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A Nonlinear Rate Control Method for Network Congestion Control

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Abstract—**Traditional congestion control algorithms exhibit low convergence rate to equilibrium when the network capacity is very large. In this paper, we present a new algorithm called Quick Kelly Control (QKC) to accelerate the convergence rate. The link utilization ratio functions are used as feedback signal and a novel nonlinear update law is constructed. The stability of this new algorithm is partly proved without considering delay. We also compare this algorithm with two classic algorithms and give simulation results. It is shown that QKC has powerful bandwidth scalability and offers fast convergence rate without sacrificing proportional fairness.**

I. INTRODUCTION

Changes in communication networks over the last decades have forced researchers to closely examine the network congestion control, particularly for the Internet. Congestion control is a distributed method to share network resources among competing sources. It consists of two components: a source algorithm that dynamically adjusts sending rate (or window size) in response to congestion in its path, and a link algorithm that updates congestion information and sends it back, implicitly or explicitly, to sources using that link.

The traditional algorithms were designed during a time when the Internet was a relatively small network compared to its size today. Therefore, researchers must reexamine the role of congestion control with the goal enhancing TCP to make it scalable to high-speed networks.

A large amount of theoretical and experimental work has been done to design stable congestion control for high-speed networks. Such examples include Fast TCP [2], Scalable TCP [3] [20], HSTCP [4], XCP [5], and BIC-TCP [6]. All of these methods aim to get quick convergence to efficiency, stable rate trajectories, fair bandwidth sharing, and low packet loss. Another different direction in congestion control is to model the network from an optimization or game theoretic point of view [7] [8] [9] [10] [11].The original work is done by Kelly [12].

In [12], a model for elastic traffic is described. In this model, each user chooses the charge per unit time that the user is willing to pay; thereafter the user's sending rate is determined by the network according to a proportional fairness criterion. In [1], two complementary congestion control algorithms are proposed: primal algorithm and dual algorithm. However, these algorithms only exhibit linear convergence to efficiency. To get an exponential convergence rate, a variant version of Kelly's algorithms which is called Max-min Kelly Control (MKC), is proposed in [13]. MKC utilizes negative network feedback which improves its convergence rate of efficiency from linear to exponential. However, MKC abandons proportional fairness and follows the max-min fairness criterion [13].

In this paper, we aim to propose a new algorithm which can achieve quick convergence rate, proportional fairness and max-min fairness. In this new algorithm, a link utilization ratio function, which is always positive, is utilized as network feedback signal. We construct a link utilization ratio based nonlinear source controller to accelerate the convergence rate. A dynamic link control law is also proposed to give corresponding link prices. The source sending rate is then dynamically updated based on these link prices. The stability and performance of the algorithm is proved by theory and simulation results.

The rest of this paper is organized as follows. In section II, we give the basic network model and review related works. In section III, we present QKC and give analysis of its stability and proportional fairness. In section IV, simulation results are given. In section V, we conclude our work and suggest directions of future research.

II. BASIC NETWORK FLOW CONTROL MODEL AND RELATED WORK

A. Network Mode

Network flows are modeled as the interconnection of information sources and communication links through the routing matrices [1] [8] [14]. Suppose we have a set of users, R , and a set of links, L . For each user $r \in R$, its route involves a set of links, which is a subset of L , denoted L_r .

For each link $l \in L$, it has a fixed capacity c_l . Based on its congestion, a link price p_l is computed. Associate each user r with a sending rate x_r . Thus, each link $l \in L_r$ has an associated link aggregate rate y_l . Suppose all links only feed back price information to the sources that utilize them. Set

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$A_{lr} = 1$, if $l \in L_r$ and set $A_{lr} = 0$ otherwise. So we have the following relationship [14]:

$$y = Ax \quad q = A^T p \tag{1}$$

where $x \in R^N$ is the source rate vector, $y \in R^L$ is the aggregate rate vector, $p \in R^L$ is the link price vector and $q \in R^N$ is the aggregate price vector. In this paper, we assume that there is no delay in the loop.

The network flow control problem can be decomposed into a static resource allocation optimization problem and a dynamic stabilization problem.

B. Static optimization problem

The static resource allocation optimization problem is to maximize the whole networks' performances. Its solution provides the desired steady state equilibrium: x^*, y^*, p^*, and q^*.

To define a fair resource allocation problem, each user *r* is associated with a utility function $U_r(x_r)$ which indicates the utility to the user *r*. Then the static resource allocation optimization problem [1] is to maximize the sum of the source utility functions $U_r(x_r)$ under the capacity constraints in the links. That is

$$\max \sum_{r=1}^{N} U_r(x_r)$$

subject to

$$Ax \le C, \; x > 0 \tag{2}$$

where C is a vector of the link capacities c_l and $U_r(x_r)$ is an increasing, strictly concave and continuous differentiable function. The Lagrangian for the system problem is:

$$L(x,p) = \sum_{r \in R} U_r(x_r) + \sum_{l \in L} p_l(c_l - y_l) \; . \tag{3}$$

The above expression associates each link l with a Lagrange multiplier p_l. This multiplier is the shadow price of the link. In many algorithms it summarizes the link congestion information.

Since $q_r = \sum_{l:\, l \in Lr} p_l$, rewriting the Lagrangian, we can get

$$L(x,p) = \sum_{r \in \mathrm{R}} (U_r(x_r) - x_r q_r) + \sum_{l \in L} p_l c_l \tag{4}$$

So the first order condition for the optimization problem (2) is

$$\frac{\partial L(x,\, p)}{\partial x_r} = 0 \, . \tag{5}$$

Since $U_r(x_r)$ is a continuous differentiable function, this gives the following set of equations:

$$U_r'(x_r) - q_r = 0 \, , \qquad r \in R \, . \tag{6}$$

Similarly, the condition for the Lagrange multiplier is [12]

$$p_l \begin{cases} = 0, & if \quad y_l < c_l \\ \ge 0, & if \quad y_l = c_l \end{cases} , \qquad r \in R \, . \tag{7}$$

C. Dynamic optimization problem

The dynamic problem is to design the source rate update law based on the aggregate price, and the link price update law based on the aggregate rate, to guarantee stability and robustness of the equilibrium.

In [1], Kelly's congestion controller is a well-known gradient algorithm, which is widely used in optimization theory [15]. For each user *r*,

$$\dot{x}_r = k_r(x_r)(U_r'(x_r) - q_r) \tag{8}$$

where $k_r(x_r)$ is an appropriately chosen scaling function. For each link *l*, the link price is computed by a static price function

$$p_l = f_l(y_l) = \frac{(y_l - c_l + \varepsilon)^+}{\varepsilon^2} \, . \tag{9}$$

where $f_l(\bullet)$ can be an increasing function and can be considered as the penalty function or the price of link l.

However, using the penalty function method, the algorithm only solves the problem (2) approximately. Clearly, the congestion controller will converge to the optimal solution of (2) if the equilibrium prices p_l^* are indeed the Lagrange multipliers of the problem (2). To this end, instead of computing p_l via the static function, the following dynamic equation is used to update p_l [14] [16] [17] [18]:

$$\dot{p}_l = g_l(p_l)(y_l - c_l)^+ \tag{10}$$

where it is assumed that $g_l(p_l) > 0$. The congestion controller (8) can be considered as computing the primal variables which are the source rates. The dynamic price update law (10) can be considered as computing the dual variables [21] or the Lagrange multipliers. Therefore, (8) and (10) can be regarded as the primal-dual algorithm. In [14], its global stability in the absence of feedback delay is proved. However, these algorithms are "too slow" for high-speed networks. In [1], let $k_r(x_r) = \kappa x_r$ and $U_r'(x_r) = w_r / x_r$.

Substituting these into (7), we can get

$$\dot{x}_r = \kappa x_r (\frac{w_r}{x_r} - q_r) \quad (11)$$

In general, no price is charged at those links which are not full utilized. Under these circumstances, the sources increase their rates by κw_r per unit time before they reach full link utilization at the slowest link. This results in linear AIMD-like probing for new bandwidth. The sources increase their rates by a constant κw_r no matter how large the links' capacities are. Thus the link utilization is very low in high-speed networks. This algorithm is un-scalable to large link capacity.

To accelerate the convergence rate of efficiency, a variant version of Kelly's algorithms which is called Max-min Kelly Control (MKC) is proposed in [13]. MKC abandons proportional fairness and utilizes negative network feedback which encourages the sources to increase their sending rates when $y_l < c_l$. Its price function can be expressed as follow:

$$p_l = \frac{y_l - c_l}{y_l} \quad (12)$$

$$q_r = \max_{l:\, l \in Lr} p_l \quad (13)$$

MKC achieves max-min fairness and exponential convergence to efficiency. However, MKC only achieves linear convergence to fairness when the network cannot provide the scale information of the number of flows [13]. In practical network, it is difficult to get the number of flows. When the link capacity is very large and the number of flows is unknown, MKC sources need a long time to converge to max-min fairness.

III. QUICK KELLY CONTROL (QKC)

In this section, a new version of Kelly control, called Quick Kelly Control (QKC) is presented.

We start our discussion with the following observations. To overcome the drawback of classic Kelly control, MKC utilizes negative feedback price which includes the links' state information when the links are not fully utilized. Negative feedback price accelerates the convergence rate. However, it leads to large overshoot to the links' capacity in the steady state if the sources rely on the sum of feedback functions [13]. To avoid large overshoot in the steady state, MKC abandons proportional fairness. MKC resources only feed back the most-congested link's state information and satisfy the max-min fairness criterion.

In [13], the author gives a conclusion: "Kelly's proportional fairness, or any mechanism that relies on the sum of feedback functions from individual routers, always exhibits linear convergence to efficiency." However, we don't think that proportional fairness or the form of "sum of feedback function" do have relationship with the algorithm's convergence rate. MKC's quick convergence rate benefits from negative feedback prices which provide more sufficient link state information. In the following part we will show that proportional fairness and quick convergence rate do not conflict. Feedback information and the source update law are the key points we should focus on.

Now, our problem is how to feedback sufficient link state information when those links are not fully utilized. In fact, negative price is only one of the possible choices and any price form which gives sufficient link state information is feasible. Note that, computing link utilization ratio is one of the most direct ways to express the link state information. So links can feed back their utilization ratios to the sources that utilize them. Then each source uses the sum of the utilization ratios that it can receive as an input of the sending rate controller. Obviously, the sum of the utilization ratios is always positive and those large links with low utilization ratios have small weights. This idea provides credible link prices even certain sources use those links with huge difference.

Let $U_r^{'}(x_r) = \frac{w_r}{x_r}$, where w_r is a positive constant. Substituting it to equation (6), we can get

$$\frac{w_r}{x_r} - q_r = 0 \quad (14)$$

Since $x_r > 0$ and $q_r > 0$, equation (14) can be rewritten as

$$\frac{w_r}{\sqrt{q_r}} - x_r\sqrt{q_r} = 0 \quad (15)$$

Consider the system of differential equations

$$\dot{x}_r = k(\frac{w_r}{\sqrt{q_r}} - x_r\sqrt{q_r}) \quad (16)$$

$$q_r = \sum_{l:\, l \in Lr} p_l \quad (17)$$

Let

$$p_l^0 = \frac{y_l}{c_l} \quad (18)$$

$$p_l = h(p_l^0) \cdot p_l^0 \cdot \exp[\lambda(p_l^0 - 1)] \quad (19)$$

where $h(\bullet)$ is an increasing function. It can be considered as a price discounter factor. Obviously the price function is a positive increasing function.

Compared to the classic Kelly control, instead of probing for new bandwidth in a linear AIMD-like manner, the resulting control algorithm (16)-(19) computes both increase component and decrease component based on the feedback information.

Similar to [1] [14], we will prove the stability of the control algorithm (16)-(19). Instead of solving the resource allocation problem (2) exactly, consider the following objective [1] [19]:

$$V(x) = \sum_r U_r(x_r) - \sum_l \int_0^{y_l} f_l(\varphi) d\varphi \qquad (20)$$

Since $f_l(\bullet)$ is an increasing function and $U_r(x_r)$ is strictly concave and continuous differentiable, it is easy to see that $V(x)$ is a strictly concave function [19]. The optimization problem (2) can be approximately replaced by the following problem [1] [19]:

$$\text{Max} \quad V(x), \quad x > 0 \qquad (21)$$

Since $y_l = \sum_{s:\, l \,\in\, Ls} x_s$, we can get [19]:

$$\begin{aligned} \frac{\partial V}{\partial x_r} &= U_r^{'}(x_r) - \sum_{l:\, l \,\in\, Ls} f_l(\sum_{s:\, l \,\in\, Ls} x_s) \\ &= U_r^{'}(x_r) - q_r \end{aligned} \qquad (22)$$

The following theorem shows that the congestion control system (16)-(19) is globally asymptotically stable. Starting from any initial state, as $t \to \infty$, the sources rates x_r will converge to the non-zero rates x_r^* which maximizes (20).

Theorem 1 Starting from any initial condition, the congestion control system (16)-(19) will converge to the unique solution x^* maximizing $V(x)$.

Proof

Since $V(x)$ is strictly concave, the maximizing value of x^* is thus unique. Let $V_1(x - x^*) = V(x^*) - V(x)$, where $V_1(0) = 0$. Because $V(x^*) \ge V(x)$, $V_1(x - x^*) \ge 0$. $V_1(x - x^*)$ is a positive definite function.

Note that

$$\frac{\partial V_1}{\partial x_r} = -U_r^{'}(x_r) + q_r = -\frac{w_r}{x_r} + q_r$$

Further

$$\begin{aligned} \frac{dV_1}{dt} &= \sum_r \frac{\partial V_1}{\partial x_r} \cdot \frac{d}{dt} x_r \\ &= (-\frac{w_r}{x_r} + q_r) k (\frac{w_r}{\sqrt{q_r}} - x_r \sqrt{q_r}) \\ &= -k(\frac{w_r}{x_r} - q_r)^2 \frac{x_r}{\sqrt{q_r}} \le 0 . \end{aligned}$$

Observe that $dV_1/dt < 0$ for $x \ne x^*$ and is equal to zero for $x = x^*$. Thus V_1 is strictly decreasing with t, unless $x = x^*$. Since the unique x^* maximizes $V(x)$ and minimizes $V_1(x - x^*)$, the function $V_1(x - x^*)$ is a Lyapunov function for the system (16)-(19).

We thus conclude that x^* is globally asymptotically stable. The theorem follows. □

It is mentioned in the above section that, if the link price is computed by a static price function, the algorithm only solves the problem (4) and it does not solve the original resource allocation optimization problem (2) exactly. To make the congestion controller converge to the optimal solution of (2), a dynamic link prices update law is needed. Instead of computing p_l via a static function, a first order dynamic link price discounter update law is given as follow:

$$\dot{\delta}_l = \theta(p_l^0 - 1) \qquad (23)$$

$$h(p_l^0) = \exp(\delta_l) \qquad (24)$$

where θ and λ are constants. Note that, an exponential function is used as price discounter. This update law is based on a very simple motivation: Subject to the capacity constraint, all links do their best to maximize their utilization. For link l, when the utilization ratio $p_l^0 < 1$, its discounter factor is decreased smoothly and a lower link price is given to encourage the sources that utilize this link. When the utilization ratio $p_l^0 > 1$, the discounter factor is increased smoothly and a larger link price is given to restrain the sources that utilize this link.

We call the resulting control algorithm (16)-(19) and (23)-(24) Quick Kelly control (QKC). It is easy to get the equilibrium condition of the QKC system:

$$x_r^* = \frac{w_r}{q_r^*}, \qquad r \in R \qquad (25)$$

$$p_l^* \begin{cases} \to 0, & if \quad y_l < c_l \\ \ge 0, & if \quad y_l = c_l \end{cases}, \qquad l \in L \qquad (26)$$

Since $U_r^{'}(x_r) = w_r / x_r$, this equilibrium is arbitrarily close to (6) and (7). So QKC solve the optimization problem (2). The stability of the equilibrium can be proved similar to the *proposition 2* and *proposition 5* in [14]. Obviously, QKC and Kelly controller have an identical form of the equilibrium. So QKC can satisfy proportional fairness criterion too.

The performances of this algorithm are confirmed by simulation results which are given in the next section.

IV. SIMULATION RESULTS

To illustrate the performance of the new congestion controller presented in this paper, we consider a simple four-source/three-link example which is presented in [14]. The corresponding routing matrix is

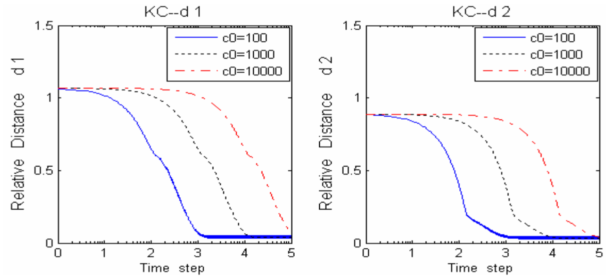

Fig. 2. d_1 and d_2 of KC.

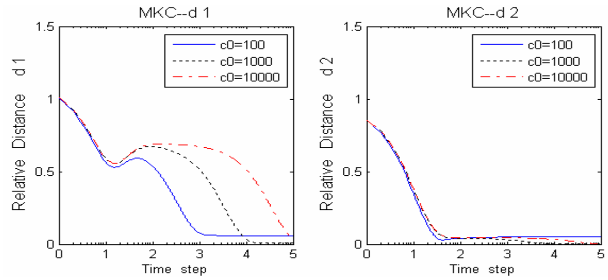

Fig. 3. d_1 and d_2 of MKC.

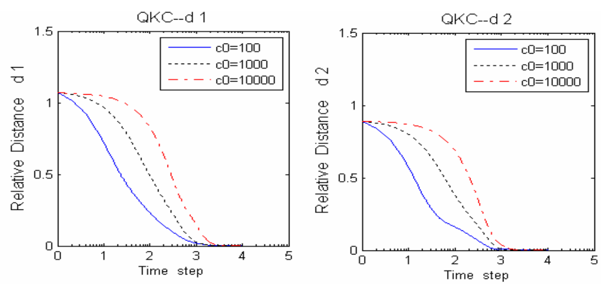

Fig. 4. d_1 and d_2 of QKC.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Its topology structure is shown in Fig.1. We assume that the link capacities are all c_0. For those algorithms (KC, QKC) which follow proportional fairness criterion, suppose that all source utility functions are $U_r(x_r) = \log(x_r)$. The solution to the optimization problem (1) [14] is

$$x^* = [0.25c_0 \quad 0.25c_0 \quad 0.5c_0 \quad 0.5c_0]^T$$
$$y^* = [0.75c_0 \quad c_0 \quad c_0]^T$$

For those algorithms (MKC, QKC) which follow Max-min fairness criterion, the solution to the optimization problem (1) is

$$x^* = [\frac{1}{3}c_0 \quad \frac{1}{3}c_0 \quad \frac{1}{3}c_0 \quad \frac{1}{3}c_0]^T,$$

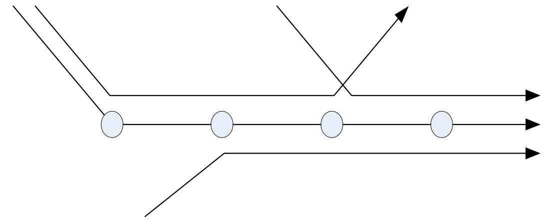
Fig. 1. A network example.

$$y^* = \left[\frac{2}{3}c_0 \quad c_0 \quad c_0\right]^T$$

The initial source rate is set to

$$x(0) = [0.1c_0 \quad 0.2c_0 \quad 0.25c_0 \quad 0.15c_0]^T$$

The time step is set to 0.2.

Consider the following control systems:

KC: (8), (9) and (17). Set: $\kappa_r = 0.5$, $w_r = 1$, $\varepsilon = 0.02c_0$.

MKC: (8), (12) and (13). Set: $\kappa_r = 0.5$, $w_r = 1$

PDQKC: (16), (17), (18), (19), (23) and (24).

Set: $\kappa_r = 0.5$, $w_r = 1$, $\theta = 0.05$, $\lambda = 2.0$, $\omega = 0.000001$, $\delta(0) = -0.8\ln c_0$.

To compare the bandwidth scalability of these controllers, two Euclidean distance functions, which describe the relative distance between the current state and the equilibrium point, are defined by (27) and (28).

$$d_1 = \left\| \frac{x - x^*}{x^*} \right\| = \sqrt{\sum_r (\frac{x_r - x_r^*}{x_r^*})^2}\ . \tag{27}$$

$$d_2 = \left\| \frac{y - y^*}{y^*} \right\| = \sqrt{\sum_l (\frac{y_l - y_l^*}{y_l^*})^2}\ . \tag{28}$$

If the equilibrium is asymptotically stable, starting from the initial state, the sources and the links will converge to x^* and y^* respectively as $t \to \infty$. It is clear that, $d_1 \to 0$ as $x \to x^*$ and $d_2 \to 0$ as $y \to y^*$. Set $c_0 = 100$, $c_0 = 1000$ and $c_0 = 10000$ respectively. By logarithmizing the time steps, Fig.2-4 show the bandwidth scalability of these controllers.

KC exhibits linear convergence to efficiency and fairness [13]. In Fig.2, both d_1 and d_2 approximately tend to zero as the time increases. The convergence time of d_1 and the convergence time of d_2 are proportional to the link capacities. In Fig.3, d_2 is sharply decreased and all the links tend to their equilibrium exponentially no matter how large the link capacities are. However, except for the early state, the process of d_1 is similar to the KC's. In Fig.4, QKC shows its powerful bandwidth scalability and fast convergence to efficiency and fairness. Both d_1 and d_2 converge to zero in a short time. As the link capacities grow from 100 to 10000, only small variation of the convergence time is observed. If we change the form of price aggregate expression, QKC can also satisfy the max-min criterion. A max-min form of QKC is obtained by replacing (17) by (13).

Although MKC offers exponential convergence to efficiency, its source convergence rate to fairness is much slower than QKC. In practical network, to avoid impacting the network, large rate acceleration is not expected at the early state. Thus, it doesn't make much sense to get pure exponential convergence to efficiency.

V. CONCLUSIONS

This paper has presented a novel nonlinear network congestion control algorithm called Quick Kelly control (QKC). In QKC, a link utilization ratio function is used as feedback signal to accelerate convergence rate and improve the bandwidth scalability. Comparing to classical Kelly control and MKC, QKC can achieve fast convergence to the equilibrium and has powerful bandwidth scalability. All the analysis and simulation results based on an assumption that there is no delay in the loop. Our future work involves the improvement of dynamic performance and delay stability.

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