2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

American options in regime-switching Lévy models with non-semibounded stochastic interest rates

Svetlana Boyarchenko and Sergei Levendorskiĭ

Abstract— **A general numerical method for pricing American options in regime-switching jump-diffusion models of stock dynamics with stochastic interest rates and/or volatility is developed. Time derivative and infinitesimal generator of the process for factors that determine the dynamics of the interest rate and/or volatility are discretized. The result is a sequence of embedded perpetual options in a Markov-modulated Lévy model. Options in this sequence are solved using an iteration method based on the Wiener-Hopf factorization. Contrary to the earlier version of the method, the interest rate may assume non-positive values. As applications, explicit algorithms for Vasicek and Black's models with jumps are derived. Numerical examples show that the option prices in these two models are very close.**

I. INTRODUCTION

There exist several groups of numerical methods for pricing of American options: finite difference methods, variational methods, Monte-Carlo simulations, various analytical approximations, reduction to an integral equation for the early exercise boundary, and the analytical method of lines (a.k.a. Carr's randomization or Canadization) in several forms. See, e.g., [3], [4], [5], [6], [9] and the bibliography therein. In [2], we suggested a general approach to construction of numerical methods for pricing of American options in models with 2-3 factors, which combines elements of finite difference methods, Carr's randomization, and the Wiener-Hopf factorization in the form standard in analysis. The basic case for our approach is a regime-switching Lévy model, which we considered in [1]. Stochastic interest rates and/or stochastic volatility are approximated by finite state Markov chains. In [1], the pricing procedures were developed under a certain technical assumption, which implied that it is optimal to exercise the option should the stochastic factor fall sufficiently low (the case of put-like options) or rise sufficiently high (the case of call-like options). In this paper, we relax this condition, and derive the rules which select the states in which the option is never exercised. After that, we modify the pricing procedure for American options with finite time horizon so that this no-exercise effect is taken into account. In applications to models with stochastic interest rates, in presence of jumps, it is possible that the American put option should not be exercised at any level of the stock price if the riskless rate is below a certain positive level. We produce numerical examples for Vasicek and Black's models with embedded jumps.

Supported in part by NSF grant #DMS-0631628

S. Boyarchenko is with Department of Economics, The University of Texas at Austin, 1 University Station, C3100, Austin, TX 78712, USA sboyarch@eco.utexas.edu

S. Levendorskiĭ is with Department of Economics, The University of Texas at Austin, 1 University Station, C3100, Austin, TX 78712, USA leven@eco.utexas.edu

II. PRELIMINARIES

A. *Lévy processes and Expected Present Value operators*

The results of the paper are valid for Lévy processes satisfying (ACP)-condition, or absolute continuity of potential measures (see Definition 41.11 in [10]). This condition is satisfied if the transition kernel has the density. As a basic example, we will use the the double-exponential model jump-diffusion model (Kou's model [7]), with the Lévy exponent

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + bz + \frac{c^+ z}{\lambda^+ - z} + \frac{c^- z}{\lambda^- - z}. \quad (1)$$

Recall that Ψ appears in the representation of the moment generating function of a Lévy process: $E\left[e^{zX_t}\right] = e^{t\Psi(z)}$ and in the formula for action of the infinitesimal generator of X_t, denoted L, on exponential functions: $Le^{zx} = \Psi(z)e^{zx}$. In Kou's model, the coefficient c^+ (respectively, c^-) characterizes the intensity of upward jumps (respectively, downward jumps). The parameter λ^+ describes the relative intensity of large jumps: the smaller the λ^+, the larger is the probability of large upward jumps as opposed to small ones. Likewise, the smaller the λ^-, the larger is the probability of large downward jumps. If one of the $c^\pm$ is zero, there are no jumps in the corresponding direction.

In this section, we use the notation T; in Sections II – VI, T will denote the maturity of the American put. Introduce the normalized resolvent or *expected present value operator (EPV-operator)* of a stochastic process X:

$$\mathcal{E}g(x) = E^x[g(X_T)] = qE^x\left[\int_0^{+\infty} e^{-qt}g(X_t)dt\right]. \quad (2)$$

This operator calculates the EPV of a stream $qg(X_t)$. Applying $\mathcal{E}$ to $g(x) = e^{zx}$ and using the equality $E\left[e^{zX_t}\right] = e^{t\Psi(z)}$, we obtain that $\mathcal{E}$ acts on exponential functions as the multiplication operator by the number $q(q - \Psi(z))^{-1}$. To ensure that the expectation is finite, it is necessary and sufficient that the real part of $q - \Psi(z)$ is positive. Since $(q - L)e^{zx} = (q - \Psi(z))e^{zx}$, we conclude that $q^{-1}(q - L)$ and $\mathcal{E}$ are mutual inverses. To make this statement precise, we need to specify function spaces between which $q^{-1}(q - L)$ and $\mathcal{E}$ act. We will also need the normalized EPV-operators $\mathcal{E}^\pm$ of the supremum process $\bar{X}_t = \sup_{0\le s\le t} X_s$ and the infimum process $\underline{X}_t = \inf_{0\le s\le t} X_s$. They are defined by replacing X_t in (2) with $\bar{X}_t$ and $\underline{X}_T$, respectively. Evidently, $\mathcal{E}^+ g(x) = E^x[g(\bar{X}_T)]$ and $\mathcal{E}^- g(x) = E^x[g(\underline{X}_T)]$, where $T \sim \mathrm{Exp}\, q$. It is straightforward to check that $\mathcal{E}^+$ and $\mathcal{E}^-$

also act on an exponential function e^{zx} as multiplication operators by numbers, which we denote $\kappa^+(z)$ and $\kappa^-(z)$, respectively:

$$\mathcal{E}^+ e^{zx} = \kappa^+(z) e^{zx}, \quad \mathcal{E}^- e^{zx} = \kappa^-(z) e^{zx}. \tag{3}$$

These numbers are $\kappa^+(z) = E\left[e^{z\bar{X}_T}\right], \kappa^-(z) = E\left[e^{z\underline{X}_T}\right]$. Note that to simplify the notation, we suppress the dependence of the EPV-operators $\mathcal{E}, \mathcal{E}^\pm$ and numbers $\kappa^\pm(z)$ on q (and on the process X).

B. *Wiener-Hopf factorization*

The Wiener-Hopf factorization formula reads:

$$E[e^{zX_T}] = E[e^{z\bar{X}_T}]E[e^{z\underline{X}_T}], \ \forall\ z \in i\mathbb{R}.$$

Equivalently, $\forall\ z \in i\mathbb{R}$,

$$q/(q - \Psi(z)) = \kappa^+(z)\kappa^-(z). \tag{4}$$

Applying $\mathcal{E}, \mathcal{E}^+$ and $\mathcal{E}^-$ to $g(x) = e^{zx}$ and using (4) and (3), we obtain the third version of the Wiener-Hopf factorization formula:

$$\mathcal{E}g(x) = \mathcal{E}^+\mathcal{E}^- g(x) = \mathcal{E}^-\mathcal{E}^+ g(x). \tag{5}$$

By linearity, (5) holds for linear combinations of exponents and integrals of exponents, hence for wide classes of functions. Equation (5) means that the normalized EPV-operator of a Lévy process admits a factorization into a product of the normalized EPV-operators of the supremum and infimum processes.

C. *Example*

For the Lévy process with the characteristic exponent (1), the Wiener-Hopf factors are

$$\kappa^+(z) = \prod_{l=1,2} \frac{\beta_l^+}{\beta_l^+ - z}, \quad \kappa^-(z) = \prod_{l=1,2} \frac{\beta_l^-}{\beta_l^- - z},$$

where $\beta_2^- < \lambda^- < \beta_1^- < 0 < \beta_1^+ < \lambda^+ < \beta_2^+$ are the roots of the "characteristic equation" $q - \Psi(\beta) = 0$. EPV-operators act as follows

$$\mathcal{E}^\pm u(x) = \sum_{l=1,2} a_l^\pm \beta_l^\pm \int_0^{\pm\infty} e^{-\beta_l^\pm y} u(x+y) dy,$$

where $a_1^\pm, a_2^\pm > 0$ come from the decomposition of $\kappa^\pm(z)$ into the sum of simple fractions.

III. CARR'S RANDOMIZATION IN REGIME-SWITCHING MODELS

Using the appropriate change of the unknown functions and variable $x \mapsto -x$, it is easy to reduce the pricing of American call options to pricing of American put options, and vice versa. We will consider the put. Let λ_{jk} be the transition rate from state j to state k. The riskless rate in state j is q_j. The infinitesimal generator of the driving Lévy process $X_t^{(j)}$ in state j is denoted by L_j, and the Lévy exponent of the process $X_t^{(j)}$ – by Ψ_j. We assume that a switch from state j to state k and a jump of $X_t^{(j)}$ do not happen simultaneously, a.s. However, we may produce simultaneous switches and jumps playing with different payoff functions in different states of the Markov chain. The payoff functions G_j are of the form $G_j(x) = K_j - B_j e^x$, where K_j and B_j are positive. We assume that

$$0 \le \sum_{k \ne j} \lambda_{jk}(B_j - B_k) + (q_j - \Psi_j(1))B_j \quad \forall\ j, \tag{6}$$

$$\exists \quad j \quad \text{such that} \quad \sum_{k \ne j} \lambda_{jk}(K_k - K_j) - q_j K_j < 0. \tag{7}$$

The meaning of conditions (6)–(7) is as follows. The RHS in (6) is non-negative iff the asset price is a local supermartingale, and zero, if it is a local martingale. Indeed, $(\Psi_j(1) - q_j)B_j$ can be interpreted as the rate of local gains in the discounted asset price while the process remains in state j, and $\sum_{k \ne j} \lambda_{jk}(B_k - B_j)$ as the rate of gains due to regime switching (since the switches are instantaneous, no discounting is involved). The total rate of gains cannot be positive; if it is zero, the asset is a local martingale. If all K_j are equal: $K_1 = K_2 = \cdots = K$, then the LHS in (7) is $-q_j K$. If the interest rate is positive in all states, then it is optimal to exercise the put option provided the stock price falls sufficiently low. However, if there are no switches and the riskless rate is zero then it is optimal not to exercise the American put until expiry, at any level of the stochastic factor. The same holds in the regime-switching case if the riskless rate is zero in all states. If the riskless rate is zero in some states and positive in the other states, then an early exercise can be optimal in some states and non-optimal in the other states, at any level of the stochastic factor; clearly, the same conclusion holds if the riskless rate is negative. In the case of different strike prices, the interpretation is similar: the LHS in (7) is the rate of gains due to waiting in state j, at very low levels of the stochastic factor X_t, where the payoff equals K_j (modulo an error of order $O(e^{X_t})$). Hence, if the LHS is non-negative, then it may be non-optimal to exercise the option in state j, even at very low levels of X_t.

We write $j \in I_{\text{ex}}$ (resp., $j \in I_{\text{no-ex}}$) if the inequality in (7) holds (resp., fails). If $I_{\text{ex}} = \emptyset$, then it is non-optimal to exercise the option till expiry, and the American put option is equivalent to the European put option. The following steps are formulated for the case $I_{\text{ex}} \ne \emptyset$.

I.Divide the time interval $[0, T]$ by points $(0 =)t_0 < t_1 < \cdots < t_N(= T)$ into small intervals of lengths $\Delta_s = t_{s+1} - t_s$, $s = 0, 1, \ldots, N-1$, set $q_j^s = \Delta_s^{-1} + q_j + \Lambda_j$, and, using q_j^s and L_j in place of q and L, define the EPV-operators $\mathcal{E}_j^s, \mathcal{E}_j^{s,-}, \mathcal{E}_j^{s,+}$ and the Wiener-Hopf factors $\kappa_j^{s,-}(1), \kappa_j^{s,+}(1)$.

II. Choose the grid in x-space and a quadrature procedure, which will be used to calculate the action of the EPV-operators $\mathcal{E}_j^s, \mathcal{E}_j^{s,-}, \mathcal{E}_j^{s,+}$. Denote by $xmin$ the lowest point on the grid.

III. For $j = 1, 2, \ldots, m$, set $v_{j,*}^N(x) = G_j(x)_+$ (here and below, calculations of function values are made at points of the chosen grid).

IV. For a fixed $s < N$, we calculate

$$w_{0j}^s = q_j^s((B_j/\kappa_j^{s,-}(1))e^x - K_j), \tag{8}$$

and set $v_j^{s0} = 0, h_j^{s0} = +\infty, j = 1, 2, \dots, m$. Then, for $n = 1, 2, \dots$, we define, step by step, in the interior cycle in $j = 1, 2, \dots, m$,

(i) functions

$$w_j^{sn} = \mathcal{E}_j^{s,+}\left(\sum_{k\neq j}\lambda_{jk}v_k^{s,n-1} + \Delta_s^{-1}v_{j,*}^{s+1}\right); \quad (9)$$

$$\tilde{w}_j^{sn} = w_j^{sn} + w_{0j}^s; \quad (10)$$

(ii) if $j \in I_{\text{no-ex}}$, set $h_j^{sn} = xmin$, otherwise, define h_j^{sn} as a unique solution of the equation

$$\tilde{w}_j^{sn}(h) = 0; \quad (11)$$

(iii) functions

$$v_{0j}^{sn} = (q_j^s)^{-1}\mathcal{E}_j^{s,-}\mathbf{1}_{(-\infty,h_j^{sn}]}(-w_{j0}^s), \quad (12)$$

$$v_{1j}^{sn} = (q_j^s)^{-1}\mathcal{E}_j^{s,-}\mathbf{1}_{(h_j^{sn},+\infty)}w_j^{sn}, \quad (13)$$

$$v_j^{sn} = v_{1j}^{sn} + v_{0j}^{sn}. \quad (14)$$

When an appropriate termination rule for the cycle in n and integration procedure for calculation of action of operators $\mathcal{E}_j^{s,+}$ and $\mathcal{E}_j^{s,-}$ are chosen, the set of formulas (8)–(14) becomes an algorithm for calculation of Carr's randomization approximations v_j^{sn} and h_j^{sn} to the put option price and exercise threshold in each state, over the time interval $[0, T]$.

IV. Models with stochastic interest rate driven by a process of the Ornstein-Uhlenbeck type and deterministic volatility

A. The model

Below, all processes are under an EMM chosen by the market. The interest rate is a deterministic function $r_t = r(Y_t)$ of a Markov process Y_t with the state space $\mathbb{R}^n$ or its subset. The basic examples are quadratic term structure models (QTSM), Vasicek model, more general affine term structure models (ATSM) of class $A_0(n)$ and Black's model. The stock price is modeled as

$$\log S_t = X_t + b^T Y_t, \quad (15)$$

where $b \in \mathbb{R}^n$. The process $(X, Y) = \{(X_t, Y_t)\}_{t\geq 0}$ is a Markov process on the Descartes product of $\mathbb{R}$ and the state space of Y. We assume that Y is independent of X but the drift of X_t depends on Y_t: $\mu = \mu(Y_t)$. (Thus, we assume that the volatility of X and its jump density are independent of Y. The generalization to the stochastic volatility case will be considered in a separate publication.) To be more specific, we assume that (X_t, Y_t) evolves as a solution to the system of stochastic differential equations

$$dY_t = \kappa(\theta - Y_t)dt + dZ_t^r, \quad (16)$$

$$dX_t = \mu(Y_t)dt + dZ_t, \quad (17)$$

where κ is an anti-stable $n \times n$ matrix, $\theta \in \mathbb{R}^n$, and Z_t^r and Z_t are independent Lévy processes in $\mathbb{R}^n$ and $\mathbb{R}$. Consider a contingent claim with the deterministic or random expiry date τ and payoff $G(X_\tau, Y_\tau) = K - \exp[X_\tau + bY_\tau]$. Denote by L^r and Ψ^r (resp., by L and Ψ) the infinitesimal generator and Lévy exponent of Z^r (resp., Z). Since we model the stock dynamics as $S_t = \exp(X_t + b^T Y_t)$, we have to assume that Ψ^r is well-defined at b, and Ψ is well-defined at 1. Applying the Feynman-Kac theorem, we obtain that in the region where the option remains alive, V is the solution of the equation

$$(\partial_t + (\kappa(\theta - y), \partial_y) + (\mu(y), \partial_x) + L^r + L - r(y))V(t, x, y) = 0 \quad (18)$$

(subject to appropriate boundary conditions), where L^r and L act w.r.t. y and x, respectively, and $(A, B) := \sum_{j=1}^m A_j B_j$ denotes the sum of ordered products of operators. If the dividend rate, $\delta(Y_t)$, is given, then

$$\mu(y) = r(y) - \delta(y) - b^T\kappa(\theta - y) - \Psi^r(b) - \Psi(1). \quad (19)$$

B. Reduction to a regime-switching model

As in the Carr's randomization procedure [4], we discretize time; in addition, we discretize the y-space. On the first step, we discretize the former, and reduce the pricing problem to the one in a regime-switching model. The state space of the modulating Markov chain is $\bar{\Delta} \cdot \mathbb{Z}$, where $\bar{\Delta}$ is a chosen step in the y-space. We truncate the state space $\bar{\Delta} \cdot \mathbb{Z}$ to obtain a finite-state Markov chain. For realistic parameter values, the short rate is of order several percent, and the probability of deviation up to a level of 20 percent is very small unless the time interval is extremely large. Thus, it is safe to keep only $y_j = j\Delta$ satisfying $r(y_j) \leq 0.2$. If the riskless rate may assume non-positive values, we also delete the states satisfying $r(y_j) < r_-(\leq 0)$, where the lower bound for the riskless rate is chosen so that the probability that r_t reaches r_- during the life of the option is small. After the discretization of y-variable and truncation, we obtain a regime-switching model with the payoffs $G_j(x) = K - e^{x+by_j}$.

This reduction involves two subtle issues related to verification of (6)–(7), one of the crucial conditions for the algorithm for regime-switching models. First, if the dividend rate is zero on a subset of non-zero measure, the RHS of (6) is negative in some states. However, if the y-step is small, the discrepancy is small as well, and it can be shown that the procedure for regime-switching models still works.

Second, in all cases, the straightforward truncation leads to a chain for which (6) fails at the boundary points. Hence, we need to modify the transition rates at the boundary. In the diffusion case, a convenient modification can be formulated as the (discrete version of the) boundary condition $V_{yy} = 0$. The same modification of the diffusion component is applied in the jump-diffusion case; for the jump component, the modification is similar in spirit albeit different. Note that this modification significantly decreases the truncation error and increases the convergence of the computational scheme. Numerical examples show that the resulting errors become fairly small for reasonable values of parameters of the numerical scheme.

C. The gap between the strike and early exercise boundary at expiry

Let $H_{**}(t,y) = \exp[h_*(t,y) + by]$ be the early exercise boundary in S-space. If the stock pays no dividends, then, in the pure diffusion case, the limit of the early exercise boundary at expiry is the strike: $\lim_{t\uparrow T} H_{**}(t,y) = K$. For processes with jumps, in many numerical examples, we observed the gap between the limit of the early exercise boundary and strike, which agrees with the general result (Theorem 2.2) in [8]. In the model under consideration, this result is as follows. Fix (x,y) in the in-the-money region for the American put, that is, $G(x,y) := K - e^{x+by} > 0$. In the pure diffusion case, if t is sufficiently close to T and the process is at (x,y): $(X_t, Y_t) = (x,y)$, then it is optimal to exercise the option. However, if jumps are present, and

$$\begin{aligned}\delta(y)S - r(y)K \quad &+ \quad \int_{\mathbb{R}} (Se^{by'} - K)_+ F^r(dy') \\ &+ \quad \int_{\mathbb{R}} (Se^{x'} - K)_+ F(dx') > 0,\end{aligned}$$

where $S = e^{x+by}$, then it is non-optimal to exercise the American put at $(X_t, Y_t) = (x,y)$ even if t is arbitrarily close to T. In in-the-money region for the put $K > S$, therefore, in the case $b < 0$, we have the condition

$$\begin{aligned}\delta(y)S \quad &- \quad r(y)K + \int_{-\infty}^{\log(K/S)/b} (Se^{by'} - K) F^r(dy') \\ &+ \quad \int_{\log(K/S)}^{+\infty} (Se^{x'} - K) F(dx') > 0. \qquad (20)\end{aligned}$$

Theorem 4.1: Let $\delta \geq 0$, $b < 0$, and either $F^r(dy')$ is non-zero in a neighborhood of $-\infty$ or $F(dx')$ is non-zero in a neighborhood of $+\infty$. Then

a) if $r(y) \leq 0$, then $H_{**}(0+, y) = 0$;
b) if $r(y) > 0$ and the condition

$$r(y) - \delta(y) < \int_{-\infty}^{0} (e^{by'} - 1) F^r(dy') + \int_{0}^{+\infty} (e^{x'} - 1) F(dx') \qquad (21)$$

fails, then $H_{**}(0+, y) = K$;

c) if $r(y) > 0$ and (21) holds, then $H_{**}(0+, y) = KR_{0+}(y)$, where $R_{0+}(y)$ is a unique solution of the equation

$$\begin{aligned}r(y) \quad &= \quad \delta(y)R + \int_{-\infty}^{-\log(R)/b} (Re^{by'} - 1) F^r(dy') \\ &+ \int_{-\log R}^{+\infty} (Re^{x'} - 1) F(dx'). \qquad (22)\end{aligned}$$

We see that if the dividend rate is non-negative and either positive jumps in X_t or negative jumps in Y_t are present, then, at sufficiently low levels of the short rate r, there must be a gap. This conclusion holds for $b < 0$. If $b > 0$, then positive jumps in Y_t matter, and if $b = 0$, then the gap is independent of jumps in Y_t.

V. Reduction to a regime-switching model for processes of Ornstein-Uhlenbeck type

Assume for simplicity that Z^r and Z are Brownian motions with embedded Poisson jumps, without drift components. The volatilities and Lévy densities are denoted σ_r, σ and $F^r(dy)$ and $F(dx)$, respectively. Let Δ be the time step, and $\bar{\Delta}$ – the y-step. We choose Δ so that $N = T/\Delta$ is an integer, and we take $\bar{\Delta}$ of order $\sqrt{\Delta}$. Set $t_s = s\Delta$, $s = 0, 1, \ldots, N$, and $y_j = j\bar{\Delta}$, $j \in \mathbb{Z}$. Further, denote $V_j^s(x) = V(t_s, x, y_j)$, and approximate

- $V_t(t_s, x, y_j)$ by $\Delta^{-1}(V_j^{s+1}(x) - V_j^s(x))$;
- $V_{yy}(t_s, x, y_j)$ by $\bar{\Delta}^{-2}\left(V_{j+1}^s - 2V_j^s + V_{j-1}^s\right)$;
- $\kappa(\theta - y_j)V_y(t_s, x, y_j)$ by $\bar{\Delta}^{-1}\kappa(\theta - y_j)(V_{j+1}^s(x) - V_j^s(x))$, if $\kappa(\theta - y_j) \geq 0$, and by $\bar{\Delta}^{-1}\kappa(\theta - y_j)(V_j^s(x) - V_{j-1}^s(x))$, if $\kappa(\theta - y_j) < 0$.

To discretize the integral part, we use the following approximation:

$$\int_{-\infty}^{+\infty} (u(y_j + y') - u(y_j)) F^r(dy') \approx \sum_{k \neq j} C_k (u(y_k) - u(y_j)),$$

where $C_k = c_{k-j} - c^0_{k-j} + c^0_{k-j-1}$, and

$$\begin{aligned}c_l \quad &= \quad \int_{y_l}^{y_{l+1}} F^r(dy'), \\ c_l^0 \quad &= \quad \bar{\Delta}^{-1} \int_{y_l}^{y_{l+1}} (y' - y_l) F^r(dy').\end{aligned}$$

Set $L_j = \mu(y_j)\partial_x + L$, $q_j = r(y_j)$, $K_j = K$, $B_j = e^{by_j}$,

$$\lambda_{j,j+1} \quad = \quad \frac{\sigma_r^2}{2\bar{\Delta}^2} + \frac{\kappa(\theta - y_j)_+}{\bar{\Delta}} + C_1, \qquad (23)$$

$$\lambda_{j,j-1} \quad = \quad \frac{\sigma_r^2}{2\bar{\Delta}^2} + \frac{\kappa(-\theta + y_j)_+}{\bar{\Delta}} + C_{-1}, \qquad (24)$$

and, for $|j - k| > 1$, set $\lambda_{jk} = C_{k-j}$. The transition rates introduced above define the regime-switching model, which approximates the model with the stochastic interest rates. The state space being $\mathbb{Z}$, we need truncate it and impose appropriate boundary conditions. In the BM case, we use the discretized version of the standard condition $V_{yy} = 0$ at the boundary, equivalently, we modify the transition rates at the boundary. In the presence of jumps, the same modification is used for the diffusion part of the infinitesimal generator. For the jump part, we use a similar non-local version. For motivation, we use the following interpretation of the discretized boundary condition $V_{yy} = 0$ in the gaussian case. It is (a special case of) the following linear extrapolation procedure: for $-m + 1 < k < 1$, $V_k = 2V_1 - V_{2-k}$, and for $m < k < 2m$, $V_k = 2V_m - V_{2m-k}$. This procedure can be applied in the case of jumps to approximate a term $\lambda_{jk}(V_k - V_j)$ in the analytical expression for the infinitesimal generator of the infinite Markov chain with $\lambda_{jk}(2V_m - V_{2m-k} - V_j)$. Our final goal being the construction of a finite Markov chain with the state space $\{1, 2, \ldots, m\}$, this approximation makes no sense for the other k. Also, we feel that it would be somewhat unnatural to approximate a jump up by a weighted sum of a jump up and jump down; in addition, such an approximation may lead to negative transition rates (consider the case of jumps in only one direction). Hence, for jumps up, we apply this extrapolation procedure if $k = m+1, \dots, 2m-j-1$, and for jumps down, if $k = 3-j, 3-j+1, \dots, 0$. For any other k, we simply replace λ_{jk} with zero. To sum up: we consider the jump part of the infinitesimal generator L^r_J, with the transition rates $\lambda^J_{jk} = C_{k-j}$. First, we truncate the sum in the formula for L^r_J and, for $1 \le j \le m$, leave only

$$\sum_{k\neq j, 1\le k\le m} \lambda^J_{jk}(V_k - V_j) + \sum_{2-j<k<1} \lambda^J_{jk}(V_k - V_j) + \sum_{m<k<2m-j} \lambda^J_{jk}(V_k - V_j).$$

After that, in the second sum, we substitute $V_k = 2V^1 - V_{2-k}$, and in the third sum, we set $V_k = 2V_m - V_{2m-k}$. These changes having being made, we compare the result with the formula for the action of the infinitesimal generator of the Markov chain with m states

$$(L^r_{J,\mathrm{tr}}V)_j = \sum_{k\neq j, 1\le k\le m} \lambda^{J,\mathrm{tr}}_{jk}(V_k - V_j), \tag{25}$$

and derive the following formula for transition rates $\lambda^{J,\mathrm{tr}}_{jk}$ of the truncated (and modified) chain: for $1 \le j \le m$,

$$\lambda^{J,\mathrm{tr}}_{jk} = \begin{cases} \lambda^J_{jk} - \lambda^J_{j,2m-k}, & j<k<m, \\ \lambda^J_{jm} + 2\sum_{j<l<m} \lambda^J_{j,2m-l}, & k=m, \\ \lambda^J_{jk} - \lambda^J_{j,2-k}, & 1<k<j, \\ \lambda^J_{j1} + 2\sum_{1<l<j} \lambda^J_{j,2-l}, & k=1 \end{cases} \tag{26}$$

For $l > 0$, set $C^+_l = C_l, C^-_l = C_{-l}$. The resulting formulas for the transition rates λ_{jk} of the truncated and modified chain are

- $\frac{\kappa(\theta-y_1)}{\Delta} + C^+_1 - C^+_{2m-3}$,, if $j=1, k=2$;
- $\frac{\sigma_r^2}{2\Delta^2} + \frac{\kappa(\theta-y_j)_+}{\Delta} + C^+_1 - C^+_{2m-2j-1}$, if $2 \le j \le m-2, k=j+1$;
- $C^+_{k-j} - C^+_{2m-k-j}$, if $1 \le j \le m-2, j+2 \le k \le m-1$;
- $C^+_{m-j} + 2\sum_{j+1\le l\le m-1} C^+_{2m-l-j}$, if $1 \le j \le m-2, k=m$;
- $\frac{\sigma_r^2}{2\Delta^2} + \frac{\kappa(\theta-y_{m-1})_+}{\Delta} + C^+_1$, if $j=m-1, k=m$;
- $\frac{\kappa(y_m-\theta)}{\Delta} + C^-_1 - C^-_{2m-3}$, if $j=m, k=m-1$;
- $\frac{\sigma_r^2}{2\Delta^2} + \frac{\kappa(-\theta+y_j)_+}{\Delta} + C^-_1 - C^-_{2j-3}$, if $3 \le j \le m-1, k=j-1$;
- $C^-_{j-k} - C^-_{j+k-2}$, $3 \le j \le m, 2 \le k \le j-2$;
- $C^-_{j-1} + 2\sum_{2\le l\le j-1} C^-_{j+l-2}$, if $3 \le j \le m, k=1$;
- $\frac{\sigma_r^2}{2\Delta^2} + \frac{\kappa(-\theta+y_2)_+}{\Delta} + C^-_1$, if $j=2, k=1$.

In the pure Gaussian case, non-negativity of the transition rates λ_{jk} is satisfied automatically. In the presence of jumps, we need an additional condition. The following condition is necessary and sufficient for $|j-k|>1$ and sufficient for $k = j\pm1$: $C^\pm_l \ge 0$ for all $l>0$. For $F^r(dy)$ given by exponential densities $c^\pm_r(\pm\lambda^\pm_r)e^{-\lambda^\pm_r y}dy$ on the positive and negative half-axes, where $c^\pm_r > 0$ and $\lambda^-_r < 0 < \lambda^+_r$, the Lévy exponent $\Psi^r(z)$ is well-defined at $z=b$ iff $\lambda^-_r < b < \lambda^+_r$, and, therefore, this will be our standing assumption. The straightforward calculations yield, for $l>0$

$$\begin{aligned} C^-_l &= c^-_r e^{\lambda^-_r y_l}(-\lambda^-_r\bar\Delta)^{-1}(e^{-\lambda^-_r\bar\Delta} + e^{\lambda^-_r\bar\Delta} - 2), \\ C^+_l &= c^+_r e^{-\lambda^+_r y_l}(\lambda^+_r\bar\Delta)^{-1}(e^{-\lambda^+_r\bar\Delta} + e^{\lambda^+_r\bar\Delta} - 2). \end{aligned}$$

Hence, $C^\pm_l \ge 0$ for exponential densities on each half-axis, and for linear combinations of such densities with positive coefficients.

VI. Numerical example: the Vasicek model and Black's model of the short rate

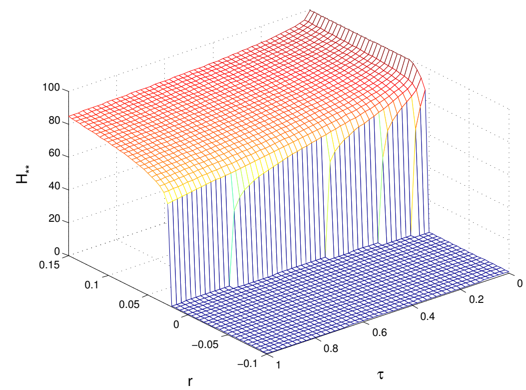

Fig. 1. Early exercise boundary at $\tau \le 1$ to expiry, in the Vasicek model

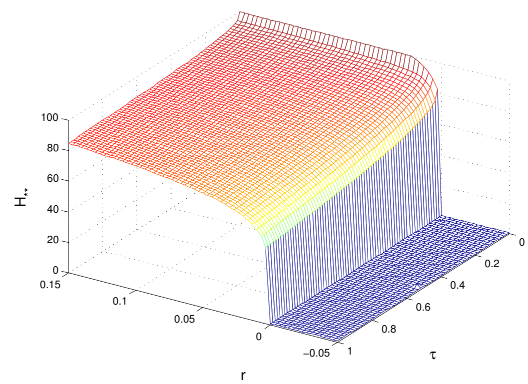

Fig. 2. Early exercise boundary at $\tau \le 1$ to expiry, in Black's model

We consider the American put on a non-dividend paying stock, with the payoff $G(X_t, Y_t) = K - \exp(X_t + bY_t)$. We consider two cases: the riskless rate is as in the Vasicek model: $r_t = Y_t$, and as in Black's model: $r_t = \max\{0, Y_t\}$. The jump components of X_t and Y_t are as in Kou's model. Parameters of the model are as follows:

- $K = 100, b = -0.2, r0 = 0.01, \kappa = 1.5, \theta = 0.2$
- $\sigma_r = 0.05, c^-_r = 0.25, \lambda^-_r = -70, c^+_r = 0.25, \lambda^+_r = 75$
- $\sigma = 0.22, c^- = 0.2, \lambda^- = -5, c^+ = 0.2, \lambda^+ = 10$

Calculations were made on a notebook PC, with the characteristics Intel(R)Core(TM)2 CPU T7200 2.00 GHz RAM

TABLE I

EARLY EXERCISE BOUNDARY FOR $\tau \leq 1$, IN THE VASICEK MODEL; $K = 100$

	r	0	0.02	0.04	0.06	0.08	0.1
	0	1	1	1	1	1	1
	0+	0	98.9433	1	1	1	1
	0.2	0	79.4772	83.7235	85.6632	87.0009	88.0336
τ	0.4	0	71.8504	79.405	82.1155	83.9386	85.3384
	0.6	0	62.2367	76.369	79.7359	81.9306	83.598
	0.8	0	6.2247	73.9338	77.8959	80.4017	82.2873
	1	0	5.7985	71.862	76.3759	79.1525	81.2244

TABLE II

OPTION VALUES AT $\tau = 1$, IN THE VASICEK MODEL; $K = 100$

	S	81.87	90.48	100	110.52	122.14
	$\log(S/K)$	-0.2	-0.1	0	0.1	0.2
	0	20.5195	14.5104	9.5162	5.7983	3.3274
	0.02	19.7047	13.7259	8.8698	5.3361	3.0359
r	0.04	19.0575	13.0361	8.2872	4.9201	2.7761
	0.06	18.5846	12.4336	7.7611	4.5448	2.5441
	0.08	18.2802	11.9076	7.2835	4.2047	2.3362
	0.1	18.1381	11.4478	6.8476	3.8953	2.1493

2046 MB. The program was written in MATLAB, and parallel computations were not used, although the structure of the iteration procedure allows for independent calculations for each $y_j, j = 1, 2, \ldots, m$, at each step of the iteration procedure. Since in the numerical example below, the number of states is 81, one can expect that with parallel computations the CPU time would had been much less. The general conclusion is that, even with this inefficient realization, relative errors smaller than 0.005 – even for out-of-the-money options – can be obtained in reasonable time, at millions at points in (t, y, x)–space. Time per point is negligible. Total CPU time is 924.2 sec; the output is the early exercise threshold at $25 \cdot 81 = 2025$ points in (t, y)-space, and at $2025\times$ several thousand points in (t, y, x)-space. Assuming that the option values at 200 points in x-space are needed, the CPU time per point is 0.0023 sec. The numerical results shown in Tables 1–3 and Fig. 1–2 demonstrate that the option prices in both models are very close. In both cases, the gap between the early exercise boundary and strike is clearly seen. The early exercise boundary differ very little above $r = 0.1$ – the relative error is less than 0.0001. Near $r = 0$, the difference can be very large because the "abysses" clearly seen in Fig. 1 and 2 are a bit different. In particular, it is seen that in the Vasicek model, the no-exercise region widens as time to expiry increases whereas in Black's model it remains approximately the same. The intuition is simple: due to the negative discounting below $r = 0$, it is advantageous to wait so that the spot rate can reach this region. The larger time to expiry, the higher is the probability of this favorable event, and the higher is the option value of waiting.

TABLE III

BLACK'S MODEL VS. VASICEK MODEL: RELATIVE DIFFERENCE OF OPTION VALUES, IN UNITS OF 10^{-4}. $K = 100$, $\tau = 1$.

	S	81.87	90.48	100	110.52	122.14
	$\log(S/K)$	-0.2	-0.1	0	0.1	0.2
	0	1.25	0.78	-0.21	-0.72	-4.8
	0.02	-0.05	-0.81	-2.2	-3.2	-2.7
r	0.04	-0.42	-1.5	-3.1	4.3	-3.6
	0.06	-0.43	-1.7	-3.4	-4.9	-4.3
	0.08	-0.27	-1.7	-3.5	-5.0	-4.3
	0.1	-0.1	-1.5	-3.5	-4.9	-4.2

VII. CONCLUSIONS AND FUTURE WORKS

In the paper, we extended a general algorithm developed in [1] for pricing of American options in regime-switching Lévy models to the case of non-positive interest rates, and applied this algorithm to the American put and call options on a stock under non-positive stochastic interest rates. We approximated the stochastic factor that determines the interest rate dynamics by a Markov chain, and derived explicit formulas for transition rates. The stock price is modelled as $S_t = e^{X_t + bY_y}$, and the short rate is modelled as a function of Y_t, where X_t is a Lévy process, and Y_t is a jump-diffusion process of the Ornstein-Uhlenbeck type, independent of X_t; drift of X_t depends on Y_t. The interest rate is assumed positive, and the characteristics of the process are independent of t. Thus, the basic examples for the interest rate dynamics are the Vasicek model and Black's model. The numerical results show that the option prices in both models are very close. The early exercise boundary differ very little above $r = 0.1$ – the relative error is less than 0.0001.

The method can be applied to more general affine term structure models, in particular, CIR model, extended Vasicek and CIR models, and to stochastic volatility models.

REFERENCES

[1] S. Boyarchenko and S. Levendorskiǐ, *American Options in Regime-Switching Models*, Working paper, September 2006, http://ssrn.com/abstract=929215

[2] S. Boyarchenko and S. Levendorskiǐ, *American options in regime-switching models with stochastic interest rates*, Working paper, 2007

[3] M. Broadie and J. Detemple, Option pricing: valuation models and applications, *Management science*, vol. 50, 2004, pp 1145-1177.

[4] P. Carr, Randomization and the American put, *Review of Financial Studies*, vol. 11, 1998, pp 597-626.

[5] J. Detemple, *American-Style Derivatives: Valuation and Computation*, Chapman & Hall/CRC Financial Mathematics Series, Volume 4, 2005

[6] P. Glasserman, *Monte Carlo Methods in Financial Engineering*, Springer-Verlag, Berlin; 2003.

[7] S.G. Kou, A jump-diffusion model for option pricing, *Management Science*, 48 (2002), pp. 1086-1101.

[8] S.Z. Levendorskiǐ, *American and European Options in Multi-Factor Jump-Diffusion Models, Near Expiry* (February 24, 2007). Available at SSRN: http://ssrn.com/abstract=965192

[9] A. Medvedev and O. Scaillet, *Pricing American Options under Stochastic Volatility and Stochastic Interest Rates* (February 2007). Available at SSRN: http://ssrn.com/abstract=966055

[10] K.-I. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, 1999.