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Partial exact linearization design for the Acrobot walking

Sergej Čelikovský, Jiří Zikmund and Claude Moog

Abstract— **A new control concept for a class of simple underactuated mechanical system, the so-called Acrobot, is presented here. Despite being seemingly a simple system, the acrobot comprises many important difficulties when controlling the most challenging underactuated system - the walking robot. This paper presents the design of the asymptotical tracking of the prescribed trajectory generated by a suitable open-loop input of the acrobot. Such a design is based on the partial exact linearization of the third order combined with a certain robust stabilization technique. The proposed control is then demonstrated by the exponential tracking of the walking-like trajectory of the acrobot. Besides theoretical proofs, our approach is supported by numerical simulations and illustrated by acrobot movement animations.**

Index Terms– **Mechanical Systems, exact feedback linearization, underactuated system, walking robots.**

I. INTRODUCTION

Efficient control of the underactuated mechanical system constitutes one of the most challenging problems of recent decades, see [12], [4] and references within there. Reliable and economic walking is the typical example of the related studies among both control and robotic community. One of the simplest underactuated mechanical systems is the acrobot. Despite being a seemingly simple system, the acrobot comprises many important features of the underactuated walking robots having the degree of underactuation equal to one. As a matter of fact, one can show that any n-link having $n-1$ actuators between its links, can be decomposed into a fully actuated system and an acrobot "disturbed" by some influence from that fully actuated (and therefore fully exact feedback linearizable) subsystem, see [10], [7]. As a consequence, the effective control of the acrobot is an important step on the route to underactuated walking. Recently, numerous papers have addressed the stabilization of its inverted position extending its domain of attraction [1], [9], [5], [11], [3], [13].

This paper aims to use a similar approach as in [3], [13] for the asymptotical tracking of a suitable target trajectory generated by an open loop control. As might have been expected, the asymptotical tracking constitutes principally a more complicated problem than the stabilization since the corresponding error dynamics has a more complex structure than the Acrobot model itself. In particular, designed tracking feedback can handle limited initial tracking error only and its performance is better when the Acrobot movement is slower. On the other hand, the mentioned approach uses a special coordinate and feedback transformation which, in general, posseses singularities. It is worth to underline that, as discussed later in more detail, these singularities may occur during the stabilization of the Acrobot inverted position, but are excluded along walking-like trajectories.

The rest of the paper is organized as follows. The next section briefly presents the model of n-link underactuated system together with various possibilities of partial exact feedback linearization. Section 3 presents in detail the main result of this contribution including proofs. Numerical simulations are described in Section 4. Final section draws briefly some conclusions and discusses some open future research outlooks toward efficient underactuated walking.

II. ACROBOT

The acrobot depicted on Figure 1 is a special case of n-link chain with $n-1$ actuators attached by one of its ends

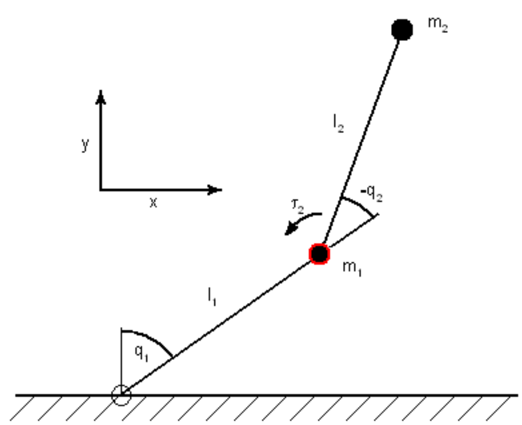

Fig. 1. Acrobot.

to a pivot point through an unactuated rotary joint. Such a system can be modelled by usual Lagrangian approach [6]. The corresponding Lagrangian is as follows

$$\mathcal{L}(q,\dot{q}) = K - V = \frac{1}{2}\dot{q}^T D(q)\dot{q} - V(q) \tag{1}$$

where q denotes a n-dimensional configuration vector on the configuration manifold Q and $D(q)$ is the inertia matrix, K is the kinetic energy and V is the potential energy of the

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Sergej Čelikovský and Jiří Zikmund are with the Faculty of Electric Engineering, Czech Technical University in Prague and with the Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Prague, CR, celikovs@utia.cas.cz, zikmuj@control.fel.cvut.cz

Jiri Zikmund and Claude Moog are with the Institut de Recherche en Communications et Cybernétique, UMR CNRS 6597, Nantes, France, Claude.Moog@irccyn.ec-nantes.fr

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system. The resulting Euler-Lagrange equation is

$$\begin{bmatrix} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}}{\partial q_1} \\ \vdots \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_n} - \frac{\partial \mathcal{L}}{\partial q_n} \end{bmatrix} = u = \begin{bmatrix} 0 \\ \tau_2 \\ \vdots \\ \tau_n \end{bmatrix}, \qquad (2)$$

where u stands for vector of external controlled forces. The system (2) is the so-called **underactuated** mechanical system having the degree of the underactuation equal to one, [10]. Moreover, the underactuated angle is at the pivot point. Equation (2) leads to a dynamic equation in the form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u \qquad (3)$$

where $D(q)$ is the inertia matrix, $C(q,\dot{q})$ contains Coriolis and centrifugal terms, $G(q)$ contains gravity terms and u stands for vector of external forces.

For the Acrobot, these computations lead to a second-order nonholonomic constraint and a kinetic symmetry, i.e. the inertia matrix depends only on the second variable q_2

$$D(q) = \left[\begin{array}{l|r} \theta_1 + \theta_2 + 2\theta_3 \cos q_2 & \theta_2 + \theta_3 \cos q_2 \\ \theta_2 + \theta_3 \cos q_2 & \theta_2 \end{array}\right], \qquad (4)$$

$$C(q,\dot{q}) = \begin{bmatrix} -\theta_3 \sin q_2 \dot{q}_2 & -(\dot{q}_2 + \dot{q}_1)\theta_3 \sin q_2 \\ \theta_3 \sin q_2 \dot{q}_1 & 0 \end{bmatrix}, \qquad (5)$$

$$G(q) = \begin{bmatrix} -\theta_4 g \sin q_1 - \theta_5 g \sin(q_1 + q_2) \\ -\theta_5 g \sin(q_1 + q_2) \end{bmatrix}, \qquad (6)$$

where the 2-dimensional configuration vector (q_1, q_2) consists of angles defined on Figure 1 and

$$\begin{array}{c} \theta_1 = (m_1 + m_2) l_1^2 + I_1, \ \theta_2 = m_2 l_2^2 + I_2, \\ \theta_3 = m_2 l_1 l_2, \ \theta_4 = (m_1 + m_2) l_1, \ \theta_5 = m_2 l_2. \end{array} \qquad (7)$$

The **partial exact feedback linearization** method is based on a system transformation into a new system of coordinates that display linear dependence between some output and new input [8]. From the theoretical point of view, the mechanical system dynamics is described by n-dimensional state space. Static state feedback linearization of the suitable output function relative degree r yields a linear subsystem of dimension r. In other words, the maximal feedback linearization problem consists in linearizing a function with maximal relative degree. In [7] it was shown that if the generalized momentum conjugate to the cyclic variable is not conserved (as it is the case of Acrobot) then there exists a set of outputs that defines a one-dimensional exponentially stable zero dynamics. That means that it is possible to find a function $\overline{y}(q,\dot{q})$ with relative degree 3 that transforms the original system (3) by a local coordinate transformation

$$z = T(q,\dot{q}), \quad z_1 = \overline{y},\ z_2 = \dot{\overline{y}},\ z_3 = \ddot{\overline{y}},\ z_4 = f(q,\dot{q}), \qquad (8)$$

into the new input/output linear system with unobservable nonlinear dynamics of dimension 1.

$$\begin{array}{c} \dot{z}_1 = z_2,\ \dot{z}_2 = z_3,\ \dot{z}_3 = \alpha(q,\dot{q})v + \beta(q,\dot{q}) = w, \\ \dot{z}_4 = \psi_1(q,\dot{q}) + \psi_2(q,\dot{q})\tau_2. \end{array} \qquad (9)$$

In the case of the Acrobot there are two independent functions with relative degree 3 transforming the system into the desired form[1] (9), namely

$$\begin{aligned} \sigma &= \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = (\theta_1 + \theta_2 + 2\theta_3 \cos q_2)\dot{q}_1 + \qquad (10) \\ &\qquad (\theta_2 + \theta_3 \cos q_2)\dot{q}_2, \\ p &= q_1 + \frac{q_2}{2} + \frac{2\theta_2 - \theta_1 - \theta_2}{\sqrt{(\theta_1 + \theta_2)^2 - 4\theta_3^2}} \arctan \\ &\qquad \left(\sqrt{\frac{\theta_1 + \theta_2 - 2\theta_3}{\theta_1 + \theta_2 + 2\theta_3}} \tan \frac{q_2}{2} \right). \qquad (11) \end{aligned}$$

The zero dynamics is used to investigate the internal stability when the corresponding output is forced to zero. For the most simple cases $\overline{y} = Cp$ or $\overline{y} = C\sigma$ the resulting zero dynamics is only critically stable. However, considering the output function $\overline{y} = C_1 p(q) + C_2 \sigma(q,\dot{q})$ one gets the following zero dynamics $\dot{p} + C_1[C_2 d_{11}(q_2)]^{-1} p = 0$ which is asymptotically stable whenever C_1/C_2 is positive, $d_{11}(q_2)$ is the corresponding part of the inertia matrix D in (3). Unfortunately, the corresponding transformations have a complex set of singularities, unless C_1 is very small, which is not suitable for practical purposes.

III. MAIN RESULT

As already noticed, the maximal linearizations recalled in the previous section are either only locally defined with many complex singular points, or yield a critically stable zero dynamics. Our main result takes advantage of other special properties of the latter one represented by auxiliary linearizing output $\overline{y} = \sigma$. The corresponding linearizing transformation will be shown to have quite limited singularities. To be more specific, using the set of functions with maximal relative degree, the following transformation

$$T: \quad \xi_1 = p, \xi_2 = \sigma, \xi_3 = \dot{\sigma}, \xi_4 = \ddot{\sigma} \qquad (12)$$

can be defined. Notice, that by (10,11) and some straightforward but laborious computations the following relation holds

$$\dot{p} = d_{11}(q_2)^{-1}\sigma, \qquad (13)$$

where $d_{11}(q_2) = (\theta_1 + \theta_2 + 2\theta_3 \cos q_2)$ is the corresponding element of the inertia matrix D in (3). Applying (12), (13) to (3) we obtain the Acrobot's dynamics in partial exact linearized form

$$\begin{aligned} \dot{\xi}_1 &= d_{11}(q_2)^{-1}\xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= \xi_4 \\ \dot{\xi}_4 &= \alpha(q,\dot{q})\tau_2 + \beta(q,\dot{q}) = w \end{aligned} \qquad (14)$$

with the new coordinates ξ and the input w being well defined wherever $\alpha(q,\dot{q})^{-1} \neq 0$. To determine the region

[1] Actually, by (2) $\dot{\sigma} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \frac{\partial \mathcal{L}}{\partial q_1}$ and therefore by (1) $\dot{\sigma} = -\frac{\partial V(q)}{\partial q_1}$ as $D(q) \equiv D(q_2)$ by (4). In other words, $\dot{\sigma}$ has relative degree 2, i.e. σ has the relative degree 3. Moreover, by the straightforward differentiation it holds $\dot{p} = d_{11}(q_2)^{-1}\sigma$, i.e. $\dot{p}$ has relative degree 2, i.e. p should have relative degree 3 as well.

where such a transformation can be applied, let us express it in an explicit way. Namely, the straightforward computations show that

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = T(q_1, q_2, \dot{q}_1, \dot{q}_2) := \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}, \tag{15}$$

$$\begin{bmatrix} T_1 \\ T_3 \\ T_2 \\ T_4 \end{bmatrix} = \begin{bmatrix} p(q_1, q_2) \\ \theta_4 g \sin q_1 + \theta_5 g \sin(q_1 + q_2) \\ \Phi_2(q_1, q_2) \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \end{bmatrix} \tag{16}$$

where p, σ are given by (10,11) and Φ_2 by (21) later on. Further, denote

$$\phi = \begin{bmatrix} \phi_1(\xi_1, \xi_3) \\ \phi_2(\xi_1, \xi_3) \end{bmatrix}, \text{ such that} \tag{17}$$

$$\begin{aligned} T_1(\phi_1(\xi_1, \xi_3), \phi_2(\xi_1, \xi_3)) &= \xi_1, \\ T_3(\phi_1(\xi_1, \xi_3), \phi_2(\xi_1, \xi_3)) &= \xi_3. \end{aligned} \tag{18}$$

It holds by (15-16) that

$$\frac{\partial[\xi_1, \xi_3, \xi_2, \xi_4]^\top}{\partial[q^\top, \dot{q}^\top]^\top} = \begin{bmatrix} \Phi_1(q_1, q_2) & 0 \\ \Phi_3(q, \dot{q}) & \Phi_2(q_1, q_2) \end{bmatrix} \tag{19}$$

where $q := [q_1, q_2]^\top$, $\Phi_3(q, \dot{q})$ is a certain (2×2) matrix of smooth functions while

$$\Phi_1(q_1, q_2) = \begin{bmatrix} 1 & \frac{\theta_2 + \theta_3 \cos q_2}{\theta_1 + \theta_2 + 2\theta_3 \cos q_2} \\ \theta_4 g \cos q_1 + \theta_5 g \cos(q_1 + q_2) & \theta_5 g \cos(q_1 + q_2) \end{bmatrix}, \tag{20}$$

$$\Phi_2(q_1, q_2) = \begin{bmatrix} \theta_1 + \theta_2 + 2\theta_3 \cos q_2 & \theta_2 + \theta_3 \cos q_2 \\ \theta_4 g \cos q_1 + \theta_5 g \cos(q_1 + q_2) & \theta_5 g \cos(q_1 + q_2) \end{bmatrix}. \tag{21}$$

Further, it obviously holds for (17,18)

$$\frac{\partial \phi(\xi_1, \xi_3)}{\partial[\xi_1, \xi_3]^\top} = \Phi_1^{-1}(q_1, q_2) =$$

$$\frac{1}{s(q)} \begin{bmatrix} \theta_5 g \cos(q_1 + q_2) & -\frac{\theta_2 + \theta_3 \cos q_2}{\theta_1 + \theta_2 + 2\theta_3 \cos q_2} \\ -\theta_4 g \cos q_1 - \theta_5 g \cos(q_1 + q_2) & 1 \end{bmatrix}, \tag{22}$$

$$s(q) := \det \Phi_1 = \frac{\det \Phi_2}{d_{11}(q)} = g d_{11}^{-1}(q) \times \tag{23}$$

$$\big((\theta_1 + \theta_3 \cos q_2)\theta_5 \cos(q_1 + q_2) - (\theta_2 + \theta_3 \cos q_2)\theta_4 \cos q_1\big).$$

Moreover, the coordinate change (15,16) is locally invertible at each point where

$$s(q) \neq 0. \tag{24}$$

Indeed, $D(q) > 0$ and the above $\alpha(q, \dot{q})$, $\beta(q, \dot{q})$ from (14) are given as

$$\alpha(q, \dot{q}) = \frac{\det \Phi_2}{\det D(q)}, \quad \begin{bmatrix} 0 \\ \beta(q, \dot{q}) \end{bmatrix} = \Phi_3(q, \dot{q})\dot{q}, \tag{25}$$

where Φ_2 is given by (21). As a matter of fact, by virtue of [2] and the references therein, the coordinate change (16) is globally invertible on any open set where (24) holds and which is both connected and simply connected. In other words, the acrobot model is state and feedback equivalent on any such set to the system (14). Figure 2 depicts some of these sets. Moreover, for possible walking application, the following lemma is useful.

Lemma 1: The relation (24) holds if the Acrobot center of mass is strictly above the surface and

$$\begin{gathered} (m_1 + m_2)l_1^2 + I_1 > m_2 l_1 l_2, \ m_2 l_2^2 + I_2 > m_2 l_1 l_2, \\ q_1 \in (-\pi/2, \pi/2), \ q_1 + q_2 \in (3\pi/2, \pi/2). \end{gathered}$$

Proof Instead of performing tedious computations, let us give the following mechanics motivated proof. First, notice that (24) means that the matrix (21) is regular. Secondly, one can easily see that the first assumption of Lemma 1 is equivalent to $\theta_1 > \theta_3, \theta_2 > \theta_3$, cf. (7), therefore the entries of the first row of the matrix (21) are always strictly positive. At the same time, the first entry of the second row of the matrix (21) is the overall Acrobot potential energy with respect to the ground surface while the second entry of the second row is the potential energy of the second link only with respect to the actuated joint position. Notice, that $q_1 \in (-\pi/2, \pi/2)$ means that the first link points upward while $q_1 + q_2 \in (3\pi/2, \pi/2)$ means that the second link points downward. Therefore, the first entry of the second row of the matrix (21) is positive, while its second entry is negative. Taking into the account, as shown above, that the entries of the first row of the matrix (21) are strictly positive, one concludes that both rows of that matrix are always linearly independent, i.e. the matrix (21) is regular and therefore $s(q) \neq 0$. □

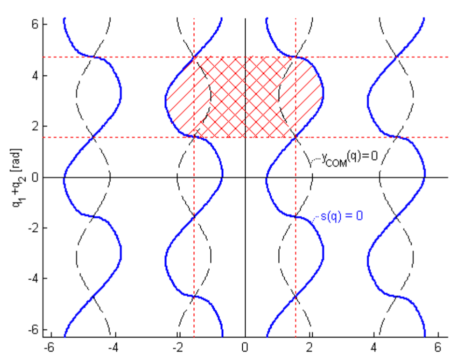

Fig. 2. Singularities and possible regular set of the coordinate change (16).

Remark 1: Notice, that the second condition of Lemma 1 $q_1 \in (-\pi/2, \pi/2)$, $q_1 + q_2 \in (3\pi/2, \pi/2)$ is quite natural in case of possible future walking-like movement of the Acrobot as its violation means that either the stance leg is bellow or fully lying on walking surface, or the swing leg points horizontally or upward. Apparently, such configurations are not likely during walking, or even have to be avoided for other practical reasons as well. Moreover, the first condition of Lemma 1 obviously holds for almost any reasonable combination of lengths and masses, e.g. it clearly holds for $l_1 = l_2$ and by continuity arguments for sufficiently small $|l_1 - l_2|$ as well. The last feature may be used to shorten slightly the swing leg during a step not to hit the ground. Again, walking robots having very different lengths of their legs links are obviously unrealistic also for many other practical reasons.

The situation is nicely demonstrated on Fig. 2, where full lines show singularities $s(q) = 0$ while dashed lines denote the configurations with center of mass lying on the walking surface. Proof of Lemma 1 applies within the crosshatched area and any reasonable walking takes place even deep inside this crosshatched area. □

In the sequel we will therefore concentrate ourselves to study the system (14). This system is almost linear, but there is a nonlinearity $d_{11}(q_2)^{-1}$ in the first row that depends on q_2 only. Instead of expressing this nonlinearity in coordinates $\xi_{1,2,3,4}$ and trying to study its exact influence one can use some favorable qualitative properties. Namely, one can easily see that

$$a_{min} \le d_{11}(q_2)^{-1} \le a_{max} \tag{26}$$

$$a_{min} := \frac{1}{m_2(l_1+l_2)^2 + m_1 l_1^2 + I_1 + I_2} \tag{27}$$

$$a_{max} := \frac{1}{m_2(l_1-l_2)^2 + m_1 l_1^2 + I_1 + I_2}. \tag{28}$$

Notice, that

$$a_{max} - a_{min} = \frac{4 l_1 l_2 m_2 (m_2(l_1+l_2)^2 + m_1 l_1^2 + I_1 + I_2)^{-1}}{(m_2(l_1-l_2)^2 + m_1 l_1^2 + I_1 + I_2)} \tag{29}$$

being quite small number and therefore the nonlinearity $d_{11}(q_2)^{-1}$ is actually varying in a quite narrow range. Therefore, its derivative also evolves in favorable way, namely

$$\frac{\partial [d_{11}(q_2)^{-1}]}{\partial q_2} = (2\theta_3 \sin q_2) d_{11}(q_2)^{-2}. \tag{30}$$

$$\left|\frac{\partial [d_{11}^{-1}]}{\partial q_2}\right| \le 2\theta_3 a_{max}^2. \tag{31}$$

The above favorable properties of the Acrobot partial linearization will be used in the sequel for the feedback design ensuring the exponentially tracking of a given walking like trajectory. Assume that an open loop control generating a suitable reference trajectory is given in partial exact linearized coordinates (14). In other words, our task is to track the following reference system

$$\dot{\xi}_1^{ref} = d_{11}^{-1}(q_2^{ref})\xi_2^{ref},\ \dot{\xi}_2^{ref} = \xi_3^{ref},$$
$$\dot{\xi}_3^{ref} = \xi_4^{ref},\ \dot{\xi}_4^{ref} = w^{ref}, \tag{32}$$

The following theorem gives a constructive way how asymptotically track the reference system (32).

Theorem 1: Consider the system (14) with the by the following feedback

$$\begin{aligned} w &= w^{ref} + \\ &\Theta^3 K_1 e_1 + \Theta^3 K_2 e_2 + \Theta^2 K_3 e_3 + \Theta K_4 e_4, \\ e &=: \xi - \xi^{ref}. \end{aligned} \tag{33}$$

Further, let $K_1 < 0$ and $K_{2,3,4}$ are such that the polynomial $\lambda^3 + K_4\lambda^2 + K_3\lambda + K_2$ is Hurwitz. Then there exist $\Theta > 0, \mathcal{R} > 0, \mathcal{B} > 0$ such that for all reference trajectories given by (32) and satisfying

$$\forall t \ge 0 \quad |s(\phi_2(\xi^{ref})(t))| \ge \mathcal{B} > 0, \tag{34}$$

$$|\xi_2^{ref}(t)| \le \mathcal{R},\ \forall t \ge 0, \tag{35}$$

where ϕ_2 is given by (17,18)) and $s(q)$ by (23), it holds locally exponentially for e given by (33) that

$$e(t) \to 0,\ t \to \infty.$$

Proof. Subtracting (32) from (14) with 33 one obtains

$$\dot{e}_1 = d_{11}^{-1}(\phi_2(\xi_1,\xi_3))\xi_2 - d_{11}^{-1}(\phi_2(\xi_1^{ref},\xi_3^{ref}))\xi_2^{ref},$$

$$\dot{e}_2 = e_3, \dot{e}_3 = e_4, \dot{e}_4 = \Theta^3 K_1 e_1 + \Theta^3 K_2 e_2 + \Theta^2 K_3 e_3 + \Theta K_4,$$

where ϕ_2 is given by (17,18). Straightforward computations based on the Taylor expansions give

$$\dot{e}_1 = \mu_2(t)e_2 + \mu_1(t)e_1 + \mu_3(t)e_3 + o(e) \tag{36}$$
$$\dot{e}_2 = e_3 \tag{37}$$
$$\dot{e}_3 = e_4 \tag{38}$$
$$\dot{e}_4 = \Theta^3 K_1 e_1 + \Theta^3 K_2 e_2 + \Theta^2 K_3 e_3 + \Theta K_4, \tag{39}$$

$$\mu_1(t) = \xi_2^{ref}(t)\frac{\partial [d_{11}^{-1}]}{\partial q_2}\frac{\partial \phi_2}{\partial \xi_1}(q_2^{ref}(t)), \tag{40}$$

$$\mu_2(t) = d_{11}^{-1}(q_2^{ref}(t)), \tag{41}$$

$$\mu_3(t) = \xi_2^{ref}(t)\frac{\partial [d_{11}^{-1}]}{\partial q_2}\frac{\partial \phi_2}{\partial \xi_3}(q_2^{ref}(t)), \tag{42}$$

$$q_2^{ref}(t) = \phi_2(\xi_1^{ref}(t), \xi_3^{ref}(t)), \quad q_2 \in [0, 2\pi). \tag{43}$$

Using the relation (22), all the estimates (26-31) and fixing some $\mathcal{R}, \mathcal{B} > 0$ in (34,35) one has that

$$|\mu_1(t)| \le 2\theta_3 a_{max}^2 (\theta_4 + \theta_5)\frac{\mathcal{R}}{\mathcal{B}} \tag{44}$$

$$|\mu_3(t)| \le 2\theta_3 a_{max}^2 \frac{\mathcal{R}}{\mathcal{B}}, \quad 0 < a_{min} \le \mu_2(t) \le a_{max}. \tag{45}$$

Further, denote

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ K_2 & K_3 & K_4 \end{bmatrix}, \quad A^\top S + SA = -I, \tag{46}$$

where a symmetric positive definite matrix S exists as A is obviously Hurwitz by the assumptions of the theorem being proved. Consider the following Lyapunov's function candidate

$$V(\xi) = \frac{1}{2}(\overline{e}_1^2) + [\overline{e}_2, \overline{e}_3, \overline{e}_4] S [\overline{e}_2, \overline{e}_3, \overline{e}_4]^\top$$

where

$$\overline{e}_1 = e_1,\ \overline{e}_2 = e_2 + \frac{K_1}{K_2}e_1,\ \overline{e}_3 = \Theta^{-1}e_3,\ \overline{e}_4 = \Theta^{-2}e_4.$$

Notice, that the system (36-39) can be written as

$$\dot{\overline{e}}_1 = \mu_2(t)\overline{e}_2 - (\mu_1(t) + \mu_2(t)\frac{K_1}{K_2})\overline{e}_1 + \Theta\mu_3(t)\overline{e}_3 + o(\overline{e}),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\overline{e}_2\\ \overline{e}_3\\ \overline{e}_4\end{bmatrix} = \Theta A\begin{bmatrix}\overline{e}_2\\ \overline{e}_3\\ \overline{e}_4\end{bmatrix} + \begin{bmatrix}\frac{K_1\mu_2(t)}{K_2}\left(\overline{e}_2 - \frac{K_1}{K_2}\overline{e}_1\right)\\ 0\\ 0\end{bmatrix}.$$

With all the above notations, the full derivation of the Lyapunov's function candidate can be expressed as follows

$$\dot{V} = \Theta\left(\mu_3\overline{e}_1\overline{e}_3 - \overline{e}_2^2 - \overline{e}_3^2 - \overline{e}_4^2\right) - \frac{K_1\mu_2 + K_2\mu_1}{K_2}\overline{e}_1^2 + o(\overline{e}^2)$$

$$+\mu_2\overline{e}_1\overline{e}_2 + 2[\overline{e}_2, \overline{e}_3, \overline{e}_4]S\begin{bmatrix}\frac{K_1\mu_2}{K_2}\left(\overline{e}_2 - \frac{K_1}{K_2}\overline{e}_1\right)\\ 0\\ 0\end{bmatrix}.$$

Recall, that in the formulation of the theorem being proved it is claimed that "there exist $\Theta, \mathcal{R}, \mathcal{B} > 0$". Therefore, one can in the sequel put always

$$\frac{\mathcal{B}}{\mathcal{R}}\Theta = 2\theta_3 a_{max}^2(\theta_4 + \theta_5), \qquad (47)$$

to obtain after some straightforward computations that

$$\dot{V} \leq o(\overline{e}^2) - \Theta\left(\overline{e}_2^2 + \overline{e}_3^2 + \overline{e}_4^2\right) - \frac{K_1 a_{min} + K_2\frac{\theta_4+\theta_5}{\Theta}}{K_2}\overline{e}_1^2$$

$$\overline{e}_1(\overline{e}_3 + a_{max}\overline{e}_2) + 2[\overline{e}_2, \overline{e}_3, \overline{e}_4]S\begin{bmatrix}\frac{K_1 a_{max}}{K_2}\left(\overline{e}_2 - \frac{K_1}{K_2}\overline{e}_1\right)\\ 0\\ 0\end{bmatrix}.$$

Now, the higher order terms $o(\overline{e}^2)$ may be ignored as they do not affect the **local exponential stability** which is actually is investigated. Furthermore, notice that by (46), matrix S is uniquely given by selection of gains $K_{2,3,4}$. Therefore, the cross terms of the above expression for $\dot{V}$ are independent of the design parameter Θ and they depend only of the fixed pre-selected gains $K_{1,2,3,4}$ and a given systems physical parameters. Furthemore, as the parameter Θ multiplies the negative quadrates of $\overline{\xi}_2, \overline{\xi}_3, \overline{\xi}_4$, by enhancing parameter Θ one can make the full derivative of Lyapunov's function candidate less than a strictly negative quadratic form. The last property, together with quadratic character of the Lyapunov's function guarantees global exponential stability of the corresponding closed loop system. □

Remark 2: One can easily see that using the constructive character of the above proof it is possible in a straightforward though tedious manner, to find reasonable estimates for the parameter Θ giving the exponentially tracking feedback (33). Then, based on (47) one can obtain relation between estimates and $\mathcal{R}, \mathcal{B}$. Notice, that first of them characterize the speed during the reference step while the second one closeness to singular point of the linearizing transformations. In such a way, the set of all reference walking trajectories where our approach is viable may be clearly described. Currently, the methodology for target walking trajectory design guaranteeing safe validity of (34) and (35) is developed. Its full description is out of the scope of this short paper, its basic ideas are indicated in the simulations section. Preliminary results indicate that one can always design the step "slow enough" and therefore based on theorem just proved guarantee its local exponential tracking. It is just worth notice at this point, that this reference trajectory design is also using favourable properties of the system in those very same σ-linearized coordinates (14). □

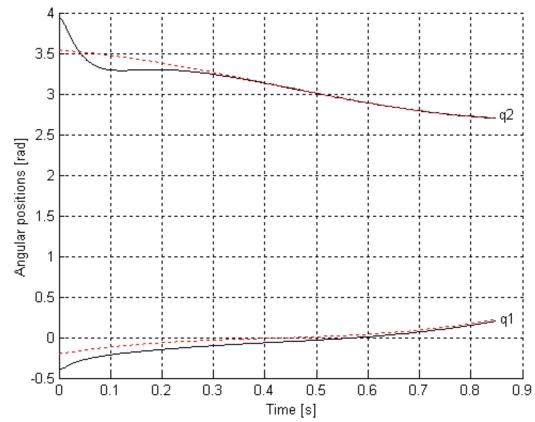

Fig. 3. Angular positions q_1, q_2 and references (dotted line)

IV. SIMULATIONS

Theorem 1 and subsequent remarks are illustrated by the simulation experiments using parameters $m_i = 1$, $l_{1,2} = 1$ and $I_i = 0.1$, see (7). First, the walking target trajectory generated by the reference model (32) having $\dot{\xi}_4^{ref} = 0$ and some sufficient initial conditions of the acrobot is obtained. Such a trajectory will be called in the sequel a the *pseudo-passive* one. The pseudo-passive above trajectory is passive in the linearized coordinates, but after recomputing their virtual input variable $w = 0$ into the original torques $\tau_2, \tau_3, \ldots$, a certain static state feedback maintaining constant velocity of the Acrobot center of mass is obtained. With

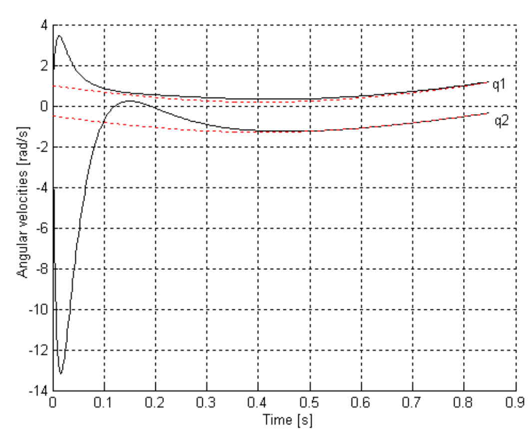

Fig. 4. Angular velocities $\dot{q}_1, \dot{q}_2$ and references (dotted line)

such a choice, the resulting trajectory depends merely on the choice of initial angular velocities. These velocities can be selected to ensure that at the given time the tip of the swing leg reaches exactly the walking surface at the given distance from the stance leg support point. After having designed the target walking trajectory in the above way, Theorem 1 can be used to achieve its exponential tracking from fairly different initial positions of the Acrobot. Choosing shorter and slower steps one can always comply with the appropriate assumptions of Theorem 1. These conclusions are supported by numerous simulations experiments, some of them are collected in figures bellow to illustrate our approach.

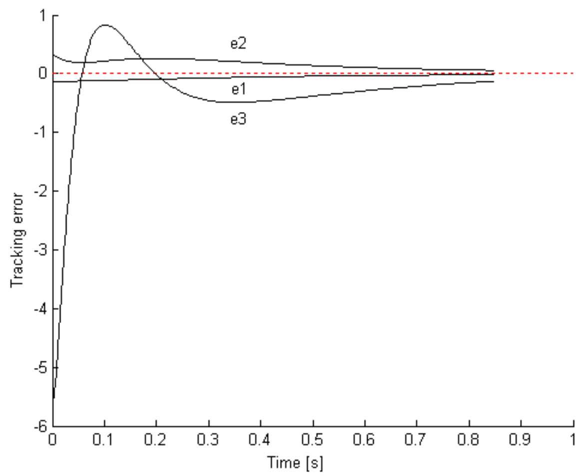

Fig. 5. Tracking errors e_1, e_2, e_3.

Fig. 3, 4 present the tracking of the pseudo-passive target trajectory generated by $(q_1(0), q_2(0)) = (-0.4, \pi - 2q_1(0))$ $(q_1^{ref}(0), q_2^{ref}(0)) = (-0.2, \pi - 2q_1^{ref}(0))$ and $(\dot{q}_1, \dot{q}_2) = (1, -0.5)$. Notice that the initial positions error may be 100%, or even more, if unlimited actuators torques are allowed. The gains $(K_1, K_2, K_3, K_4) = (-10, 6, 12, 8)$ and factor $\Theta = 20$ are selected as the coefficients of proposed control law fulfilling Theorem 1.

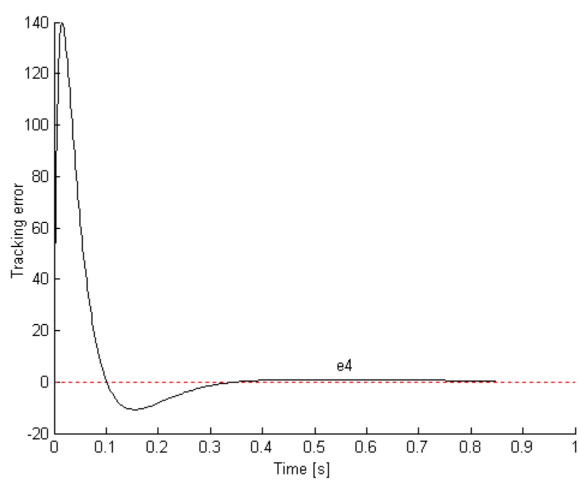

Fig. 6. Tracking error e_4

V. Conclusions

The main advantage of our new approach lies in using special coordinates both suitable for tracking feedback design and for future target walking trajectory design. Moreover, extension to more general walking like underactuated systems is possible by simple combining our approach with a very standard treatment for fully actuated systems. Summarizing, the acrobot comprises typical underactuated walking features and this helps to open new strategies for more complicated configurations being the subject of the ongoing research.

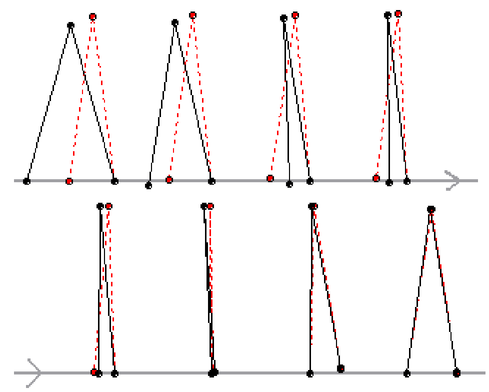

Fig. 7. The animation of the single step shown in time moments with gaps $\Delta t = 0.12s$ between them. Dotted line is the reference, the full one represents the actual Acrobot movement.

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