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Piecewise-Affine Parameter-Varying Control of Wheeled Mobile Robots

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***Abstract*— This paper defines a new class of systems and presents a novel controller synthesis method. This new methodology is motivated by and applied to the problem of path following of a wheeled mobile robot (WMR). The new class of systems proposed in this paper is called piecewise-affine parameter-varying (PWAPV), which is a combination of piecewise-affine (PWA) and linear parameter-varying (LPV) systems. The synthesis of PWAPV controllers for uncertain PWAPV systems can be cast as a parameterized set of matrix inequalities, which can be approximated by a finite set of LMIs and solved efficiently using available software. As an application, actuator input voltage laws are designed to guarantee that a WMR follows a desired path that is parameterized by a time-varying curvature. Simulation results show the effectiveness of the new control law.**

I. INTRODUCTION

The problem of path following for autonomous vehicles is very important and has received a great deal of attention in the past ten years. The importance of path following is made evident by the vast amount of work carried out in the area of path parameterization for the motion control of unicycle-type land robots [19], [23], marine vehicles [8], [11], and aerial vehicles (UAVs) [1], [22].

Initial research focused primarily on path following and trajectory tracking for nonholonomic vehicles using only kinematic models. A good survey of the work done up until 1995 was conducted by Kolmanovsky and McClamroch [16]. An increasing amount of research is now examining the combination of kinematic and dynamic models. One of the first publications to use backstepping to include dynamics is the work by Fierro and Lewis [13]. Since then, different control methods have been examined, including adaptive backstepping [5], [7], [14], discontinuous backstepping [24], dynamic feedback linearization [20], approximate feedback linearization [15], and sliding mode control [6]. More recently, research has focused on the robustness of controllers to unmodeled dynamics and parameter uncertainty [9], [10], [23]. However, little research has been conducted in order to include the dynamics of the actuators in the controller design process [2], [4], [17].

The new path following control method proposed in this paper consists of a three-step procedure. In the first step, a kinematic steering control law is designed assuming the path curvature is time-varying (for the case where the path curvature is constant see [18]). Two curvature limits and a curvature rate of change limit are first defined for the desired path and then the nonlinear WMR parameterized path kinematics described in [23] are approximated by an uncertain PWAPV system, while assuming that the WMR forward velocity is constant. Then, a PWAPV steering control law is designed using a parameter-dependent quadratic Lyapunov function. In the second step, a backstepping-type approach is used to include the vehicle dynamics and design the wheel control torques that guarantee convergence of the WMR forward and rotational velocities to the desired values. Finally, in the third step, the actuator dynamics are included and the input voltages are designed using backstepping.

There are four primary advantages to the new path following controller synthesis method proposed here. First, the PWAPV controller synthesis method can be formulated as a convex optimization program subject to a parameterized set of inequalities, which can be approximated by a finite set of LMIs and solved efficiently using available software. Second, it includes both the general, non-singular path parameterization proposed in [23] and the actuator dynamics. Third, the PWAPV control law can also stabilize the original nonlinear parameter-dependent system. And fourth, it is a first step toward including hard nonlinearities in the actuator dynamics, which are important PWA characteristics.

The outline of the paper is as follows. Section II defines the class of PWAPV systems introduced in this paper. Then, the new PWAPV controller synthesis method for uncertain PWAPV slab systems is derived in Section III. Section IV develops an application to path following control. Then, section V shows a numerical example, followed by the conclusions in Section VI.

II. PWAPV APPROXIMATION OF NONLINEAR SYSTEMS

A. Class of Systems

Consider a nonlinear system of the form

$$\dot{\mathbf{x}}(t) = A(\rho)\mathbf{x}(t) + a(\rho) + f(\mathbf{x}) + B(\rho)\mathbf{u}(t), \qquad (1)$$

where $\rho = \rho(t)$ is the time-varying parameter, $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, and $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is the input vector. Matrices $A(\rho) \in \mathbb{R}^{n\times n}$, $a(\rho) \in \mathbb{R}^n$ and $B(\rho) \in \mathbb{R}^{n\times n_u}$ are affine in the parameter ρ, while the vector $f(\mathbf{x}) \in \mathbb{R}^n$ is nonlinear in the state vector $\mathbf{x}(t)$.

This paper introduces a class of uncertain PWAPV systems to approximate the nonlinear system (1). The dynamics of uncertain PWAPV systems are described by

$$\begin{aligned}\dot{\mathbf{x}}(t) = &\left[A_i(\rho) + \Delta A(\mathbf{x})\right]\mathbf{x}(t)\\ &+ \left[a_i(\rho) + \Delta a(\mathbf{x})\right] + B_i(\rho)\mathbf{u}(t),\end{aligned} \qquad (2)$$

for $\mathbf{x}(t) \in \mathcal{R}_i$, where matrices $A_i(\rho) \in \mathbb{R}^{n\times n}$, $a_i(\rho) \in \mathbb{R}^n$ and $B_i(\rho) \in \mathbb{R}^{n\times n_u}$ are affine in the time-varying parameter ρ and represent the nominal PWAPV system, while matrices $\Delta A(\mathbf{x}) \in \mathbb{R}^{n\times n}$ and $\Delta a(\mathbf{x}) \in \mathbb{R}^n$ are the uncertainty terms. The polytopic regions, $\mathcal{R}_i$, $i \in \mathcal{I} = \{1, \ldots, M\}$, partition a subset of the state space $\mathcal{X} \subset \mathbb{R}^n$ such that $\cup_{i=1}^M \overline{\mathcal{R}}_i = \mathcal{X}$, $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$, $i \neq j$, where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. It is assumed that the desired closed-loop equilibrium point $\mathbf{x}^{cl}$ is the origin. The region in which $\mathbf{x}^{cl}$ lies is denoted as $\mathcal{R}_{i^\star}$. A slab is a special case of a polyhedron, and is defined as follows.

Definition 1: A slab is defined as

$$\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n \mid h_1 < H^T\mathbf{x} < h_2\}, \tag{3}$$

where $H \in \mathbb{R}^n$ and $h_1, h_2 \in \mathbb{R}$. □

Definition 2: A PWAPV slab system is a PWAPV system for which the regions are slabs. □

For PWAPV slab systems, each region $\mathcal{R}_i$ can be equivalently described by a degenerate ellipsoid $\mathcal{E}_i$, such that $\mathcal{R}_i \subseteq \mathcal{E}_i$ and $\mathcal{E}_i \subseteq \mathcal{R}_i$, where

$$\mathcal{E}_i = \{\mathbf{x} \in \mathbb{R}^n \mid \|L_i\mathbf{x} + l_i\| < 1\}. \tag{4}$$

This covering is described by

$$\begin{cases} L_i = 2H^T/(h_2 - h_1) \\ l_i = -(h_2 + h_1)/(h_2 - h_1) \end{cases}. \tag{5}$$

Finally, the following *a priori* assumptions, adapted from previous work in the robust piecewise-linear (PWL) control literature [12], are made for the uncertainty terms:

$$\begin{cases} \Delta A^T(\mathbf{x})\Delta A(\mathbf{x}) < U_{A_i}^T U_{A_i} \\ \Delta a(\mathbf{x})\Delta a^T(\mathbf{x}) < U_{a_i} U_{a_i}^T \end{cases}, \tag{6}$$

for $\mathbf{x}(t) \in \mathcal{R}_i$. As shown in the proof of the following theorem, the nonlinear parameter-dependent system (1) is equivalent to an uncertain PWAPV system of the form (2) under certain conditions.

Theorem 1: The nonlinear parameter-dependent system (1) is equivalent to the uncertain PWAPV system (2) with

$$\begin{cases} A_i(\rho) = A(\rho) + \tilde{A}_i \\ a_i(\rho) = a(\rho) + \tilde{a}_i \\ B_i(\rho) = B(\rho) \end{cases} \tag{7}$$

if

$$f(\mathbf{x}) - \tilde{A}_i\mathbf{x}(t) - \tilde{a}_i = \Delta A(\mathbf{x})\mathbf{x}(t) + \Delta a(\mathbf{x}), \tag{8}$$

for $\mathbf{x}(t) \in \mathcal{R}_i$. □

Proof: System (1) can be rewritten as

$$\begin{aligned} \dot{\mathbf{x}}(t) = A(\rho)\mathbf{x}(t) + a(\rho) + f(\mathbf{x}) + B(\rho)\mathbf{u}(t) \\ +[\tilde{A}_i\mathbf{x}(t) + \tilde{a}_i] - [\tilde{A}_i\mathbf{x}(t) + \tilde{a}_i], \end{aligned} \tag{9}$$

for $\mathbf{x}(t) \in \mathcal{R}_i$. Using the constraint (8), this last equation becomes

$$\begin{aligned} \dot{\mathbf{x}}(t) = A(\rho)\mathbf{x}(t) + a(\rho) + \tilde{A}_i\mathbf{x}(t) + \tilde{a}_i \\ +\Delta A(\mathbf{x})\mathbf{x}(t) + \Delta a(\mathbf{x}) + B(\rho)\mathbf{u}(t), \end{aligned} \tag{10}$$

for $\mathbf{x}(t) \in \mathcal{R}_i$. This can be rewritten as

$$\begin{aligned} \dot{\mathbf{x}}(t) = [A_i(\rho) + \Delta A(\mathbf{x})]\mathbf{x}(t) \\ +[a_i(\rho) + \Delta a(\mathbf{x})] + B_i(\rho)\mathbf{u}(t), \end{aligned} \tag{11}$$

for $\mathbf{x}(t) \in \mathcal{R}_i$, where

$$\begin{cases} A_i(\rho) = A(\rho) + \tilde{A}_i \\ a_i(\rho) = a(\rho) + \tilde{a}_i \\ B_i(\rho) = B(\rho) \end{cases}.$$

This concludes the proof. □

In order to obtain the uncertain PWAPV system (2), the following steps must be carried out:

1) The nonlinear system (1) is approximated by a nominal PWAPV system.
2) The uncertainty bounds (6) are determined such that the original nonlinear system is contained in the uncertain PWAPV system.

The next two subsections address these steps.

B. PWAPV Approximation

The first step in approximating the nonlinear system (1) by the uncertain PWAPV system (2) is to obtain a nominal PWAPV system of the form

$$\dot{\mathbf{x}}(t) = A_i(\rho)\mathbf{x}(t) + a_i(\rho) + B_i(\rho)\mathbf{u}(t), \tag{12}$$

for $\mathbf{x}(t) \in \mathcal{R}_i$, using (7). Matrices $A(\rho)$, $a(\rho)$ and $B(\rho)$ in (7) are obtained from (1), and matrices $\tilde{A}_i$ and $\tilde{a}_i$ come from the PWA approximation of the nonlinear function

$$f(\mathbf{x}) \approx \tilde{A}_i\mathbf{x}(t) + \tilde{a}_i, \tag{13}$$

for $\mathbf{x}(t) \in \mathcal{R}_i$.

The PWA approximation (13) of the nonlinear function $f(\mathbf{x})$ in (1) can be obtained by solving the following convex optimization problem [21]:

Problem 1: Given a partition of the state space $\mathcal{X}$, the sampling points $\mathbf{x}_k$, $k = 1, \ldots, N_s$, and matrices $\tilde{A}_L$, $\tilde{a}_L$:

$$\begin{aligned} \min \quad & \textstyle\sum_{k=1}^{N_s} \mathbf{e}^T(\mathbf{x}_k)\mathbf{e}(\mathbf{x}_k) \\ s.t. \quad & \mathbf{e}(\mathbf{x}_k) = f(\mathbf{x}_k) - \tilde{A}_i\mathbf{x}_k - \tilde{a}_i, \\ & (\tilde{A}_i - \tilde{A}_j)\mathbf{x}_{k^\star} + (\tilde{a}_i - \tilde{a}_j) = 0, \\ & \tilde{A}_{i^\star} = \tilde{A}_L,\ \tilde{a}_{i^\star} = \tilde{a}_L, \\ & i = 1, \ldots, M,\ j \in \mathcal{N}_i, \\ & k = 1, \ldots, N_s, \end{aligned}$$

where M is the number of state space partitions, $\mathbf{x}_k$ are sampling points, $\mathbf{x}_{k^\star}$ are the sampling points corresponding to the boundary between two neighbouring regions, and the linearization of the nonlinear function $f(\mathbf{x})$ in (1) at $\mathbf{x}^{cl}$ is given by $\tilde{A}_L\mathbf{x}(t) + \tilde{a}_L$. □

A numerical method for determining the PWA uncertainty bounds (6) is proposed in the next subsection.

C. PWA Uncertainty Bounds

The error function $\mathbf{e}(\mathbf{x}_k)$ defined in Problem 1, at the sampling points $\mathbf{x}_k$ is

$$\Delta A(\mathbf{x}_k)\mathbf{x}_k + \Delta a(\mathbf{x}_k) = \mathbf{e}(\mathbf{x}_k) = f(\mathbf{x}_k) - \tilde{A}_i\mathbf{x}_k - \tilde{a}_i, \quad (14)$$

for $i = 1, \dots, M$, $k = 1, \dots, N_s$, where matrices $\Delta A(\mathbf{x}_k)$ and $\Delta a(\mathbf{x}_k)$ are the uncertainty terms in (2) and are assumed to be polynomial functions in $\mathbf{x}_k$ of order n_p. Moreover, defining $W_{A_i} = U_{A_i}^T U_{A_i}$ and $W_{a_i} = U_{a_i} U_{a_i}^T$, the PWA constraints (6) can be written as the LMIs

$$\begin{bmatrix} W_{A_i} & \Delta A^T(\mathbf{x}_k) \\ \Delta A(\mathbf{x}_k) & I_{(n)} \end{bmatrix} > 0 \quad (15)$$

and

$$\begin{bmatrix} W_{a_i} & \Delta a(\mathbf{x}_k) \\ \Delta a^T(\mathbf{x}_k) & 1 \end{bmatrix} > 0, \quad (16)$$

for $\mathbf{x}_k \in \mathcal{R}_i$, $k = 1, \dots, N_s$.

The PWA uncertainty bounds (6), W_{A_i}, W_{a_i} can be obtained by solving the following convex optimization problem.

Problem 2: Given the sampling points $\mathbf{x}_k$, $k = 1, \dots, N_s$, the scalar n_p, and the nominal PWAPV system (12):

$$\begin{array}{ll} \min & \sum_{i=1}^{M} trace\left[W_{A_i} + W_{a_i}\right] \\ s.t. & (14),\ (15),\ (16) \\ & i = 1, \dots, M, \\ & k = 1, \dots, N_s, \end{array}$$

where $\Delta A(\mathbf{x}_k)$ and $\Delta a(\mathbf{x}_k)$ are assumed to be polynomial functions in $\mathbf{x}_k$ of order n_p. □

Remark 1: Note that the PWA uncertainty bounds (6) resulting from the solution of Problem 2 are dependent on the sampling points $\mathbf{x}_k$. Therefore, it cannot be guaranteed in general that the uncertainties satisfy the bounds (6) for $\forall \mathbf{x}(t) \in \mathbb{R}^n$. However, sampling methods can be used to offer this guarantee for certain cases. □

III. PWAPV Controller Synthesis

This section states and solves the PWAPV controller synthesis problem for the class of uncertain PWAPV slab systems defined in the previous section. The synthesis of a PWAPV controller will be used in the first step of the path following method proposed in the next section.

A. Design Objectives

The objective is to design a feedback control law that globally exponentially stabilizes the uncertain PWAPV system (2) to the origin. The PWAPV state feedback control law proposed here is of the form

$$\mathbf{u}(t) = K_i(\rho)\mathbf{x}(t) + k_i(\rho), \quad (17)$$

for $\mathbf{x}(t) \in \mathcal{R}_i$. Substituting (17) into (2) yields the closed-loop uncertain PWAPV system

$$\dot{\mathbf{x}}(t) = \bar{A}_i^{cl}(\rho)\mathbf{x}(t) + \bar{a}_i^{cl}(\rho), \quad (18)$$

for $\mathbf{x}(t) \in \mathcal{R}_i$, where

$$\begin{cases} \bar{A}_i^{cl}(\rho) = \{[A_i(\rho) + \Delta A(\mathbf{x})] + B_i(\rho)K_i(\rho)\} \\ \bar{a}_i^{cl}(\rho) = \{[a_i(\rho) + \Delta a(\mathbf{x})] + B_i(\rho)k_i(\rho)\} \end{cases} . \quad (19)$$

Some background material must be reviewed before the PWAPV controller synthesis problem is stated.

B. Mathematical Preliminaries

This subsection presents two lemmas and one theorem to be used in the ensuing development.

Lemma 1: [3] Let X and Y be real constant matrices of compatible dimensions, then the following equation

$$X^T Y + Y^T X \le \epsilon X^T X + \epsilon^{-1} Y^T Y$$

holds for any $\epsilon > 0$. □

Lemma 2: If $M \ge 0$, then

$$BMB^T \le \mathrm{trace}(M) BB^T$$

for any matrix B with appropriate dimensions. □

Proof: It suffices to show that $\mathrm{trace}(M)I - M \ge 0$. This is true because the eigenvalues $\lambda_i(\beta I - M)$ are equal to $\beta - \lambda_i(M)$ for any β. Therefore, since the trace is the sum of all eigenvalues and since $M \ge 0$, the eigenvalues of $[\mathrm{trace}(M)I - M]$ are all greater than or equal to zero, which finishes the proof. □

In the following theorem, ρ_{min} and ρ_{max} are limits on ρ, and $\dot{\rho}_{max}$ is a limit on the magnitude of $\dot{\rho}$.

Theorem 2: Consider the closed-loop system (18), and let $\mathbf{x}^{cl} = 0$ be an equilibrium point of this system. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

$$V(\mathbf{x}, \rho) = \mathbf{x}^T P(\rho)\mathbf{x} \quad (20)$$

with $P(\rho) = P^T(\rho) > 0$, and

$$\begin{aligned} \dot{V}(\mathbf{x}, \rho, \dot{\rho}) &= \dot{\mathbf{x}}^T P(\rho)\mathbf{x} + \mathbf{x}^T \dot{P}(\rho)\mathbf{x} + \mathbf{x}^T P(\rho)\dot{\mathbf{x}} \\ &= \dot{\mathbf{x}}^T P(\rho)\mathbf{x} + \mathbf{x}^T \dot{\rho}\tfrac{dP}{d\rho}\mathbf{x} + \mathbf{x}^T P(\rho)\dot{\mathbf{x}} \\ &< -\alpha V(\mathbf{x}, \rho) \end{aligned} \quad (21)$$

for $\forall \rho \in [\rho_{min}, \rho_{max}]$, $|\dot{\rho}| \le \dot{\rho}_{max}$ and $\forall t \ge 0$, where $\alpha > 0$. Then the system (18) is globally exponentially stable to $\mathbf{x}^{cl}$ and the function $V(\mathbf{x}, \rho)$ is called a parameter-dependent quadratic Lyapunov function, where $\alpha > 0$ is an upper bound on the decay rate of the magnitude of the state vector $\mathbf{x}(t)$.

Proof: The proof is based on the LPV arguments presented in the work of Wu [25]. It is omitted here because of space constraints. □

C. PWAPV Controller Synthesis

The synthesis of PWAPV controllers for uncertain PWAPV slab systems is now presented as a theorem.

Theorem 3: Consider the closed-loop PWAPV system (18). Let $I_{(n)}$ be an identity matrix of dimension n, where n is the number of state variables. If, given $\alpha > 0$, $k_{lim} > 0$, $\mu_i < 0$, $i = 1, \dots, M$, $\epsilon_j > 0$, $j = 1, \dots, 6$ and $\lambda_{max} > 0$, for $\forall \rho \in [\rho_{min}, \rho_{max}]$ and $|\dot{\rho}| \le \dot{\rho}_{max}$, it is true that

$$\begin{bmatrix} \Omega_i & I_{(n)} & I_{(n)} & Q(\rho)U_{A_i}^T \\ I_{(n)} & -\dot{\rho}_{max}^{-1}|\lambda_{max}|^{-1}I_{(n)} & 0 & 0 \\ I_{(n)} & 0 & -\epsilon_1 I_{(n)} & 0 \\ U_{A_i}Q(\rho) & 0 & 0 & -\epsilon_1^{-1}I_{(n)} \end{bmatrix} < 0, \tag{22}$$

$$\begin{bmatrix} \bar{\Omega}_i & I_{(n)} & I_{(n)} & Q(\rho)U_{A_i}^T & Q(\rho)L_i^T \\ I_{(n)} & -\dot{\rho}_{max}^{-1}|\lambda_{max}|^{-1}I_{(n)} & 0 & 0 & 0 \\ I_{(n)} & 0 & -\epsilon_1 I_{(n)} & 0 & 0 \\ U_{A_i}Q(\rho) & 0 & 0 & -\epsilon_1^{-1}I_{(n)} & 0 \\ L_iQ(\rho) & 0 & 0 & 0 & -\Gamma_i^{-1} \end{bmatrix} < 0, \tag{23}$$

$Q(\rho) = Q^T(\rho) > 0$, $\frac{dQ}{d\rho} - \lambda_{max}I_{(n)} \leq 0$, inequality (22) is verified for $i = i^\star$, and inequality (23) is verified for $i \neq i^\star$ with $1 - l_i^2 < 0$, where

$$\Omega_i = A_i(\rho)Q(\rho) + B_i(\rho)Y_i(\rho) + Q(\rho)A_i^T(\rho) + Y_i^T(\rho)B_i^T(\rho) + \alpha Q(\rho)$$

for $i = i^\star$ and

$$\begin{aligned} \bar{\Omega}_i = {} & A_i(\rho)Q(\rho) + B_i(\rho)Y_i(\rho) + Q(\rho)A_i^T(\rho) \\ & + Y_i^T(\rho)B_i^T(\rho) + \alpha Q(\rho) \\ & + \mu_i[1 + l_i^2(1-l_i^2)^{-1}]* \\ & \quad [(1+\epsilon_2^{-1})a_i(\rho)a_i^T(\rho) + (1+\epsilon_2)U_{a_i}U_{a_i}^T] \\ & + \mu_i[1 + l_i^2(1-l_i^2)^{-1}]* \\ & \quad [a_i(\rho)k_i^T(\rho)B_i^T(\rho) + B_i(\rho)k_i(\rho)a_i^T(\rho)] \\ & + [\mu_i\epsilon_3 k_{lim}^2 + \mu_i\epsilon_4 k_{lim}^2 + \epsilon_5]U_{a_i}U_{a_i}^T \\ & + l_i(1-l_i^2)^{-1}[a_i(\rho)L_iQ(\rho) + Q(\rho)L_i^T a_i^T(\rho)] \\ & + [\mu_i\epsilon_3^{-1} + \mu_i\epsilon_4^{-1}l_i^4(1-l_i^2)^{-2} + \epsilon_6 \\ & \quad + \mu_i k_{lim}^2\{1 + l_i^2(1-l_i^2)^{-1}\}]B_i(\rho)B_i^T(\rho) \end{aligned}$$

with $\Gamma_i = [\epsilon_5^{-1}l_i^2(1-l_i^2)^{-2} + \epsilon_6^{-1}l_i^2(1-l_i^2)^{-2}k_{lim}^2 + \mu_i^{-1}(1-l_i^2)^{-1}]$ for $i \neq i^\star$ and if

$$\begin{bmatrix} k_{lim}^2 & k_i(\rho) \\ k_i^T(\rho) & 1 \end{bmatrix} > 0, \ i = 1, \ldots, M, \tag{24}$$

then the closed-loop uncertain PWAPV system (18) is globally exponentially stable to the origin. □

Proof: The proof is omitted for lack of space. It is based on Lemmas 1 and 2 and on Theorem 2.

D. Numerical Solution

To approximate the inequalities in Theorem 3 by a finite set of LMIs, it will be assumed that $Q(\rho)$, $Y_i(\rho)$ and $k_i(\rho)$ are affinely dependent on the time-varying parameter $\rho(t)$, that is,

$$\begin{cases} Q(\rho) = \overline{Q0} \cdot \rho(t) + \overline{Q1} \\ Y_i(\rho) = \overline{Y0}_i \cdot \rho(t) + \overline{Y1}_i \\ k_i(\rho) = \overline{k0}_i \cdot \rho(t) + \overline{k1}_i \end{cases} . \tag{25}$$

Based on these assumptions, the PWAPV controller synthesis problem can be formulated as follows.

Problem 3: Given scalars $\alpha > 0$, $\epsilon_j > 0$, $j = 1, \ldots, 6$, $k_{lim} > 0$, $\mu_i < 0$, ρ_{min}, ρ_{max}, $\dot{\rho}_{max}$, $\lambda_{max} > 0$, vectors Y_i^{lim}, and a set of grid points ρ_k, $k = 1, \ldots, N_\rho$:

$$\begin{aligned} \text{find} \quad & \overline{Q0},\ \overline{Q1},\ \overline{Y0}_i,\ \overline{Y1}_i,\ \overline{k0}_i,\ \overline{k1}_i \\ s.t. \quad & Q(\rho_k) = \overline{Q0} \cdot \rho_k + \overline{Q1}, \\ & Q(\rho_k) = Q^T(\rho_k) > 0, \\ & \overline{Q0} - \lambda_{max}I_{(n)} \leq 0, \\ & Y_i(\rho_k) = \overline{Y0}_i \cdot \rho_k + \overline{Y1}_i, \\ & -Y_i^{lim} \prec Y_i(\rho_k) \prec Y_i^{lim}, \\ & k_i(\rho_k) = \overline{k0}_i \cdot \rho_k + \overline{k1}_i, \\ & \begin{bmatrix} k_{lim}^2 & k_i(\rho_k) \\ k_i(\rho_k)^T & 1 \end{bmatrix} > 0, \\ & i = 1, \ldots, M,\ k = 1, \ldots, N_\rho, \end{aligned}$$

and inequality (22) evaluated at $\rho = \rho_k$ for $i = i^\star$ and inequality (23) evaluated at $\rho = \rho_k$ for $i \neq i^\star$ with $1 - l_i^2 < 0$. The symbols $\succ$ and $\prec$ denote component-wise inequalities. □

The gains for the PWAPV control law (17) are then

$$\begin{cases} K_i(\rho) = Y_i(\rho)Q^{-1}(\rho) \\ k_i(\rho) = \overline{k0}_i \cdot \rho(t) + \overline{k1}_i \end{cases} \tag{26}$$

and the Lyapunov function (20) is parameterized by $P(\rho) = Q^{-1}(\rho)$, where $Y_i(\rho)$ and $Q(\rho)$ are given in (25).

IV. Path Following Controller Synthesis

This section proposes an application of the methodology of this paper to the path following problem for a WMR.

A. Kinematic Controller Synthesis

We begin by considering only the WMR parameterized path kinematics taken from [23] with constant velocity $u = u_{des}$, rewritten as

$$\begin{bmatrix} \dot{\theta} \\ \dot{s}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c_c(t)\dot{s} \\ 0 & -c_c(t)\dot{s} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ s_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} -c_c(t)\dot{s} \\ -\dot{s} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ u_{des}\cos\theta \\ u_{des}\sin\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r(t), \tag{27}$$

which is a nonlinear parameter-dependent system of the form (1), where the state vector is $\mathbf{x}_1(t) = [\theta,\ s_1,\ y_1]^T$, the input vector is $\mathbf{u}_1(t) = r(t)$, and $c_c = c_c(t)$ is the time-varying parameter $\rho(t)$. The desired closed-loop equilibrium point is $\mathbf{x}_1^{cl} = [0,\ 0,\ 0]^T$.

Partitioning the state variable θ into M regions and using the methods described in Section II, the nonlinear parameter-dependent system (27) is first approximated by an uncertain PWAPV system of the form (2). A PWAPV control law of the form

$$r(t) = K_i(c_c)\mathbf{x}_1(t) + k_i(c_c), \tag{28}$$

for $\mathbf{x}_1(t) \in \mathcal{R}_i$, can then be designed such that the closed-loop uncertain PWAPV system (18) satisfies the Lyapunov conditions (20) and (21) for $\forall c_c \in [c_{c_{min}}, c_{c_{max}}]$, $|\dot{c}_c| \leq \dot{c}_{c_{max}}$, and $\forall t \geq 0$, where $\alpha > 0$. This PWAPV control law can be obtained by solving Problem 3 yielding

$$\begin{cases} K_i(c_c) = \underline{Y_i(c_c)Q_1^{-1}(c_c)} \\ k_i(c_c) = \overline{k0}_i \cdot c_c(t) + \overline{k1}_i \end{cases} \tag{29}$$

and the Lyapunov function (20) is parameterized by $P_1(c_c) = Q_1^{-1}(c_c)$.

In the proof of the following theorem, it will be shown that the PWAPV steering control law (28), designed for the uncertain PWAPV system, can also globally exponentially stabilize the nonlinear system (27) to $\theta = 0$, $s_1 = 0$, and $y_1 = 0$.

Theorem 4: The PWAPV control law (28) resulting from the solution of Problem 3 globally exponentially stabilizes the nonlinear parameter-dependent system (27) to the origin if the bounds (6) are such that the difference between system (27) and the nominal PWAPV system (12) satisfies these bounds $\forall \mathbf{x}_1(t) \in \mathbb{R}^n$. □

Proof: It was shown in Section III that the PWAPV control law (28) resulting from the solution of Problem 3 globally exponentially stabilizes the uncertain PWAPV system (2) for all uncertainties verifying the bounds (6). Using the numerical methods of Section II, if uncertainty bounds (6) are obtained such that the difference between system (27) and the nominal PWAPV system (12) satisfies these bounds for $\forall \mathbf{x}_1(t) \in \mathbb{R}^n$, it can be concluded from Theorem 1 that the closed-loop nonlinear parameter-dependent system

$$\begin{aligned} \dot{\mathbf{x}}_1(t) = [A(c_c) + B(c_c)K_i(c_c)]\mathbf{x}_1(t) \\ +[a(c_c) + B(c_c)k_i(c_c)] + f(\mathbf{x}_1) \end{aligned} \tag{30}$$

satisfies

$$\begin{aligned} \dot{V}_1(\mathbf{x}_1, c_c, \dot{c}_c) \quad &= \dot{\mathbf{x}}_1^T P_1(c_c)\mathbf{x}_1 + \mathbf{x}_1^T \dot{c}_c \tfrac{dP_1}{dc_c}\mathbf{x}_1 + \mathbf{x}_1^T P_1(c_c)\dot{\mathbf{x}}_1 \\ &< -\alpha V_1(\mathbf{x}_1, c_c), \end{aligned} \tag{31}$$

for $\forall c_c \in [c_{c_{min}}, c_{c_{max}}]$ and $|\dot{c}_c| \leq \dot{c}_{c_{max}}$. □

B. Dynamic Controller Synthesis

We now use an integrator backstepping-type approach to include the WMR dynamics. Consider the WMR kinematics

$$\begin{bmatrix} \dot{\theta} \\ \dot{s}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} -c_c(t)\dot{s} \\ c_c(t)\dot{s}y_1 - \dot{s} \\ -c_c(t)\dot{s}s_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r \tag{32}$$

or

$$\dot{\mathbf{x}}_1 = g(\mathbf{x}_1, c_c) + g_u(\mathbf{x}_1)u + g_r(\mathbf{x}_1)r. \tag{33}$$

Combining the kinematics with the WMR dynamics yields

$$\begin{cases} \dot{\mathbf{x}}_1 = g(\mathbf{x}_1, c_c) + g_u(\mathbf{x}_1)u + g_r(\mathbf{x}_1)r \\ \dot{u} = \frac{1}{Mr_w}(T_R + T_L) \\ \dot{r} = \frac{c}{Ir_w}(T_R - T_L) \end{cases}. \tag{34}$$

We now consider r as a *virtual control*, and call the PWAPV steering control law (28) a *stabilizing function* and denote it by r_{des}. The new control inputs are the torques

$$\begin{cases} T_R = \frac{Mr_w}{2}\left\{\dot{u}_{des} - \frac{\alpha}{2}\xi_u - \left(\frac{\partial V_1}{\partial \mathbf{x}_1}\right)^T g_u(\mathbf{x}_1)\right\} \\ \qquad + \frac{Ir_w}{2c}\left\{\dot{r}_{des} - \frac{\alpha}{2}\xi_r - \left(\frac{\partial V_1}{\partial \mathbf{x}_1}\right)^T g_r(\mathbf{x}_1)\right\} \\ T_L = \frac{Mr_w}{2}\left\{\dot{u}_{des} - \frac{\alpha}{2}\xi_u - \left(\frac{\partial V_1}{\partial \mathbf{x}_1}\right)^T g_u(\mathbf{x}_1)\right\} \\ \qquad - \frac{Ir_w}{2c}\left\{\dot{r}_{des} - \frac{\alpha}{2}\xi_r - \left(\frac{\partial V_1}{\partial \mathbf{x}_1}\right)^T g_r(\mathbf{x}_1)\right\} \end{cases}, \tag{35}$$

for $\mathbf{x}_1(t) \in \mathcal{R}_i$.

Theorem 5: Consider the system (34). Let there exist a constant forward velocity u_{des} and a stabilizing function r_{des} given by (28), as well as a parameter-dependent quadratic Lyapunov function $V_1(\mathbf{x}_1, c_c)$ that satisfies (20) and (21). Then the wheel control torques T_R and T_L given by (35) render the system (34) globally exponentially stable to $u = u_{des}$, $r = r_{des}$, $\theta = 0$, $s_1 = 0$, and $y_1 = 0$ for $\forall c_c \in [c_{c_{min}}, c_{c_{max}}]$ and $|\dot{c}_c| \leq \dot{c}_{c_{max}}$. □

Proof: It follows from [18]. □

C. Actuator Controller Synthesis

Consider the WMR kinematics and dynamics rewritten as

$$\begin{bmatrix} \dot{u} \\ \dot{r} \\ \dot{\theta} \\ \dot{s}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r - c_c(t)\dot{s} \\ u\cos\theta + c_c(t)\dot{s}y_1 - \dot{s} \\ u\sin\theta - c_c(t)\dot{s}s_1 \end{bmatrix} + \begin{bmatrix} \frac{1}{Mr_w} \\ \frac{c}{Ir_w} \\ 0 \\ 0 \\ 0 \end{bmatrix} T_R + \begin{bmatrix} \frac{1}{Mr_w} \\ \frac{-c}{Ir_w} \\ 0 \\ 0 \\ 0 \end{bmatrix} T_L \tag{36}$$

or as

$$\dot{\mathbf{x}}_2 = g(\mathbf{x}_2, c_c) + g_{T_R}(\mathbf{x}_2)T_R + g_{T_L}(\mathbf{x}_2)T_L. \tag{37}$$

Combining the WMR kinematics and dynamics with the actuator dynamics results in

$$\begin{cases} \dot{\mathbf{x}}_2 = g(\mathbf{x}_2, c_c) + g_{T_R}(\mathbf{x}_2)T_R + g_{T_L}(\mathbf{x}_2)T_L \\ \dot{T}_R = -\frac{K_m R_a}{L_a} i_R - \frac{K_m K_b}{L_a}\dot{\phi}_R + \frac{K_m}{L_a}V_R \\ \dot{T}_L = -\frac{K_m R_a}{L_a} i_L - \frac{K_m K_b}{L_a}\dot{\phi}_L + \frac{K_m}{L_a}V_L \end{cases}. \tag{38}$$

We now consider T_R and T_L as *virtual controls*, and call the control torque laws (35) *stabilizing functions* denoted by $T_{R_{des}}$ and $T_{L_{des}}$. The new control inputs to the augmented system (38) are the actuator voltages V_R and V_L. Using a similar reasoning as before, it can be shown that the control laws

$$\begin{cases} V_R = \frac{L_a}{K_m}\dot{T}_{R_{des}} - \frac{\alpha}{2}\frac{L_a}{K_m}\xi_{T_R} \\ \qquad - \frac{L_a}{K_m}\left(\frac{\partial V_2}{\partial \mathbf{x}_2}\right)^T g_{T_R}(\mathbf{x}_2) + K_b\dot{\phi}_R + R_a i_R \\ V_L = \frac{L_a}{K_m}\dot{T}_{L_{des}} - \frac{\alpha}{2}\frac{L_a}{K_m}\xi_{T_L} \\ \qquad - \frac{L_a}{K_m}\left(\frac{\partial V_2}{\partial \mathbf{x}_2}\right)^T g_{T_L}(\mathbf{x}_2) + K_b\dot{\phi}_L + R_a i_L \end{cases}, \tag{39}$$

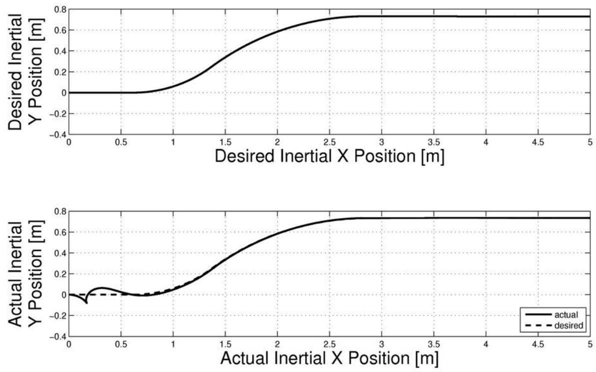

Fig. 1. Plot of desired and actual trajectories

for $\mathbf{x}_1(t) \in \mathcal{R}_i$, globally exponentially stabilize the system (38) to $T_R = T_{R_{des}}$, $T_L = T_{L_{des}}$, $u = u_{des}$, $r = r_{des}$, $\theta = 0$, $s_1 = 0$, and $y_1 = 0$ for $\forall c_c \in [c_{c_{min}}, c_{c_{max}}]$ and $|\dot{c}_c| \leq \dot{c}_{c_{max}}$, where $\left(\frac{\partial V_2}{\partial \mathbf{x}_2}\right)^T = 2P_2(c_c)(\mathbf{x}_2 - \mathbf{x}_2^{cl})$ with

$$P_2(c_c) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & P_1(c_c) \end{bmatrix}. \quad (40)$$

V. Simulation Results

For the synthesis of the PWAPV steering control law (28), Problem 3 is solved with $\alpha = 0.01$, $\mu_i = -100$, $k_{lim} = 1$, $c_{c_{min}} = -1.0\ m^{-1}$, $c_{c_{max}} = +1.0\ m^{-1}$, $\dot{c}_{c_{max}} = +2.0\ m^{-1}s^{-1}$, $Y_i^{lim} = [20,\ 20,\ 20]^T$, $\lambda_{max} = 10^{-6}$, a set of $N_{c_c} = 20$ grid points over the interval $c_c = [-1.0,\ +1.0]$,

$$\epsilon_1 = 7.2 \times 10^{+6}, \quad \epsilon_2 = 5.0 \times 10^{+1}, \quad \epsilon_3 = 1.0 \times 10^{+2},$$
$$\epsilon_4 = 5.0 \times 10^{+2}, \quad \epsilon_5 = 5.0 \times 10^{+1}, \quad \epsilon_6 = 5.0 \times 10^{-1}.$$

Simulations were performed with all initial conditions equal to zero, except for the heading error, which was set to $\theta = \pi\ rad$. The desired and actual trajectories are shown in Figure 1, where it can be seen that the actual trajectory converges to the desired one.

VI. Conclusions

This paper presented a PWAPV controller synthesis method that can be formulated as a convex optimization program subject to a parameterized set of inequalities. These inequalities were approximated by a finite set of LMIs and then solved efficiently using available software. The method was applied to a non-singular path parameterization model for a wheeled mobile robot including the actuator dynamics.

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