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Improving performance analysis with structures of outputs and constraints*

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Abstract— **This paper is devoted to the analysis of output performances for linear systems with state/output constraints. Analysis methods are obtained by incorporating the structures of outputs and state constraints into non-quadratic Lyapunov functions. Output ranges and overshoots are estimated for given sets of initial conditions and bounded inputs. Attempts are also made to detect a large invariant set within state/output constraints. Numerical examples demonstrate the effectiveness of the non-quadratic Lyapunov functions and the proposed methods. In particular, it is shown that, even for linear time invariant systems, the output bound can be sharpened by using a simple non-quadratic Lyapunov function that incorporates the structure of the output.**

Keywords: Output performance, state constraints, Lyapunov functions, invariant set, linear matrix inequalities

I. INTRODUCTION

Every physical quantity is subject to a certain constraint. In many situations, the violation of the constraints may cause severe damages or destroy a device. Thus it is important to have a clear understanding of the range of the key quantities during the operation of the device under various circumstances. Consider a dynamical system described by the following equations:

$$\dot{x} = f(x, w, t), \;\; y = h(x, w), \tag{1}$$

where x is the state, y the output representing some key quantities, and w the external input or disturbance. Assume internal stability of the system in the absence of disturbance (i.e., $w = 0$). Suppose that the initial condition $x(0)$ belongs to a set X_0 and the external input $w(t)$ belongs to a set W for all t. Our main concern is to determine the range of the output y during the operation of the system under all possible initial conditions and external input. Another related issue is to find a safety set of initial conditions and input range so that certain output constraints will not be violated.

For the system (1) under fixed initial condition and input, the range of output can be easily obtained via simulation if the model is exactly known and accurate. However, for practical systems which usually have uncertain models, uncertain time-varying parameters, uncertain external inputs, and/or variable initial conditions, it is quite insufficient to use simulation to determine a range for the outputs.

Practical ways to estimate the range of state and output in the presence of model and input uncertainties are given in [2], where quadratic Lyapunov functions and invariant ellipsoids are used as major tools for various analysis and design problems. In [5], invariant polytopes are used to deal with state and input constraints. Although quadratic Lyapunov functions may lead to conservative results, they are still popular in practice since they may yield optimization problems with linear matrix inequality (LMI) constraints which are easily tractable.

In recent years, efforts have been made towards the development of non-quadratic Lyapunov functions to improve stability and performance analysis (see e.g.,[1], [3], [4], [6], [8], [10], [11], [12], [14], [15]). Most of the Lyapunov functions in these works pertain to or are composed from several quadratic functions and thus lead to optimization problems with matrix inequality constraints, generally a mixture of LMIs and bilinear matrix inequalities (BMIs). Although the optimization problems are generally non-convex, suboptimal solutions can be obtained with LMI-based algorithms.

In this paper, we will consider Lyapunov functions that are constructed by using the structures of the output and the structures of the state/output constraints. They are similar to the max of quadratics used in [4], [10], but are easier to handle numerically, since they effectively incorporate the structure of the output and constraints. The algorithms are a little more complicated than those arising from quadratic functions but the improvement is significant.

This paper is organized as follows. In Section II, we use one example to demonstrate how the estimation of output bound can be sharpened by using the structure of the output in the construction of invariant sets. This motivates the development of non-quadratic Lyapunov functions that reflect the structure of the output and constraints. Section III derives methods for the estimation of output bounds and overshoots under a set of given initial conditions. Section IV considers the problem of estimating output bound under a set of norm-bounded persistent disturbances and Section V derives methods for detecting a large safe operating range (an invariant set) within state constraint. Section VI concludes the paper.

Notation:

co$S$: The convex hull of a set S. For $P \in \mathbb{R}^{n\times n}, P = P^{\mathrm{T}} > 0$, $\mathcal{E}(P) := \{x \in \mathbb{R}^n : \; x^{\mathrm{T}} P x \leq 1\}$. For a positive definite function V, $L_V := \{x \in \mathbb{R}^n : V(x) \leq 1\}$. For $H \in \mathbb{R}^{r\times n}$, $\mathcal{L}(H) := \left\{x \in \mathbb{R}^n : |H_\ell x| \leq 1, \ell \in I[1,r]\right\}$, where H_ℓ is the ℓth row of H. About the relationship between $\mathcal{E}(P)$ and $\mathcal{L}(H)$, we have (e.g., see [7]),

$$\mathcal{E}(P) \subseteq \mathcal{L}(H) \Longleftrightarrow H_\ell P^{-1} H_\ell^{\mathrm{T}} \leq 1 \quad \forall\, \ell \in I[1,r]. \tag{2}$$

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II. Motivation and Preliminaries

This work arose from an attempt to improve the estimation of the maximal output and the overshoot for linear time invariant systems and linear differential inclusions. For a linear system

$$\dot{x} = Ax, \quad y = Cx, \quad x(0) = x_0, \tag{3}$$

the output response $y(t) = Ce^{At}x_0$ can be readily computed with simple Matlab functions. However, it is hard to extend this method to the situation where there are time-varying uncertainties in the matrix A and/or uncertainties in the initial condition $x(0)$.

A powerful tool to deal with time varying uncertainties and nonlinearities is the Lyapunov function. Quadratic Lyapunov functions are known to be conservative but still popular because they are empowered by the numerically efficient LMI technique. In [2] (page 88), the output bounds for linear time invariant systems and linear differential inclusions are estimated via LMIs under a unified framework. Here we use the linear time invariant system (3) with a given initial condition to illustrate the main idea.

Suppose that the output y is a scalar and C is a row vector. An upper bound for the output response of (3) can be estimated by solving the following generalized eigenvalue problem (gevp):

$$\begin{array}{ll} \inf_{P>0} & \delta, \\ s.t. & A^{\mathrm{T}}P + PA \leq 0, \;\; x_0^{\mathrm{T}}Px_0 \leq 1, \\ & C^{\mathrm{T}}C \leq \delta^2 P. \end{array} \tag{4}$$

The explanation is as follows. Let (P, δ) be a feasible pair satisfying the constraints in (4). Then $V(x) = x^{\mathrm{T}}Px$ is a quadratic Lyapunov function whose 1-level set is the ellipsoid

$$\mathcal{E}(P) = \{x : x^{\mathrm{T}}Px \leq 1\}.$$

The LMI $A^{\mathrm{T}}P + PA \leq 0$ ensures that $\mathcal{E}(P)$ is an invariant set. The area between the two planes $Cx = \pm\delta$ is denoted by the set

$$\mathcal{L}(C/\delta) = \{x : |Cx| \leq \delta\}.$$

The inequality $C^{\mathrm{T}}C \leq \delta^2 P$ ensures that the ellipsoid $\mathcal{E}(P)$ lies between the two planes $Cx = \pm\delta$, i.e.,

$$\mathcal{E}(P) \subset \mathcal{L}(C/\delta).$$

And the inequality $x_0^{\mathrm{T}}Px_0 \leq 1$ simply means that the initial condition belongs to the invariant ellipsoid $\mathcal{E}(P)$. Thus the state $x(t)$ will stay inside $\mathcal{E}(P) \subset \mathcal{L}(C/\delta)$ for all t and we have $|y(t)| \leq \delta$ for all $t > 0$.

Is there any room for improvement? Consider an example with

$$A = \begin{bmatrix} 0 & 1 \\ -0.1 & -1 \end{bmatrix}, \;\; C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \;\; x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{5}$$

The minimal δ for (4) is $\delta^* = 1.2252$ and the optimal P^* is

$$P^* = \begin{bmatrix} 1.3753 & 0.7505 \\ 0.7505 & 1.5010 \end{bmatrix}. \tag{6}$$

The invariant ellipsoid $\mathcal{E}(P^*)$ and the two straight lines $Cx = \pm 1.2252$ are plotted in Fig. 1. The initial condition is marked with "$*$". The trajectory starting from x_0 is plotted

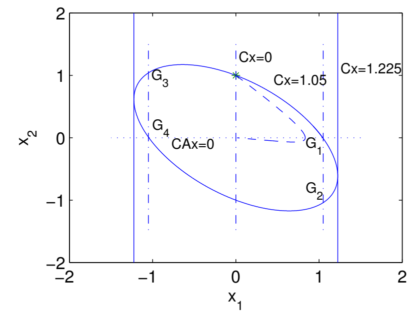

Fig. 1. Invariant set clipped from an invariant ellipsoid.

with dashed curve. In an attempt to improve the estimation, we observed that as the maximal output is reached, we must have $\dot{y}(t) = 0$, i.e., $C\dot{x} = CAx = 0$. This means that the maximal output can only be reached when x is in the straight line $CAx = 0$ (a hyperplane for high-order systems). For this particular example, $CAx = 0$ overlaps the horizontal x_1 axis. Above this line, $CAx > 0$, i.e., $y(t)$ increases ($x(t)$ goes rightward); and below this line, $y(t)$ decreases ($x(t)$ goes leftward). Let the intersections of the boundary of the ellipsoid with $CAx = 0$ be G_1 and G_4. Here we have $G_1 = (1.05, 0), G_4 = (-1.05, 0)$. If we draw two straight lines $Cx = \delta_1$ and $Cx = -\delta_1$ passing G_1 and G_4 respectively, then $\delta_1 = 1.05$. Let the other intersections of the straight lines with the ellipsoid boundary be G_2 and G_3. Then the line segment G_1G_2 is below $CAx = 0$, hence for every point x in this segment, y decreases along the trajectory, i.e., the vector Ax points to the left; Similarly, for every point x in the segment G_3G_4, the vector Ax points to the right. Since the ellipsoid is invariant, Ax points inward of the ellipsoid along the boundary. Thus the intersection of the ellipsoid $\mathcal{E}(P^*)$ and the strip $\mathcal{L}(C/\delta_1)$ is also an invariant set. Therefore all trajectories starting from within this intersection $\mathcal{E}(P^*) \cap \mathcal{L}(C/\delta_1)$ will stay in it and the output will be restricted to $|y(t)| \leq \delta_1 = 1.05 < 1.2252 = \delta^*$. This gives a smaller bound for the output response.

Suppose that the initial condition is outside of the intersection $\mathcal{E}(P^*) \cap \mathcal{L}(C/\delta_1)$ but inside the ellipsoid. Since the ellipsoid is invariant and the system is exponentially stable, the trajectory will enter $\mathcal{E}(P^*) \cap \mathcal{L}(C/\delta_1)$ before a local maximum/minimum of $y(t)$ is reached, and once entering the intersection, it will stay there. This fact can be used to estimate the overshoot for a system under a step input by shifting the origin to the steady state.

This example shows that we can sharpen the estimation of the output bound and overshoot simply by utilizing the plane $CAx = 0$ (a straight line for second order systems), or by using the planes $Cx = \pm\delta$ to form part of the boundary of an invariant set. The resulting invariant set is a clipped ellipsoid. In this paper, we will develop methods for sharpening the output bound or overshoot by optimizing such invariant set. The optimization problem will involve LMIs and simple BMIs. For the above example, the output bound can be further reduced to 0.9161. This is because the optimized P for the new optimization problem is different from P^* in (6). As we can see from Fig. 2, the new ellipsoid $\mathcal{E}(P)$ forms a much tighter bound for the same trajectory as in Fig. 1 before it reaches the line $CAx = 0$, i.e., before the output reaches the maximal value.

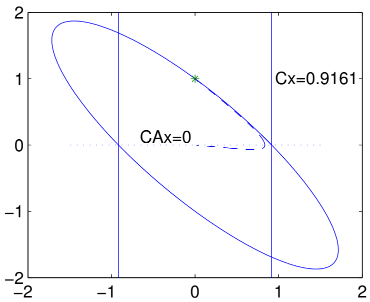

Fig. 2. Clipped ellipsoid after optimization.

In this paper, we are going to use Lyapunov functions of the form

$$V(x) = \max\{x^{\mathrm{T}} P_i x, i = 1, 2, \cdots, m\},$$

where some of the P_i's are $C_i^{\mathrm{T}} C_i$ with C_i taken from the output matrix. This type of functions are not everywhere differentiable and we need to use directional derivative to quantify the time derivative of $V(x(t))$ along a trajectory. A general result about the directional derivative of this type of functions can be found in [10].

For a function $V(x)$, the one sided directional derivative is defined ([13], page 213) with respect to two variables: x and a vector ζ specifying the direction of motion. In particular, the one-sided directional derivative of V, at x along ζ is defined as

$$\dot{V}(x;\zeta) := \lim_{\Delta t \to 0, \Delta t > 0} \frac{V(x + \zeta \Delta t) - V(x)}{\Delta t}.$$

For $x \in \mathbb{R}^n$, let

$$I_{\max}(x) := \{i : x^{\mathrm{T}} P_i x = V(x)\} = \{i : x^{\mathrm{T}} P_i x \geq x^{\mathrm{T}} P_j x \ \forall j\}.$$

Then by [10], the directional derivative of V at x along ζ is

$$\dot{V}(x;\zeta) = \max\{2x^{\mathrm{T}} P_i \zeta : i \in I_{\max}(x)\}. \tag{7}$$

III. Output Bound and Overshoot

A. Output bound for given set of initial conditions

1) Linear time-invariant systems: We first consider the linear system

$$\dot{x} = Ax, \quad y = Cx, \tag{8}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. For systems with multiple outputs, each output can be examined individually. With a positive definite matrix $P \in \mathbb{R}^{n \times n}$, a Lyapunov function $V(x)$ and its 1-level set are defined as follows,

$$V(x) := \max\{x^{\mathrm{T}} P x, x^{\mathrm{T}} C^{\mathrm{T}} C x\},$$

$$L_V := \{x \in \mathbb{R}^n : V(x) \leq 1\}.$$

It is easy to see that the level set is the intersection of the ellipsoid

$$\mathcal{E}(P) = \{x \in \mathbb{R}^n : x^{\mathrm{T}} P x \leq 1\}$$

and the strip

$$\mathcal{L}(C) = \{x \in \mathbb{R}^n : |Cx| \leq 1\}.$$

Since $V(x)$ is homogeneous of degree 2, for a positive number γ, we have

$$\gamma L_V = \{x \in \mathbb{R}^n : V(x) \leq \gamma^2\},$$

and the maximal $y = Cx$ for $x \in \gamma L_V$ is γ.

Let $X_0 \subset \mathbb{R}^n$ be a set of initial conditions. If γL_V is an invariant set and $X_0 \subset \gamma L_V$, then all trajectories starting from X_0 will stay inside γL_V and the output $y(t)$ remains bounded by γ. Thus the problem of estimating an upper bound for the output can be transformed into one of searching for a matrix P such that the resulting L_V is invariant and $X_0 \subset \gamma L_V$ with the minimal γ. In what follows, we translate the condition for set invariance and that for set inclusion into matrix inequalities.

Proposition 1: The level set γL_V is invariant if and only if there exist $\alpha \geq 0$ and $\beta \geq 0$ such that

$$\alpha(C^{\mathrm{T}} C A + A^{\mathrm{T}} C^{\mathrm{T}} C) \leq P - C^{\mathrm{T}} C \tag{9}$$

$$PA + A^{\mathrm{T}} P \leq \beta(C^{\mathrm{T}} C - P). \tag{10}$$

The set inclusion condition $X_0 \subset \gamma L_V$ can be stated as matrix inequalities if X_0 is a polygon or an ellipsoid. If $X_0 = \mathrm{co}\{x_k : k = 1, 2, ...K\}$, then $X_0 \subset \gamma L_V$ if and only if

$$x_k^{\mathrm{T}} P x_k \leq \gamma^2, \quad x_k^{\mathrm{T}} C^{\mathrm{T}} C x_k \leq \gamma^2 \quad \forall k. \tag{11}$$

If $X_0 = \{x \in \mathbb{R}^n : x^{\mathrm{T}} R x \leq 1\}$, then $X_0 \subset \gamma L_V$ if and only if

$$P \leq \gamma^2 R, \quad C^{\mathrm{T}} C \leq \gamma^2 R. \tag{12}$$

Combining the matrix conditions for set invariance and set inclusion, it is straight forward to formulate an optimization problem for minimizing the bound on output response. The following is for the case where X_0 is a polygon,

$$\inf_{P > 0, \alpha, \beta \geq 0} \quad \gamma^2, \tag{13}$$

$$s.t. \ A^{\mathrm{T}} P + PA \leq \beta(C^{\mathrm{T}} C - P), \tag{14}$$

$$\alpha(C^{\mathrm{T}} C A + A^{\mathrm{T}} C^{\mathrm{T}} C) \leq P - C^{\mathrm{T}} C, \tag{15}$$

$$x_k^{\mathrm{T}} P x_k \leq \gamma^2, \ \ x_k^{\mathrm{T}} C^{\mathrm{T}} C x_k \leq \gamma^2 \quad \forall k. \tag{16}$$

If α and β are set to zero, the problem reduces to (4) (with P scaled by γ^2). The additional parameters α and β relaxe the constraint and may lead to tighter estimate of the output bound. The results in Fig. 2 are generated by solving (13) for the second-order example (5). As we can see, the output bound is reduced from 1.2252 to 0.9161.

The above result can be easily extended to linear differential inclusions. Consider a polytopic linear differential inclusion with a fixed C,

$$\dot{x} \in \mathrm{co}\{A_i x : i = 1, 2, \cdots, K\}, \quad y = Cx. \tag{17}$$

Corollary 1: The level set γL_V is invariant if and only if there exist $\alpha_i \geq 0$ and $\beta_i \geq 0$, $i = 1, 2 \cdots, K$, such that

$$\alpha_i(C^{\mathrm{T}}CA_i + A_i^{\mathrm{T}}C^{\mathrm{T}}C) \leq P - C^{\mathrm{T}}C \ \ \forall i \tag{18}$$
$$PA_i + A_i^{\mathrm{T}}P \leq \beta_i(C^{\mathrm{T}}C - P) \ \ \forall i. \tag{19}$$

B. Estimation of the overshoot

Consider the system

$$\dot{z} = Az + Bu, \quad y = Cz, \quad z_0 = 0, \tag{20}$$

where $y \in \mathbb{R}$ is a scalar output and $u \in \mathbb{R}^m$ is a step input with final value u_f. Assume A is Hurwitz. We'd like to estimate the maximal y that will be reached during the transient response. To do this, we may shift the origin to the steady state value of z by defining $x = z + A^{-1}Bu_f$. Then

$$\dot{x} = Ax, \quad y = Cx - CA^{-1}Bu_f, \quad x_0 = A^{-1}Bu_f.$$

Let $y_1(t) = Cx(t)$. Then the maximum/minimum of $y(t)$ is the sum of the maximum/minimum of $y_1(t)$ and $-CA^{-1}Bu_f$. In what follows, we try to estimate the maximum of $y_1(t)$.

Proposition 2: Suppose there exist a matrix $P > 0$, a scalar $\gamma > 0$ and a real number α such that

$$A^{\mathrm{T}}P + PA < 0 \tag{21}$$
$$\alpha A^{\mathrm{T}}C^{\mathrm{T}}CA \leq P - C^{\mathrm{T}}C \tag{22}$$
$$x_0^{\mathrm{T}}Px_0 \leq \gamma^2 \tag{23}$$

1) If $y_1(0) \leq \gamma$, then $y_1(t) \leq \gamma$ for all t.
2) If $y_1(0) \geq -\gamma$, then $y_1(t) \geq -\gamma$ for all t.
3) $\mathcal{E}(P) \cap \mathcal{L}(C)$ is an invariant set.

The same conclusions hold if (22) is replaced with

$$KCA + A^{\mathrm{T}}C^{\mathrm{T}}K^{\mathrm{T}} \leq P - C^{\mathrm{T}}C. \tag{24}$$

for certain $K \in \mathbb{R}^{n\times 1}$.

Suppose that the original system (20) has a positive final output value $y(\infty) = -CA^{-1}Bu_f$ and an overshoot, then $y_1(0) = CA^{-1}Bu_f < 0$ and the maximal $y_1(t)$ is the amount of overshoot for $y(t)$ above its final value. Using item 1) in Proposition 2, this maximal $y_1(t)$ is bounded by γ. Thus we can formulate an optimization problem to minimize γ under the constraints (21) to (23). The method of using clipped ellipsoid to estimate the overshoot is demonstrated in Example 1 and Fig. 3.

The proof of Proposition 2 is not based on any Lyapunov function. So it can not be directly extended to differential inclusions. Note that if $K = \alpha C^{\mathrm{T}}$ for certain $\alpha \geq 0$ in (24), then we obtain (15). Thus (15) is stronger than (24), and (22) can be replaced with (15) to ensure the same results. The advantage of using (15) instead of (22) or (24) is that this constraint can be extended to differential inclusions, since it ensures $\dot{V}(x; Ax) \leq 0$ for x such that $x^{\mathrm{T}}Px \leq x^{\mathrm{T}}C^{\mathrm{T}}Cx$.

Consider the linear differential inclusion

$$\dot{x} \in \mathrm{co}\{A_i x : i = 1, 2, \cdots, K\}, \quad y_1 = Cx. \tag{25}$$

Corollary 2: Suppose there exist a matrix $P > 0$ and real numbers $\alpha_i \geq 0, i = 1, 2, \cdots, K$ such that

$$\alpha_i(C^{\mathrm{T}}CA_i + A_i^{\mathrm{T}}C^{\mathrm{T}}C) \leq P - C^{\mathrm{T}}C, \quad \forall i \tag{26}$$
$$PA_i + A_i^{\mathrm{T}}P \leq 0, \quad \forall i. \tag{27}$$

Then both $\mathcal{E}(P)$ and $\mathcal{E}(P)\cap\mathcal{L}(C)$ are invariant sets for (25). For $x_0 \in \gamma\mathcal{E}(P)$ and $y_1(0) < -\gamma$ (or $y_1(0) > \gamma$), $y_1(t)$ will reach $-\gamma$ (resp. γ) at certain time instant and stay below γ (resp. above $-\gamma$) afterwards.

Example 1: A second-order system is described as

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad y = [1 \ \ 0]z, \quad z(0) = 0.$$

Under a step input, the steady state output is $y(\infty) = 0.6667$. When transformed to the state $x = z + A^{-1}B$, we have $x_0 = \begin{bmatrix} -0.6667 \\ 1 \end{bmatrix}$ and $y_1(0) = -0.6667$. The estimated overshoot is 0.5971 (89.5%) by minimizing γ under constraints (21) to (23). The actual overshoot determined from simulation is 0.3718 (55.8%). Fig. 3 plots the resulting ellipsoid clipped by $Cx = \pm 0.5971$ and the trajectory x starting from the initial condition x_0. Fig. 4 plots the time response of the output y.

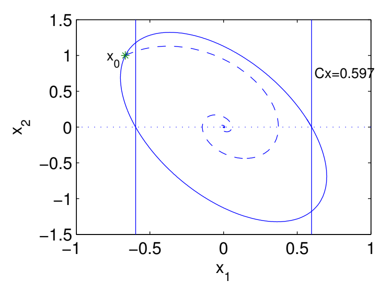

Fig. 3. Clipped invariant ellipsoid for estimating overshoot.

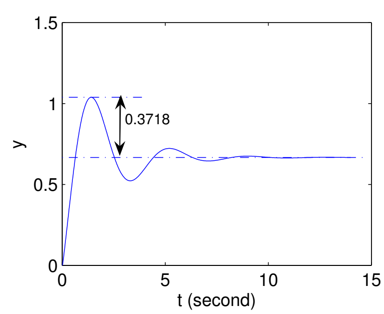

Fig. 4. Output response and the overshoot.

IV. Bound on output under persistent disturbances

A linear system subject to persistent disturbance is described as follows,

$$\dot{x} = Ax + Bw, \quad y = Cx, \tag{28}$$

where $x \in \mathbb{R}^n, w \in \mathbb{R}^m$ and $y \in \mathbb{R}$. Assume that $w^{\mathrm{T}}(t)w(t) \leq 1$ for all t and $x(0) = 0$. An upper bound for $|y(t)|$ can be estimated by sets which are invariant under all possible persistent disturbances. These invariant sets contain the actual reachable set. If an invariant set exists inside the strip $\gamma\mathcal{L}(C)$, then the output is bounded by $|y(t)| \leq \gamma$. Optimization problems can then be formulated for minimizing γ over all possible invariant sets. In [2] (page 82-83), invariant ellipsoids are considered and the optimization problem has LMI constraints except for a scalar variable within $[0, \infty)$.

In what follows, we would like to consider invariant sets clipped from ellipsoids by planes $Cx = \pm\gamma$. They are level sets of the Lyapunov function $V(x) = \max\{x^{\mathrm{T}}Px, x^{\mathrm{T}}C^{\mathrm{T}}Cx\}$, i.e., $\gamma(\mathcal{E}(P) \cap \mathcal{L}(C))$. As a result, the optimization problem has LMI constraints except for two scalar variables within $[0, \infty]$. These two variables can be optimized via plane search or with Matlab function "fminsearch".

Proposition 3: The level set γL_V is invariant for system (28) if there exist $\alpha_1, \alpha_2 \geq 0$ and $\beta_1, \beta_2 \geq 0$ such that

$$\begin{bmatrix} \beta_1 C^{\mathrm{T}}C & 0 \\ 0 & 0 \end{bmatrix} \leq \gamma^2 \begin{bmatrix} -\alpha_1(C^{\mathrm{T}}CA + A^{\mathrm{T}}C^{\mathrm{T}}C) + P - C^{\mathrm{T}}C & \alpha_1 C^{\mathrm{T}}CB \\ \alpha_1 B^{\mathrm{T}}C^{\mathrm{T}}C & \beta_1 \end{bmatrix} \tag{29}$$

$$\begin{bmatrix} \beta_2 P & 0 \\ 0 & 0 \end{bmatrix} \leq \gamma^2 \begin{bmatrix} -PA - A^{\mathrm{T}}P + \alpha_2(C^{\mathrm{T}}C - P) & PB \\ B^{\mathrm{T}}P & \beta_2 \end{bmatrix} \tag{30}$$

To estimate the bound on the output, an optimization problem can be formulated to minimize γ under the constraints (29) and (30). Note that the two matrices in (29) are linear with respect to all variables and the two matrices in (30) are linear in P for fixed α_2 and β_2. When α_2, β_2 are fixed, the minimal γ can be obtained by solving a "gevp" problem. If we define the minimal γ for the "gevp" problem as a function of α_2 and β_2, $\gamma_1(\alpha_2, \beta_2)$, we may use "fminsearch" in Matlab to find the minimal γ_1 over $\alpha_2, \beta_2 \in [0, \infty)$.

Also note that the optimization problem reduces to the corresponding problem in [2] if $\alpha_1 = \alpha_2 = \beta_1 = 0$.

We use a simple third order system to demonstrate the improvement.

Example 2: Consider the system (28) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and $C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$. The bound on the output obtained via invariant ellipsoid is 0.9023. The bound obtained via the constraints (29) and (30) is 0.6789.

V. Maximal invariant set under state constraint

Consider the linear system

$$\dot{x} = Ax, \quad y = Cx, \tag{31}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Each output is constrained within a given bound. For simplicity, assume that the bound for each output is 1, i.e., $|y_i(t)| \leq 1$ for all $i = 1, 2, \cdots, m$. Denote the corresponding state constraint set as

$$X_c = \{x \in \mathbb{R}^n : \ |C_i x| \leq 1, \ \forall\, i\}.$$

A set $X_0 \subset X_c$ is said to be admissible if all trajectories starting from X_0 will stay within X_c for all $t > 0$. We would like to determine an admissible set which is as large as possible.

A simple method is to find a maximal invariant ellipsoid inside the constraint set X_c, which can be formulated as an LMI problem (see e.g., [9]). Such an estimation can be improved by considering invariant sets which incorporate the structure of the constraints. Here we would like to consider invariant set as the level set of the Lyapunov function

$$V(x) = \max\{x^{\mathrm{T}}Px, x^{\mathrm{T}}C_i^{\mathrm{T}}C_i x, i = 1, 2, \cdots, m\}. \tag{32}$$

The 1-level set $L_V = \{x \in \mathbb{R}^n : V(x) \leq 1\}$ is formed by clipping the ellipsoid $\mathcal{E}(P)$ with planes $C_i x = \pm 1$ and thus lies within the polytope X_c. We first give a condition for the set L_V to be invariant.

Proposition 4: Given $P > 0$. Let V be defined in (32). The set L_V is invariant for (31) if there exist $a_i > 0, b_{ij} \geq 0$, $\alpha_0 \geq 0, \alpha_i \geq 0$, $i, j = 1, 2, \cdots, m$ such that $\Sigma_{i=1}^m \alpha_i = \alpha_0$, and

$$a_i(C_i^{\mathrm{T}}C_i A + A^{\mathrm{T}}C_i^{\mathrm{T}}C_i) \leq P - C_i^{\mathrm{T}}C_i + \sum_{j=1}^{m} b_{ij}(C_j^{\mathrm{T}}C_j - C_i^{\mathrm{T}}C_i), \quad i = 1, 2, \cdots, m, \tag{33}$$

$$PA + A^{\mathrm{T}}P \leq -\alpha_0 P + \sum_{j=1}^{m} \alpha_j C_j^{\mathrm{T}}C_j. \tag{34}$$

The proof easily follows from applying S-procedure to subsets where $V(x) = x^{\mathrm{T}}C_i^{\mathrm{T}}C_i x$ or $V(x) = x^{\mathrm{T}}Px$. For example, consider x such that $x^{\mathrm{T}}C_1^{\mathrm{T}}C_1 x > x^{\mathrm{T}}Px$ and $x^{\mathrm{T}}C_1^{\mathrm{T}}C_1 x > x^{\mathrm{T}}C_i^{\mathrm{T}}C_i x$ for all $i > 1$. Then $\dot{V}(x; Ax) = x^{\mathrm{T}}C_1^{\mathrm{T}}C_1 Ax$ and condition (33) with $i = 1$ ensures that

$$\begin{aligned} &\dot{V}(x; Ax) \\ &\leq \frac{1}{a_1}x^{\mathrm{T}}(P - C_1^{\mathrm{T}}C_1)x + \sum_{j=1}^{m} \frac{b_{1j}}{a_1}x^{\mathrm{T}}(C_j^{\mathrm{T}}C_j - C_1^{\mathrm{T}}C_1)x \\ &\leq 0. \end{aligned}$$

The invariant set L_V can be maximized with respect to certain shape reference set X_R such that $\eta X_R \subset L_V$ for the maximal η. The set inclusion condition $\eta X_R \subset L_V$ can be stated as LMIs if X_R is a polygon or ellipsoid. For example, consider $X_R = \mathrm{co}\{x_k : k = 1, 2, \cdots, K\}$. Then $\eta X_R \subset L_V$ if and only if

$$x_k^{\mathrm{T}}Px_k \leq 1/\eta^2, \ \ x_k^{\mathrm{T}}C_i^{\mathrm{T}}C_i x_k \leq 1/\eta^2, \ \ \forall\, i, k. \tag{35}$$

An optimization problem can be formulated to maximize η satisfying (35), (33) and (34).

Note that all the conditions in (33) are LMIs and the condition (34) is LMI for a fixed α_0.

Proposition 4 can be easily extended to linear differential inclusions by duplicating the matrix inequalities for each vertex matrix A_k, with respective coefficients $a_{ik}, b_{ijk}, \alpha_{0k}, \alpha_{jk}$.

This is because L_V is a convex set. It is invariant for the linear differential inclusion if and only if it is invariant for each vertex system.

Example 3: Consider a second order system with $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, $C = I$, and X_R contains a single point $(1, 1)$. The maximal γ such that γX_R is inside an invariant ellipsoid is 0.7071. The maximal γ such that γX_R is inside an invariant L_V is 0.8090. Fig. 5 compares the resulting invariant ellipsoid and invariant L_V (within the thick boundary). Directions of Ax are plotted along the boundary of L_V to verify that it is invariant.

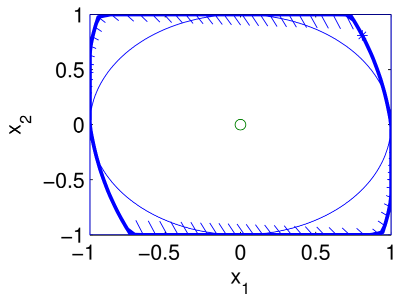

Fig. 5. Enlarged invariant set within state constraints

Example 4: Consider the balance beam system in [9]. The open-loop system is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -0.04 \end{bmatrix} x + k \begin{bmatrix} 0 \\ -0.3796 \end{bmatrix} u$$

where x_1 is the gap between the beam and the stator, x_2 its velocity, and u the control current in the electromagnets. $k \in [0.75, 1.2]$ is an uncertain gain arising from the nonlinearity of the electromagnets. Under the state feedback $u = \begin{bmatrix} 336 & 44 \end{bmatrix} x$, the closed-loop system is

$$\dot{x} \in \text{co}\{A_1 x, A_2 x\},$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -153 & -20 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 1 \\ -95 & -12 \end{bmatrix}.$$

The state x_1 is restricted to $|x_1| \leq 0.004$ due to the maximal gap. This corresponds to $|Cx| \leq 1$ with $C = \begin{bmatrix} 250 & 0 \end{bmatrix}$. We would like to determine the maximal γ such that for initial condition $\gamma \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the state constraint is not violated. This corresponds to the maximal speed starting from the horizontal position (balanced position) so that the beam will not touch the stator. Using invariant ellipsoid, the maximal γ is 0.0642; using invariant set clipped from ellipsoid by $Cx = \pm 1$, the maximal γ is 0.0715. The resulting invariant sets are plotted in Fig 6.

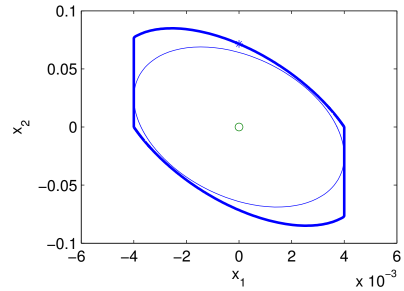

Fig. 6. Enlarged admissible set and invariant ellipsoid.

VI. Conclusions

This paper develops some methods for the estimation of output range and for the detection of a large invariant set within the state constraints. Existing methods use quadratic Lyapunov functions for these purposes. We use non-quadratic Lyapunov functions which incorporate the structures of the output and constraints. The resulting LMI-based algorithms are a little more complicated than the existing algorithms based on quadratic functions but may lead to significant improvement as demonstrated by examples. The Lyapunov functions in this paper involve one quadratic function. Further improvements are expected if we use more quadratic functions to compose the Lyapunov functions.

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