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Zero Dynamics of Sampled-Data Models for Nonlinear Systems

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Abstract— **One of the approaches to sampled-data controller design for nonlinear continuous-time systems consists of obtaining an appropriate model and then proceeding to design a controller for the model. Then, it is important to derive a good approximate sampled-data model because the exact sampled-data model for nonlinear systems is often unavailable to the controller designers. Recently, Yuz and Goodwin have proposed an accurate sampled-data model which includes extra zero dynamics, so-called the sampling zero dynamics, corresponding to the relative degree of the continuous-time nonlinear system. This paper shows that a more accurate sampled-data model is required for a controlled Van der Pol system with the relative degree two. The reason is that the closed-loop system becomes unstable when a controller design method based on cancellation of the zero dynamics is applied, and the phenomenon seems related to the instability of the sampling zero dynamics of the more accurate sampled-data model. Further, this paper derives a more accurate model than that of Yuz and Goodwin for continuous-time nonlinear systems with the relative degree two, and presents a condition which assures the stability of the sampling zero dynamics of the obtained model.**

I. INTRODUCTION

Advances in digital electronics that occurred in the second half of the 20th century have led to a rapid development in computer technology and this has had a great impact on engineering areas, including control engineering. Since recent control systems usually employ digital technology for controller implementation, the study of sampled-data control systems has become an important issue in control fields. Significant progress has been achieved in this area during this decade.

There are two distinct approaches to sampled-data controller design for nonlinear systems [1]. The first one, so-called controller emulation, involves digital implementation of a continuous-time stabilizing control law at a sufficient high sampling rate. The second approach consists of obtaining a sampled-data model and then proceeding to design a controller for the model. Emulation is regarded as the simple method, whereas it is typically inferior to the second in terms of stability and achievable performance. On the other hand, the second approach requires a good approximate sampled-data model because the exact sampled-data model for nonlinear systems is often unavailable to the controller designers.

Therefore, the accuracy of the approximate sampled-data model has proven to be a key issue in the context of control design, where a controller designed to stabilize an approximate model may fail to stabilize the exact discrete-time model [2].

Recently, Yuz and Goodwin have proposed an accurate sampled-data model [3]. The resulting model includes extra zero dynamics which are called sampling zero dynamics. It has been shown explicitly that they have no counterpart in the underlying continuous-time system and are the same as those for the linear case [4], although an implicit characterization has been given in [5]. It is worth noting here that Yuz and Goodwin's model has a mode corresponding to the sampling zero dynamics just on the unit circle when the relative degree of a continuous-time nonlinear system is two.

This paper shows that a more accurate sampled-data model is required for a controlled Van der Pol system with the relative degree two. The reason is that the closed-loop system becomes unstable when a controller design method based on cancellation of the zero dynamics is applied, and the phenomenon seems related to the instability of the sampling zero dynamics of the more accurate sampled-data model.

Further, this paper derives a more accurate model than that of Yuz and Goodwin for continuous-time nonlinear systems with the relative degree two, and shows a condition which assures the stability of the sampling zero dynamics of the obtained model.

For linear systems, the properties of the sampling zeros corresponding to the sampling zero dynamics for nonlinear systems are discussed in many papers [4], [6]-[11].

II. SYSTEM DESCRIPTION AND PREVIOUS RESULTS

Consider a class of the following single-input single-output nth-order nonlinear system

$$\begin{cases} \dot{\boldsymbol{x}} &=& \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u \\ y &=& h(\boldsymbol{x}) \end{cases} \tag{1}$$

where $\boldsymbol{x}$ is the state evolving on an open subset $\mathcal{M} \subset R^n$, and where the vector fields $\boldsymbol{f}(\cdot)$ and $\boldsymbol{g}(\cdot)$, and the output function $h(\boldsymbol{x})$ are analytic on $\mathcal{M}$.

First, the following assumptions are introduced.

Assumption 1: The unique equilibrium point lies on the origin.

Assumption 2: The continuous-time nonlinear system (1) has the uniform relative degree $r(\leq n)$ and is minimum phase in the open subset $\mathcal{M}$, where the state $\boldsymbol{x}$ evolves.

Then, the system can be expressed in its so-called normal

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form [12], [13].

$$\begin{cases} \dot{\boldsymbol{\zeta}} &= \begin{bmatrix} \mathbf{0}_{r-1} & I_{r-1} \\ 0 & \mathbf{0}_{r-1}^T \end{bmatrix} \boldsymbol{\zeta} \\ & \quad + \begin{bmatrix} \mathbf{0}_{r-1} \\ 1 \end{bmatrix} (b(\boldsymbol{\zeta},\ \boldsymbol{\eta}) + a(\boldsymbol{\zeta},\ \boldsymbol{\eta})u) \\ \dot{\boldsymbol{\eta}} &= \boldsymbol{c}(\boldsymbol{\zeta},\ \boldsymbol{\eta}) \\ y &= z_1 \end{cases} \tag{2}$$

$$\boldsymbol{\zeta} = \begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix},\ \boldsymbol{\eta} = \begin{bmatrix} z_{r+1} \\ \vdots \\ z_n \end{bmatrix}, \tag{3}$$

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\eta} \end{bmatrix},\ \boldsymbol{c} = \begin{bmatrix} c_{r+1}(\boldsymbol{\zeta},\ \boldsymbol{\eta}) \\ \vdots \\ c_n(\boldsymbol{\zeta},\ \boldsymbol{\eta}) \end{bmatrix} \tag{4}$$

where $a(\mathbf{0},\mathbf{0}) \neq 0$, $b(\mathbf{0},\mathbf{0}) = 0$ and $\boldsymbol{c}(\mathbf{0},\mathbf{0}) = \mathbf{0}$.

Under the assumptions 1 and 2, the zero dynamics of (2) is determined by

$$\dot{\boldsymbol{\eta}} = \boldsymbol{c}(\mathbf{0},\ \boldsymbol{\eta}) \tag{5}$$

and is asymptotically stable in $\mathcal{M}$.

We are interested in the sampled-data model for the nonlinear system (2) when the input is a piecewise constant signal generated by a zero-order hold (ZOH); i.e.,

$$u(t) = u(kT),\ kT \leq t < (k+1)T,\quad k = 0,\ 1, \cdots \tag{6}$$

where T is a sampling period.

For small sampling periods, Yuz and Goodwin have derived a sampled-data model of the following form

$$\begin{cases} \boldsymbol{\zeta}_{k+1} &= F_s \boldsymbol{\zeta}_k + \boldsymbol{g}_s\,(b_k + a_k u_k) \\ \boldsymbol{\eta}_{k+1} &= \boldsymbol{\eta}_k + T\boldsymbol{c}(\boldsymbol{\zeta}_k,\ \boldsymbol{\eta}_k) \\ y_k &= [1\ \mathbf{0}_{r-1}^T]\boldsymbol{\zeta}_k \end{cases} \tag{7}$$

$$F_s = \begin{bmatrix} 1 & T & \frac{T^2}{2} & \cdots & \frac{T^{r-1}}{(r-1)!} \\ & 1 & T & & \frac{T^{r-2}}{(r-2)!} \\ & & \ddots & \ddots & \vdots \\ & O & & 1 & T \\ & & & & 1 \end{bmatrix},\ \boldsymbol{g}_s = \begin{bmatrix} \frac{T^r}{r!} \\ \frac{T^{r-1}}{(r-1)!} \\ \vdots \\ \frac{T^2}{2!} \\ T \end{bmatrix} \tag{8}$$

$$b_k \equiv b(\boldsymbol{\zeta}_k,\ \boldsymbol{\eta}_k),\ a_k \equiv a(\boldsymbol{\zeta}_k,\ \boldsymbol{\eta}_k) \tag{9}$$

where the subscripts k and $k+1$ denote the time instants kT and $(k+1)T$, respectively.

Then, the zero dynamics of the sampled-data model (7) consist of the sampled counterpart of the continuous-time zero dynamics and the additional zero dynamics produced by the sampling process [3]. The latter are called the sampling zero dynamics, and equivalent to the same as those which appear asymptotically for the linear case when the sampling period tends to zero, namely, the roots of the following equations.

$$\begin{aligned} & z+1=0,\ r=2 \\ & z^2+4z+1=0,\ r=3 \\ & (z+1)(z^2+10z+1)=0,\ r=4 \\ & z^4+26z^3+66z^2+26z+1=0,\ r=5 \\ & \quad \vdots \end{aligned}$$

The zeros are given as follows.

$$\begin{aligned} & -1,\ r=2 \\ & -3.732,\ -1/3.732,\ r=3 \\ & -1,\ -9.899,\ -1/9.899,\ r=4 \\ & -2.322,\ -23.20,\ -1/2.322,\ -1/23.20,\ r=5 \\ & \quad \vdots \end{aligned}$$

The result above has been derived by the following relation.

$$\begin{aligned} z_{i+1,k+1} &= y_{k+1}^{(i)} \\ &\approx y_k^{(i)} + Ty_k^{(i+1)} + \frac{T^2}{2}y_k^{(i+2)} + \\ &\quad \cdots + \frac{T^{r-i}}{(r-i)!}y_k^{(r)} \\ &\approx z_{i+1,k} + Tz_{i+2,k} + \frac{T^2}{2}z_{i+3,k} + \\ &\quad \cdots + \frac{T^{r-i}}{(r-i)!}(a_k + b_k u_k), \\ &\quad i = 0, \cdots, r-1 \end{aligned} \tag{10}$$

Here, if a higher-order Taylor expansion such as

$$\begin{aligned} z_{i+1,k+1} &= y_{k+1}^{(i)} \\ &\approx y_k^{(i)} + Ty_k^{(i+1)} + \frac{T^2}{2}y_k^{(i+2)} + \\ &\quad \cdots + \frac{T^{r-i}}{(r-i)!}y_k^{(r)} + \frac{T^{r-i+1}}{(r-i+1)!}y_k^{(r+1)} \\ &\quad i = 0, \cdots, r-1 \end{aligned} \tag{11}$$

is applied, then a more accurate sampled-data model is obtained.

We define here the nonlinear sampling zeros σ_{ri} $(i = 1, \cdots, r-1)$ as the eigenvalues of the matrix which determines the sampling zero dynamics of a sampled-data model proposed by Yuz and Goodwin [3]. Then, it is expected that the nonlinear sampling zeros ρ_{ri} $(i = 1, \cdots, r-1)$ of the more accurate sampled-data model (11) is expressed as

$$\rho_{ri}(\boldsymbol{\zeta}_k, \boldsymbol{\eta}_k) = \sigma_{ri} + \overline{\sigma}_{ri}(\boldsymbol{\zeta}_k, \boldsymbol{\eta}_k)T \tag{12}$$

From the results of Yuz and Goodwin, at least one of the nonlinear sampling zeros lies strictly outside of the unit circle; i.e., $|\sigma_{ri}| > 1$ for $r \geq 3$. Hence, from the view point of the stability of the sampling zero dynamics, it is not important to derive a more accurate sampled-data model for nonlinear systems with the relative degree $r \geq 3$. However, when the relative degree r of a nonlinear system is two, the nonlinear sampling zero σ_{ri} is equal to -1. Namely, the stability of the sampling zero dynamics is marginal. Thus, in order to analyze the stability of the nonlinear sampling zero dynamics, it is useful to derive the relation (12) of nonlinear systems with $r = 2$. Notice here that the relative degree of many nonlinear mechanical systems in the practical field is two.

In the remainder of this section, an interesting example is shown to motivate the derivation of the more accurate nonlinear sampling zero (12) for nonlinear systems with $r = 2$ mentioned above.

Consider a controlled Van der Pol system with the following equation [13].

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \epsilon(1 - x_1^2)x_2 + u, \ \epsilon > 0 \\ y &= x_1 \end{cases} \tag{13}$$

It is obvious that the relative degree of the system (13) is two, and that the system does not have zero dynamics. A sampled-data model by Yuz and Goodwin for (13) is represented as

$$\begin{cases} x_{1,k+1} &= x_{1,k} + Tx_{2,k} \\ &\quad + \dfrac{T^2}{2}\left[-x_{1,k} + \epsilon(1 - x_{1,k}^2)x_{2,k} + u_k\right] \\ x_{2,k+1} &= x_{2,k} + T\left[-x_{1,k} + \epsilon(1 - x_{1,k}^2)x_{2,k} + u_k\right] \\ y_k &= x_{1,k} \end{cases} \tag{14}$$

The zero dynamics of the sampled-data model (14) are given by the equation $z + 1 = 0$ [3].

Consider here model following control such that the output converges to the origin. When the input u_k is designed as

$$\begin{aligned} u_k &= x_{1,k} - \epsilon(1 - x_{1,k}^2)x_{2,k} \\ &\quad + \frac{2}{T^2}\left[-Tx_{2,k} + (\alpha - 1)x_{1,k}\right], \\ &\quad 0 < \alpha < 1 \end{aligned} \tag{15}$$

then, the closed-loop sampled-data model (14) generates the output relation

$$y_{k+1} - \alpha y_k = 0 \tag{16}$$

and the internal dynamics

$$x_{2,k+1} + x_{2,k} = 0 \tag{17}$$

Thus, the convergence of the output to the origin is achieved and all the variables remain bounded [14].

However, in simulation, when the input (15) is imposed through a ZOH on the original continuous-time system (13), the output does not converge to the origin. In other words, the closed-loop system is unstable.

The reason is explained below. A more accurate sampled-data model using (11) is expressed as

$$\begin{cases} x_{1,k+1} &= x_{1,k} + Tx_{2,k} \\ &\quad + \dfrac{T^2}{2}\left[-x_{1,k} + \epsilon(1 - x_{1,k}^2)x_{2,k} + u_k\right] \\ &\quad + \dfrac{T^3}{3!}\left[-x_{2,k} - 2\epsilon x_{1,k}x_{1,k}^2 + \epsilon(1 - x_{1,k}^2)\right. \\ &\quad \left. \times \{-x_{1,k} + \epsilon(1 - x_{1,k}^2)x_{2,k} + u_k\}\right] \\ x_{2,k+1} &= x_{2,k} + T\left[-x_{1,k} + \epsilon(1 - x_{1,k}^2)x_{2,k} + u_k\right] \\ &\quad + \dfrac{T^2}{2}\left[-x_{2,k} - 2\epsilon x_{1,k}x_{1,k}^2 + \epsilon(1 - x_{1,k}^2)\right. \\ &\quad \left. \times \{-x_{1,k} + \epsilon(1 - x_{1,k}^2)x_{2,k} + u_k\}\right] \\ y_k &= x_{1,k} \end{cases} \tag{18}$$

The local truncation error between the output of the sampled-data model (18) and the true output is of order T^4. This fact means that the sampled-data model (18) is closer to the true system than that by Yuz and Goodwin.

Now, when the input (15) is applied to the sampled-data model (18), the closed-loop system is given by

$$y_{k+1} - \alpha y_k = o(T) \tag{19}$$

and the internal dynamics

$$\begin{aligned} x_{2,k+1} &= -\{1 + \epsilon(1 - x_{1,k}^2)T\}x_{2,k} \\ &\quad - \left(\frac{1}{2}x_{2,k} + \epsilon x_{1,k}x_{2,k}^2\right)T^2 + \delta \end{aligned} \tag{20}$$

where the term δ does not include the variable $x_{2,k}$. The relation (20) implies that the internal dynamics is unstable in the neighborhood of $x_{1,k} = 0$ for sufficiently small sampling periods. As a result, the convergence of the output to the origin is not achieved.

Next, it is easy to obtain the zero dynamics of the more accurate sampled-data model (18) as

$$x_{2,k+1} = -\left(1 + \frac{\epsilon}{3}T\right)x_{2,k} \tag{21}$$

Hence, it is found that the internal dynamics (20) is related directly to the zero dynamics of the more accurate sampled-data model (18).

III. MAIN RESULTS

Consider a class of the following single-input single-output nth-order nonlinear system with the relative degree two which is expressed in the normal form such as

$$\begin{cases} \dot{\boldsymbol{\zeta}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{\zeta} + \\ &\quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}(b(\boldsymbol{\zeta},\ \boldsymbol{\eta}) + a(\boldsymbol{\zeta},\ \boldsymbol{\eta})u) \\ \dot{\boldsymbol{\eta}} &= \boldsymbol{c}(\boldsymbol{\zeta},\ \boldsymbol{\eta}) \\ y &= z_1 \end{cases} \tag{22}$$

$$\zeta = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},\ \boldsymbol{\eta} = \begin{bmatrix} z_3 \\ \vdots \\ z_n \end{bmatrix}, \tag{23}$$

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\eta} \end{bmatrix},\ \boldsymbol{c} = \begin{bmatrix} c_3(\boldsymbol{\zeta},\ \boldsymbol{\eta}) \\ \vdots \\ c_n(\boldsymbol{\zeta},\ \boldsymbol{\eta}) \end{bmatrix} \tag{24}$$

where $a(\mathbf{0},\mathbf{0})\neq 0$, $b(\mathbf{0},\mathbf{0})=0$ and $\boldsymbol{c}(\mathbf{0},\mathbf{0})=\mathbf{0}$.

Now, it will be shown below when a more accurate model has the sampling zero dynamics inside or outside of the unit circle.

Before proceeding, the following assumption is needed here for the preservation of an affine property in the process of sampling.

Assumption 3:

$$\frac{\partial a(\boldsymbol{\zeta},\ \boldsymbol{\eta})}{\partial z_2} = 0 \tag{25}$$

Assumption 3 ensures that a new sampled-data system is also an affine one.

Note here that

$$\dot{u}(t) = 0,\ \ddot{u}(t) = 0,\ t \in [kT,\ (k+1)T] \tag{26}$$

and that

$$\dot{y} = \dot{z}_1 = z_2 \tag{27}$$

$$\ddot{y} = \dot{z}_2 = b + au \tag{28}$$

$$\begin{aligned} y^{(3)} &= \dot{b} + \dot{a}u \\ &= \sum_{i=1}^{n} \frac{\partial b}{\partial z_i}\dot{z}_i + \sum_{i=1}^{n} \frac{\partial a}{\partial z_i}\dot{z}_i u \\ &= \frac{\partial b}{\partial z_1} z_2 + \frac{\partial b}{\partial z_2}(b + au) \\ &\quad + \sum_{i=3}^{n} \frac{\partial b}{\partial z_i} c_i + \left(\frac{\partial a}{\partial z_1} z_2 + \sum_{i=3}^{n} \frac{\partial a}{\partial z_i} c_i \right) u \\ &= \overline{b} + \overline{a}u \end{aligned} \tag{29}$$

where

$$\overline{b} = \overline{b}(\boldsymbol{\zeta},\ \boldsymbol{\eta}) \equiv \frac{\partial b}{\partial z_1} z_2 + \frac{\partial b}{\partial z_2} b + \sum_{i=3}^{n} \frac{\partial b}{\partial z_i} c_i \tag{30}$$

$$\overline{a} = \overline{a}(\boldsymbol{\zeta},\ \boldsymbol{\eta}) \equiv \frac{\partial b}{\partial z_2} a + \frac{\partial a}{\partial z_1} z_2 + \sum_{i=3}^{n} \frac{\partial a}{\partial z_i} c_i \tag{31}$$

then, a more accurate model than that of Yuz and Goodwin is derived by substituting (29) as well as (27) and (28) into the Taylor's expansion forms of $y((k+1)T)$ and $\dot{y}((k+1)T)$ and neglecting higher order terms as follows

$$\begin{aligned} y_{k+1} &= y_k + T\dot{y}_k + \frac{T^2}{2}\ddot{y}_k + \frac{T^3}{6} y_k^{(3)} \\ &= y_k + T\dot{y}_k + \frac{T^2}{2}(b_k + a_k u_k) \\ &\quad + \frac{T^3}{6}\left(\overline{b}_k + \overline{a}_k u_k\right) \\ &= y_k + T\dot{y}_k + \frac{T^2}{2} b_k + \frac{T^3}{6}\overline{b}_k \\ &\quad + \left(\frac{T^2}{2} a_k + \frac{T^3}{6}\overline{a}_k \right) u_k \end{aligned} \tag{32}$$

$$\begin{aligned} \dot{y}_{k+1} &= \dot{y}_k + T\ddot{y}_k + \frac{T^2}{2} y_k^{(3)} \\ &= \dot{y}_k + T(b_k + a_k u_k) + \frac{T^2}{2}\left(\overline{b}_k + \overline{a}_k u_k\right) \\ &= \dot{y}_k + Tb_k + \frac{T^2}{2}\overline{b}_k + T\left(a_k + \frac{T}{2}\overline{a}_k \right) u_k \end{aligned} \tag{33}$$

$$\boldsymbol{\eta}_{k+1} = \boldsymbol{\eta}_k + T\boldsymbol{c}(\boldsymbol{\zeta}_k,\ \boldsymbol{\eta}_k) \tag{34}$$

where

$$\overline{b}_k \equiv \overline{b}(\boldsymbol{\zeta}_k,\ \boldsymbol{\eta}_k),\ \overline{a}_k \equiv \overline{a}(\boldsymbol{\zeta}_k,\ \boldsymbol{\eta}_k) \tag{35}$$

Now, the following assumption is further imposed.

Assumption 4:

$$\frac{\partial \boldsymbol{c}(\boldsymbol{\zeta},\ \boldsymbol{\eta})}{\partial z_2} = \mathbf{0} \tag{36}$$

Since $\boldsymbol{c}(\boldsymbol{\zeta},\ \boldsymbol{\eta})$ is independent of z_2 under the assumption 4, the sampled-data system (32)-(34) has the sampled counterpart of the continuous-time zero dynamics given by

$$\boldsymbol{\eta}_{k+1} = \boldsymbol{\eta}_k + T\boldsymbol{c}(\mathbf{0},\ \boldsymbol{\eta}_\mathbf{k}) \tag{37}$$

On the basis of the result in [3], the sampling zero dynamics of the model (32)-(34) are calculated below. First, when we set $y_{k+1} = y_k = 0$, then (32) leads to

$$\begin{aligned} &T\dot{y}_k + \frac{T^2}{2} b_{k0} + \frac{T^3}{6}\overline{b}_{k0} \\ &\quad + \left(\frac{T^2}{2} a_{k0} + \frac{T^3}{6}\overline{a}_{k0} \right) u_k = 0 \end{aligned} \tag{38}$$

where $b_{k0},\ a_{k0},\ \overline{b}_{k0},\ \overline{a}_{k0}$ denote the values of $b_k,\ a_k,\ \overline{b}_k,\ \overline{a}_k$ with $y_k = 0$ and $\boldsymbol{\eta}_k = \boldsymbol{\eta}^S$ where $\boldsymbol{\eta}^S$ is the state vector of the sampled counterpart (37) of the continuous-time zero dynamics [3]. Deleting u_k in (33) by (38) yields

$$\begin{aligned} \dot{y}_{k+1} &= \dot{y}_k + Tb_{k0} + \frac{T^2}{2}\overline{b}_{k0} \\ &\quad - \frac{6}{(3a_{k0} + T\overline{a}_{k0})T^2}\left(Ta_{k0} + \frac{T^2}{2}\overline{a}_{k0} \right) \\ &\quad \times \left(T\dot{y}_k + \frac{T^2}{2} b_{k0} + \frac{T^3}{6}\overline{b}_{k0} \right) \\ &= \dot{y}_k + \frac{T}{2}\left(2b_{k0} + T\overline{b}_{k0}\right) \\ &\quad - \frac{2a_{k0} + T\overline{a}_{k0}}{2(3a_{k0} + T\overline{a}_{k0})}\left(6\dot{y}_k + 3Tb_{k0} + T^2\overline{b}_{k0}\right) \end{aligned} \tag{39}$$

Here, the coefficient of (39) is approximated as

$$\begin{aligned} \frac{2a_{k0} + T\overline{a}_{k0}}{3a_{k0} + T\overline{a}_{k0}} &= \frac{2a_{k0}\left(1 + \dfrac{T\overline{a}_{k0}}{2a_{k0}}\right)}{3a_{k0}\left(1 + \dfrac{T\overline{a}_{k0}}{3a_{k0}}\right)} \\ &\approx \frac{2}{3}\left(1 + \frac{T\overline{a}_{k0}}{2a_{k0}}\right)\left(1 - \frac{T\overline{a}_{k0}}{3a_{k0}}\right) \\ &\approx \frac{2}{3}\left\{1 + \left(\frac{\overline{a}_{k0}}{2a_{k0}} - \frac{\overline{a}_{k0}}{3a_{k0}}\right)T\right\} \\ &= \frac{2}{3}\left(1 + \frac{\overline{a}_{k0}}{6a_{k0}}T\right) \end{aligned} \tag{40}$$

Hence, the relation (39) is rewritten as

$$\begin{aligned}\dot{y}_{k+1} &\approx \dot{y}_k + \frac{T}{2}\left(2b_{k0} + T\overline{b}_{k0}\right) \\ &\quad -\frac{1}{3}\left(1 + \frac{\overline{a}_{k0}}{6a_{k0}}T\right)(6\dot{y}_k + 3Tb_{k0}) \\ &\approx -\dot{y}_k + \left\{b_{k0} - \frac{1}{3}\left(\frac{\overline{a}_{k0}}{a_{k0}}\dot{y}_k + 3b_{k0}\right)\right\}T \\ &= -\dot{y}_k - \frac{\overline{a}_{k0}}{3a_{k0}}T\dot{y}_k \qquad (41)\end{aligned}$$

As a result, the sampling zero dynamics are given by (41). From (41), the main result is obtained as follows.

Theorem 1. Consider an affine nonlinear system (22) with the relative degree two. Then, for sufficiently small sampling periods T, the sampling zero dynamics of the sampled-data model (32)-(34) are given approximately by

$$\dot{y}_{k+1} + \dot{y}_k + \frac{\overline{a}_{k0}}{3a_{k0}}T\dot{y}_k = 0 \qquad (42)$$

where a_{k0} and $\overline{a}_{k0}$ are the values of a_k and $\overline{a}_k$, respectively, with $y_k = 0$ and $\boldsymbol{\eta} = \boldsymbol{\eta}^S$ where $\boldsymbol{\eta}^S$ is the state vector of the sampled counterpart of the continuous-time zero dynamics. Further, a_k and $\overline{a}_k$ are defined by (9) and (35), respectively.

It is straightforward to obtain the following theorem from Theorem 1.

Theorem 2. For sufficiently small sampling periods, the sampling zero dynamics of the sampled-data model (32)-(34) are stable if

$$\frac{\overline{a}_{k0}}{3a_{k0}} < 0 \qquad (43)$$

and they are unstable if

$$\frac{\overline{a}_{k0}}{3a_{k0}} > 0 \qquad (44)$$

IV. EXAMPLES

Two examples of Theorems 1 and 2 are shown.

Example 1. Consider a pendulum system with the following equation.

$$\left\{\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -cx_2 - d\sin x_1 + au \\ y &= x_1\end{aligned}\right. \qquad (45)$$

It is easy to see that

$$a(\boldsymbol{\zeta},\ \boldsymbol{\eta}) = a, \qquad (46)$$
$$b(\boldsymbol{\zeta},\ \boldsymbol{\eta}) = -cx_2 - d\sin x_1, \qquad (47)$$
$$\overline{a}(\boldsymbol{\zeta},\ \boldsymbol{\eta}) = \frac{\partial b}{\partial x_2}a + \frac{\partial a}{\partial x_1}x_2 = -ac \qquad (48)$$

Hence, we have

$$\frac{\overline{a}_{k0}}{3a_{k0}} = -\frac{c}{3} < 0 \qquad (49)$$

From Theorem 1, for sufficiently small sampling periods T, the sampling zero dynamics of the sampled-data model (32)-(34) corresponding to the pendulum system are given approximately by

$$\dot{y}_{k+1} + \dot{y}_k - \frac{c}{3}T\dot{y}_k = 0 \qquad (50)$$

Further, from Theorem 2, the sampled-data model corresponding to the pendulum system has stable sampling zero dynamics for sufficiently small sampling periods.

Example 2. Consider a Van der Pol system (13). Note here that

$$a(\boldsymbol{\zeta},\ \boldsymbol{\eta}) = 1, \qquad (51)$$
$$b(\boldsymbol{\zeta},\ \boldsymbol{\eta}) = -x_1 + \epsilon(1 - x_1^2)x_2, \qquad (52)$$
$$\overline{a}(\boldsymbol{\zeta},\ \boldsymbol{\eta}) = \frac{\partial b}{\partial x_2}a + \frac{\partial a}{\partial x_1}x_2 = \epsilon(1 - x_1^2) \qquad (53)$$

then, it holds that

$$\frac{\overline{a}_{k0}}{3a_{k0}} = \left.\frac{\epsilon(1 - x_1^2)}{3}\right|_{x_1=0} = \frac{\epsilon}{3} > 0 \qquad (54)$$

From Theorem 1, for sufficiently small sampling periods T, the sampling zero dynamics of the sampled-data model (32)-(34) corresponding to the Van der Pol system are given approximately by

$$\dot{y}_{k+1} + \dot{y}_k + \frac{\epsilon}{3}T\dot{y}_k = 0 \qquad (55)$$

in the neighborhood of the origin. Further, from Theorem 2, the sampled-data model corresponding to the Van der Pol system has unstable sampling zero dynamics for sufficiently small sampling periods.

V. CONCLUSIONS

Recently, Yuz and Goodwin have proposed an accurate sampled-data model which includes extra zero dynamics, so-called the sampling zero dynamics, corresponding to the relative degree of the continuous-time nonlinear system. This paper shows that a more accurate sampled-data model is required for a controlled Van der Pol system with the relative degree two. The reason is that the closed-loop system becomes unstable when a controller design method based on cancellation of the zero dynamics is applied, and the phenomenon seems related to the instability of the sampling zero dynamics of the more accurate sampled-data model. Further, this paper derives a more accurate model than that of Yuz and Goodwin for continuous-time nonlinear systems with the relative degree two, and presents a condition which assures the stability of the sampling zero dynamics of the obtained model.

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