76, 2003, pp. 678-686.

[20] W. Michiels, K. Engelborghs, P. Vansevenant and D. Roose, "Continuous Pole Placement for Delay Equations", *Automatica*, vol. 38, 2002, pp. 747-761.

[21] S. I. Niculescu, "H-infinity Memoryless Control with an Alpha-Stability Constraint for Time-Delay Systems: an LMI Approach", *IEEE Trans. Auto. Cont.*, vol. 43, 1998, pp. 739-743.

[22] A. C. Tsoi, "Recent Advances in the Algebraic System Theory of Delay DiffErential Equations." in *Recent Theoretical Developments in Control* M. J. Gregson, Ed. New York: Academic Press, 1978, pp. 67-127.

[23] Z. Shafiei and A. T. Shenton, "Frequency-domain Design of PID Controllers for Stable and Unstable Systems with Time Delay", *Automatica*, vol. 33, 1997, pp. 2223-2232.

[24] H. Shinozaki and T. Mori, "Robust Stability Analysis of Linear Time-Delay Systems by Lambert W Function: Some Extreme Point Results", *Automatica*, vol. 42, 2006, pp. 1791-1799.

[25] B. S. Suh and J. H. Yang, "A Tuning of PID Regulators via LQR Approach", *J. Chem. Eng. Jpn.*, vol. 38, 2005, pp. 344-356.

[26] L. Qiu, B. Bernhardsson, A. Rantzer, E. J. Davison, P. M. Young and J. C. Doyle, "A Formula for Computation of the Real Stability Radius", *Automatica*, vol. 31, 1995, pp. 879-890.

[27] S. Yi, P. W. Nelson and A. G. Ulsoy, "Feedback Control via Eigenvalue Assignment for Time Delayed Systems Using the Lambert W Function," *J. Vib. Control*, 2008 (in press).

[28] S. Yi and A. G. Ulsoy, "Solution of a System of Linear Delay Differential Equations Using the Matrix Lambert Function," in *Proceedings of 25th ACC*, Minnearpolis, MN, Jun. 2006, pp. 2433-2438.

[29] S. Yi, A. G. Ulsoy, and P. W. Nelson, "Solution of Systems of Linear Delay Differential Equations via Laplace Transformation," in *Proceedings of 45th IEEE CDC*, San Diego, CA, Dec. 2006, pp. 2535-2540.

[30] S. Yi, P. W. Nelson and A. G. Ulsoy, "Survey on analysis of time delayed systems via the Lambert W function," *Dyn. Contin. Discret. Impuls. Syst. Ser. A-Math Anal.*, vol. 14 (s2), pp. 296-301, 2007.

[31] S. Yi, P. W. Nelson and A. G. Ulsoy, "Delay Differential Equations via the Matrix Lambert W Function and Bifurcation Analysis: Application to Machine Tool Chatter", *Math. Biosci. Eng.*, vol. 4, 2007, pp. 355-368.

[32] S. Yi, P. W. Nelson, and A. G. Ulsoy, "Controllability and observability of systems of linear delay differential equations via the matrix Lambert W function," *IEEE Trans. Autom. Control*, 2008 (in press).