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Reduction-based Control with Application to Three-Dimensional Bipedal Walking Robots

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***Abstract*— This paper develops the concept of reduction-based control, which is founded on a controlled form of geometric reduction known as *functional Routhian reduction*. We introduce a geometric property of general serial-chain robots termed *recursive cyclicity*, leading to our presentation of the *Subrobot Theorem*. This shows that reduction-based control can arbitrarily reduce the dimensionality of any serial-chain robot, so that it may be controlled as a simpler "subrobot" while separately controlling the divided coordinates through their conserved momenta. This method is applied to construct stable directional 3-D walking gaits for a 4-d.o.f. hipped bipedal robot. The walker's sagittal-plane subsystem can be decoupled from its yaw and lean modes, and on this planar subsystem we use passivity-based control to construct limit cycles on flat ground. Due to the controlled reduction, the unstable yaw and lean modes are separately controlled to 2-periodic orbits. We numerically verify the existence of stable 2-periodic limit cycles and demonstrate turning capabilities for the controlled biped.**

I. Introduction

The implications of understanding bipedal locomotion are great due to its human application. The potential for improving prosthetic limbs, navigating uneven terrestrial surfaces, and creating efficient locomotive mechanisms are among the many incentives that drive research in this field of robotics. Most theoretical results in dynamic bipedal walking pertain to planar robot models, where concepts such as passive dynamics [4], hybrid zero dynamics [8], and passivity-based control [10]-[12] have been quite successful. However, there has been limited success in extending these ideas to the 3-D case due to the complexity of biped dynamics (although progress has been made with hybrid zero dynamics [3]).

In [1] and [2] it was observed that a bipedal robot's gait dynamics are dominated by its sagittal plane of motion. This suggested that reduction be used to isolate the sagittal subsystem, where traditional forms of passivity-based control can ensure forward walking, and then separately control motion in the lateral and axial planes. In particular, geometric reduction requires that a physical system, typically modeled by a Lagrangian, have certain symmetries that are invariant under the action of a Lie group on the configuration space. A few such forms of reduction are discussed in [7], such as Lie-Poisson, Euler-Poincaré, and Routh. In *Routhian reduction*, a Lagrangian L has configuration space $Q = \mathbb{G} \times S$, where $\mathbb{G}$ is the *symmetry group* (usually a torus) and $S \cong Q\backslash\mathbb{G}$ is the *shape space*. Then, symmetries of L are characterized by *cyclic* variables $q_i \in \mathbb{G}$, i.e., variables where

$$\frac{\partial L}{\partial q_i} = 0, \quad i \in \{1, \dim(\mathbb{G})\}. \tag{1}$$

We divide out the symmetry group $\mathbb{G}$ to project the full-order phase space TQ onto the reduced-order space TS. By requiring that the dynamics evolve on level-sets of conserved momentum maps, these symmetries allow us to directly relate the behavior of the full-order system and the reduced system.

In this case, reduced system stability corresponds to stability in phase space TQ modulo $T\mathbb{G}$. This says nothing about the stability of the divided coordinates (which in the context of bipedal walking will be the unstable yaw and lean modes), so we use energy-shaping control to break the symmetry of group $\mathbb{G}$ in order to stabilize orbits of $T\mathbb{G}$. However, we must impose symmetry-breaking in a specific manner so as to preserve the projection map, i.e., we still must be able to divide the group $\mathbb{G}$ of "almost-cyclic" variables.

Accordingly, a controlled variant of Routhian reduction is introduced in [1] and [2]: *functional Routhian reduction*. This imposes controlled reduction of the lean dynamics from 3-d.o.f. bipeds, but without modeling yaw this does not result in completely 3-D walking. Since we wish additionally to reduce and control the yaw dynamics from a 4-d.o.f. biped, we present a generalized multistage form of this controlled reduction method. We then propose a geometric property of general serial-chain robots termed *recursive cyclicity*, leading to our introduction of the *Subrobot Theorem* to show that multistage reduction can be applied to any serial-chain robot.

We exploit these results by designing a reduction-based control law for a 4-d.o.f. bipedal walker with a hip and splayed legs to impose a 2-stage controlled reduction to the sagittal-plane subsystem, on which we use passivity-based control to build robust limit cycles on flat ground. As a result of the controlled reduction, we control lean to vertical and yaw to the desired heading angle. We show that this control method results in stable directional limit cycles on flat ground, which are 2-periodic due to the side-to-side lateral swaying and axial turning motions induced by the robot's hip. The authors are unaware of any other results in dynamic walking that allow for directional control – the closest would be the quasi-static Honda Asimo biped.

II. Controlled Reduction

In this section, we introduce the k-stage variant of functional Routhian reduction for a n-d.o.f. robot, $1 \leq k < n$. This provides for recursive reduction of a dynamical system, obtained from a special Lagrangian, to the lower-dimensional

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$$M(q_2^n) = M_{q_1^n}(q_2^n) = \begin{pmatrix} m_{q_1}(q_2^n) & M_{q_1,q_2^n}(q_2^n) \\ M^T_{q_1,q_2^n}(q_2^n) & M_{q_2^n}(q_3^n) \end{pmatrix} = \begin{pmatrix} m_{q_1}(q_2^n) & — & M_{q_1,q_2^n}(q_2^n) & —— \\ | & \ddots & & \vdots \\ M^T_{q_1,q_2^n}(q_2^n) & & m_{q_k}(q_{k+1}^n) & M_{q_k,q_{k+1}^n}(q_{k+1}^n) \\ | & \cdots & M^T_{q_k,q_{k+1}^n}(q_{k+1}^n) & M_{q_{k+1}^n}(q_{k+2}^n) \end{pmatrix} \tag{2}$$

$$\begin{aligned} L_{\lambda_j^k}(q_j^n, \dot q_j^n) &= \frac{1}{2}\dot q_j^{n^T} M_{\lambda_j^n}(q_{j+1}^n)\dot q_j^n - W_{\lambda_j^k}(q_j^n, \dot q_{j+1}^n) - V_{\lambda_j^k}(q_j^n) && \text{(3a)} \\ &= L_{\lambda_{j+1}^k}(q_{j+1}^n, \dot q_{j+1}^n) + \frac{1}{2} m_{q_j}(q_{j+1}^n)(\dot q_j)^2 + \dot q_j M_{q_j,q_{j+1}^n}(q_{j+1}^n)\dot q_{j+1}^n && \text{(3b)} \\ &\quad + \frac{1}{2}\dot q_{j+1}^{n^T} \frac{M^T_{q_j,q_{j+1}^n}(q_{j+1}^n) M_{q_j,q_{j+1}^n}(q_{j+1}^n)}{m_{q_j}(q_{j+1}^n)} \dot q_{j+1}^n - \frac{\lambda_j(q_j)}{m_{q_j}(q_{j+1}^n)} M_{q_j,q_{j+1}^n}(q_{j+1}^n)\dot q_{j+1}^n + \frac{1}{2}\frac{\lambda_j(q_j)^2}{m_{q_j}(q_{j+1}^n)} \end{aligned}$$

$$M_{\lambda_j^k}(q_{j+1}^n) = M_{q_j^n}(q_{j+1}^n) + \sum_{i=j}^{k} \begin{pmatrix} 0_{i\times i} & 0_{i\times(n-i)} \\ 0_{(n-i)\times i} & \frac{M^T_{q_i,q_{i+1}^n}(q_{i+1}^n) M_{q_i,q_{i+1}^n}(q_{i+1}^n)}{m_{q_i}(q_{i+1}^n)} \end{pmatrix} \tag{4}$$

$$W_{\lambda_j^k}(q_j^n, \dot q_{j+1}^n) = \sum_{i=j}^{k} \frac{\lambda_i(q_i)}{m_{q_i}(q_{i+1}^n)} M_{q_i,q_{i+1}^n}(q_{i+1}^n)\dot q_{i+1}^n, \qquad V_{\lambda_j^k}(q_j^n) = V_{\mathrm{fct}}(q_{k+1}^n) - \frac{1}{2}\sum_{i=j}^{k}\frac{\lambda_i(q_i)^2}{m_{q_i}(q_{i+1}^n)} \tag{5}$$

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system of a "subrobot," while separately controlling the divided variables. We begin by describing a robot's typical Lagrangian dynamics.

Lagrangian Dynamics. A mechanical system with configuration space Q is described by elements $(q, \dot q)$ of the tangent bundle TQ (space of configuration and velocities) and the Lagrangian function $L : TQ \to \mathbb{R}$, given in coordinates by

$$L(q, \dot q) = K(q, \dot q) - V(q) = \frac{1}{2}\dot q^T M(q)\dot q - V(q),$$

where $K(q, \dot q)$ is the robot's kinetic energy, $V(q)$ is the robot's potential energy, and $M(q)$ is the $n \times n$ symmetric, positive-definite inertia matrix. Since the Lagrangian satisfies the n-dimensional controlled Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = Bu,$$

the dynamics for the controlled robot are given by

$$M(q)\ddot q + C(q, \dot q)\dot q + N(q) = Bu, \tag{6}$$

where $C(q, \dot q)$ is the $n \times n$ Coriolis matrix, $N(q) = \frac{\partial}{\partial q}V(q)$ is the vector of potential (e.g., gravitational) torques, $n \times n$-matrix B is assumed invertible (for full actuation), and control input u is an n-dimensional vector of joint torques.

These equations yield the dynamical control system (f, g):

$$\begin{pmatrix} \dot q \\ \ddot q \end{pmatrix} = f(q, \dot q) + g(q)u, \tag{7}$$

with vector field

$$f(q, \dot q) = \begin{pmatrix} \dot q \\ M(q)^{-1}\left(-C(q, \dot q)\dot q - N(q)\right) \end{pmatrix},$$

and matrix of control vector fields

$$g(q) = \begin{pmatrix} 0_{n\times n} \\ M(q)^{-1}B \end{pmatrix}.$$

We now describe a special class of Lagrangians that have k "almost-cyclic" variables, which can be controlled through reduction, as opposed to uncontrollable cyclic variables.

k-Almost-Cyclic Lagrangians. Consider the general case when the configuration space $Q = \mathbb{T}^k \times S$, where shape space $S \cong Q \backslash \mathbb{T}^k$ is constructed by copies of $\mathbb{R}$ and circle $\mathbb{S}^1$, and $\mathbb{T}^k = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ is the group to be divided one copy of $\mathbb{S}^1$ at a time. We denote an element $q \in Q$ by $q = (q_1^{k^T}, q_{k+1}^{n^T})^T$, with k-dimensional vector $q_1^k \in \mathbb{T}^k$ and $(n-k)$-dimensional vector $q_{k+1}^n \in S$. To begin, we suppose there exists an inertia matrix $M(q_2^n)$ of (2) that is recursively cyclic.

Definition 1: An $n \times n$-matrix M is *recursively cyclic* if each lower-right $(n-i) \times (n-i)$ submatrix is cyclic in $q_1, \ldots, q_{i+1}$ for $i \in \{0, n-1\}$, i.e., it has the form of (2) with the base case $M_{q_n^n}(q_{n+1}^n) = m_{q_n}(q_{n+1}^n) \in \mathbb{R}$ and $q_{n+1}^n = \emptyset$.

In order to control k divided variables, each stage of reduction must project from an "almost-cyclic" Lagrangian to a "functional Routhian," which is the Lagrangian of a lower-dimensional system that is also almost-cyclic for the next stage of reduction (except the final stage to be discussed later). In other words, each subrobot's parent almost-cyclic Lagrangian must contain a nested almost-cyclic Lagrangian (except again for the base case). Therefore, we are interested in a generalized form of almost-cyclic Lagrangians from [2]. We say that a Lagrangian $L_{\lambda_1^k} : T\mathbb{T}^k \times TS \to \mathbb{R}$ is *k-almost-cyclic* if, in coordinates, it has the form of (3) with $j = 1$, for some functions $\lambda_i : \mathbb{S}^1 \to \mathbb{R}$, $i \in \{1, k\}$. The closed-form definition of (3a) explicitly shows all the shaping terms necessary for k stages of controlled reduction, whereas the last three terms in the recursive definition of (3b) impose the controlled reduction to stage-1 functional Routhian $L_{\lambda_2^k}$, which is the target $(k-1)$-almost-cyclic Lagrangian of the first reduction. Here, for all q_{i+1}^n, $M_{q_{i+1}^n}(q_{i+1}^n) \in \mathbb{R}^{(n-i)\times(n-i)}$ and $m_{q_i}(q_{i+1}^n) \in \mathbb{R}$ are symmetric and positive-definite, which follows from the symmetric and positive-definite $M(q_2^n)$.

Given a k-almost-cyclic Lagrangian $L_{\lambda_1^k}$, the n-dimensional fully-actuated Euler-Lagrange equations are

$$M_{\lambda_1^k}(q_2^n)\ddot q + C_{\lambda_1^k}(q, \dot q)\dot q + N_{\lambda_1^k}(q) = Bv, \tag{8}$$

where $C_{\lambda_1^k}$ is the k-almost-cyclic Coriolis matrix, $N_{\lambda_1^k} = \frac{\partial}{\partial q} V_{\lambda_1^k}$ is the vector of k-almost-cyclic potential torques, B is an invertible matrix, and v is the n-dimensional control input vector. Then, we have the control system on TQ associated with $L_{\lambda_1^k}$, $(f_{\lambda_1^k}, g_{\lambda_1^k})$, defined as usual:

$$\begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = f_{\lambda_1^k}(q, \dot{q}) + g_{\lambda_1^k}(q) v. \tag{9}$$

Letting v_{k+1}^n be the subsystem control law on TS, we incorporate this into the full-order k-almost-cyclic system by defining the new control system $(\hat{f}_{\lambda_1^k}, \hat{g}_{\lambda_1^k})$ with input v_1^k:

$$\begin{aligned} \hat{f}_{\lambda_1^k}(q, \dot{q}) &:= f_{\lambda_1^k}(q, \dot{q}) + g_{\lambda_1^k}(q) \begin{pmatrix} 0_{k \times (n-k)} \\ I_{(n-k)\times(n-k)} \end{pmatrix} v_{k+1}^n \\ \hat{g}_{\lambda_1^k}(q) &:= g_{\lambda_1^k}(q) \begin{pmatrix} I_{k \times k} \\ 0_{(n-k) \times k} \end{pmatrix}, \end{aligned} \tag{10}$$

where v_1^k is the k-dimensional vector containing the first k elements of input v, and v_{k+1}^n is the $(n-k)$-dimensional vector containing elements $k+1, \ldots, n$ of vector v. Moreover, vector field $\hat{f}_{\lambda_1^k}$ corresponds to the v_{k+1}^n-controlled Euler-Lagrange equations (absent of control v_1^k), which will be relevant to the reduction theorem to be discussed later.

Momentum Maps. Through each stage of reduction, the system conserves a quantity corresponding to the divided degree-of-freedom, which does not explicitly appear in the lower-dimensional system but can be uniquely reconstructed. Therefore, fundamental to multistage reduction are the momentum maps $J_j : T(Q \backslash \mathbb{T}^{j-1}) \to \mathbb{R}$, $j \in \{1, k\}$, which make explicit each conserved quantity in the system:

$$\begin{aligned} J_j(q_j^n, \dot{q}_j^n) &= \frac{\partial}{\partial \dot{q}_j} L_{\lambda_j^k}(q_j^n, \dot{q}_j^n) \\ &= M_{q_j, q_{j+1}^n}(q_{j+1}^n)\dot{q}_{j+1}^n + m_{q_j}(q_{j+1}^n)\dot{q}_j. \end{aligned} \tag{11}$$

Here, $L_{\lambda_j^k}$ of (3) is the k-almost-cyclic Lagrangian for $j = 1$ or the stage-$(j-1)$ functional Routhian, to be defined next, for $j \in \{2, k\}$. Since we wish to control each cyclic variable through the corresponding momentum, the energy shaping terms in $L_{\lambda_j^k}$ break each conservative map J_j (typically constant) and force it equal to a desirable function $\lambda_j(q_j)$.

Functional Routhians. A *Routhian* is the Lagrangian of a reduced system, i.e., it characterizes the reduction's projection from the full-order system to the reduced-order system. In the framework of multistage functional Routh reduction, k stages of reduction yield k functional Routhians. For $j \in \{2, k\}$, the functional Routhian corresponding to the $(j-1)^{\text{th}}$ stage of reduction is a $(k-j+1)$-almost-cyclic Lagrangian on the tangent bundle of reduced configuration space $Q \backslash \mathbb{T}^{j-1}$. In other words, the first $k-1$ functional Routhians are generalized almost-cyclic Lagrangians that allow for further stages of controlled reduction. Therefore, given a k-almost-cyclic Lagrangian $L_{\lambda_1^k}$, the *stage-*$(j-1)$ *functional Routhian* $L_{\lambda_j^k} : T(Q \backslash \mathbb{T}^{j-1}) \to \mathbb{R}$ is obtained through a partial Legendre transformation in variable q_{j-1}:

$$L_{\lambda_j^k}(q_j^n, \dot{q}_j^n) := L_{\lambda_{j-1}^k}(q_{j-1}^n, \dot{q}_{j-1}^n) - \lambda_{j-1}(q_{j-1})\dot{q}_{j-1},$$

constrained to the momentum map

$$J_{j-1}(q_{j-1}^n, \dot{q}_{j-1}^n) = \lambda_{j-1}(q_{j-1}).$$

It follows that for $j \in \{2, k\}$, $L_{\lambda_j^k}$ has the form of (3).

The final stage of reduction, stage-k, is a functional Routhian $L_{\text{fct}} = L_{\lambda_{k+1}^k}$ of the form presented in [1] and [2]. This is obtained from the 1-almost-cyclic Routhian $L_{\lambda_k^k}$ of stage-$(k-1)$ through a partial Legendre transformation in the variable q_k with the constraint $J_k(q_k^n, \dot{q}_k^n) = \lambda_k(q_k)$. It follows that the corresponding stage-k functional Routhian $L_{\text{fct}} : TS \cong T(Q \backslash \mathbb{T}^k) \to \mathbb{R}$, which is the Lagrangian of the k-reduced subsystem, is given in coordinates by

$$\begin{aligned} &L_{\text{fct}}(q_{k+1}^n, \dot{q}_{k+1}^n) = \\ &\quad \tfrac{1}{2}\dot{q}_{k+1}^{n\,T} M_{q_{k+1}^n}(q_{k+1}^n)\dot{q}_{k+1}^n - V_{\text{fct}}(q_{k+1}^n). \end{aligned} \tag{12}$$

Therefore, the $(n-k)$-dimensional controlled Euler-Lagrange equations of L_{fct} are

$$\begin{aligned} &M_{q_{k+1}^n}(q_{k+1}^n)\ddot{q}_{k+1}^n \\ &\quad + C_{q_{k+1}^n}(q_{k+1}^n, \dot{q}_{k+1}^n)\dot{q}_{k+1}^n + N_{q_{k+1}^n}(q_{k+1}^n) = B_{q_{k+1}^n} v_{k+1}^n, \end{aligned} \tag{13}$$

where $C_{q_{k+1}^n}$ and $N_{q_{k+1}^n}$ are defined as usual, and $B_{q_{k+1}^n}$ is the invertible $(n-k) \times (n-k)$ lower-right submatrix of B corresponding to vector q_{k+1}^n. Then, we have the control system on TS associated with L_{fct}, $(f_{\text{fct}}, g_{\text{fct}})$:

$$\begin{pmatrix} \dot{q}_{k+1}^n \\ \ddot{q}_{k+1}^n \end{pmatrix} = f_{\text{fct}}(q_{k+1}^n, \dot{q}_{k+1}^n) + g_{\text{fct}}(q_{k+1}^n) v_{k+1}^n. \tag{14}$$

From this, we can define the vector field corresponding to the k-reduced, controlled Euler-Lagrange equations:

$$\hat{f}_{\text{fct}}(q_{k+1}^n, \dot{q}_{k+1}^n) := f_{\text{fct}}(q_{k+1}^n, \dot{q}_{k+1}^n) + g_{\text{fct}}(q_{k+1}^n) v_{k+1}^n. \tag{15}$$

Controlled Reduction Theorem. In the absence of control v_1^k, we can relate solutions of reduced-order vector field $\hat{f}_{\text{fct}}$ to solutions of full-order vector field $\hat{f}_{\lambda_1^k}$ and vice versa (in a generalized form of the functional Routhian reduction result of [2]). The divided variables, which are transformed into conserved momentum quantities, evolve according to a dynamic momentum map as described earlier.

Theorem 1: Let $L_{\lambda_1^k}$ be a k-almost-cyclic Lagrangian and L_{fct} the corresponding stage-k functional Routhian. Then $(q_1^k(t), q_{k+1}^n(t), \dot{q}_1^k(t), \dot{q}_{k+1}^n(t))$ is a solution to the v_{k+1}^n-controlled vector field $\hat{f}_{\lambda_1^k}$ on $[t_0, t_F]$ with

$$\begin{aligned} \dot{q}_j(t_0) &= \frac{1}{m_{q_j}(q_{j+1}^n(t_0))} \Big(\lambda_j(q_j(t_0)) \\ &\qquad - M_{q_j, q_{j+1}^n}(q_{j+1}^n(t_0))\dot{q}_{j+1}^n(t_0)\Big), \end{aligned} \tag{16}$$

for all $j \in \{1, k\}$, if and only if $(q_{k+1}^n(t), \dot{q}_{k+1}^n(t))$ is a solution to the controlled vector field $\hat{f}_{\text{fct}}$ and for all $j \in \{1, k\}$, $(q_j(t), \dot{q}_j(t))$ satisfies

$$\begin{aligned} \dot{q}_j(t) &= \frac{1}{m_{q_j}(q_{j+1}^n(t))} \Big(\lambda_j(q_j(t)) \\ &\qquad - M_{q_j, q_{j+1}^n}(q_{j+1}^n(t))\dot{q}_{j+1}^n(t)\Big). \end{aligned} \tag{17}$$

We want to apply this form of controlled reduction, proven in [5], to general robots that are not k-almost-cyclic, so we must show that any serial-chain robot has a recursively-cyclic inertia matrix and that a feedback control law exists that yields a shaped k-almost-cyclic Lagrangian.

Subrobot Theorem. The following are proven in [5]:

Lemma 1: Any n-d.o.f. serial-chain manipulator's kinetic energy $K_n(q,\dot{q}) = \frac{1}{2}\dot{q}^T M_n(q)\dot{q}$ is cyclic in its first degree-of-freedom, i.e., $\frac{\partial}{\partial q_1}K_n(q,\dot{q}) = 0$, where q_1 is the first element of the n-vector q of nongeneralized coordinates. Equivalently, $M_n(q)$ is cyclic in variable q_1.

Lemma 2: Any n-d.o.f. serial-chain manipulator's $n \times n$ inertia matrix $M_n(q)$ contains a lower-right $(n-1)\times(n-1)$ submatrix $M_{n-1}(q_2,\dots,q_n)$, which is the inertia matrix of the $(n-1)$-d.o.f. "subrobot" corresponding to the original manipulator with its first degree-of-freedom fixed.

Lemma 3: Any n-d.o.f. serial-chain manipulator's inertia matrix is recursively cyclic as in Definition 1.

Theorem 2: Any n-d.o.f. serial-chain manipulator with a potential energy that is cyclic in the first k coordinates can be reduced to its corresponding $(n-k)$-d.o.f. subrobot, where $1 \le k < n$.

Corollary 1: For any fully actuated n-d.o.f. serial-chain manipulator with a potential energy that is cyclic in the first k coordinates, there exists a feedback control law that shapes the system to the k-almost-cyclic form, where $1 \le k < n$.

Theorem 1 and Corollary 1 imply that by utilizing a reduction-based control law, initial conditions satisfying (16) allow the shaped dynamics of a serial-chain n-d.o.f. robot to project onto the dynamics of the corresponding $(n-k)$-d.o.f. subrobot, which is entirely decoupled from the first k coordinates and thus behaves and can be controlled as a typical $(n-k)$-d.o.f. robot. Moreover, the dynamics of the first k d.o.f. evolve according to constraint (17), showing that the subsystem dynamics do affect the first k coordinates but in a controlled manner. In particular, we can force these coordinates to set-points or periodic orbits with our choice of functional momentum maps $\lambda(q_j) = -\alpha_j(q_j - \bar{q}_j)$, where α_j is a gain constant and $\bar{q}_j$ is a desired angle constant, for $j \in \{1,k\}$. We now present the application of reduction-based control to bipedal walking robots.

III. Modeling Bipedal Walkers

A simple bipedal walking robot has two phases that are naturally modeled by a hybrid system: a continuous swing phase and an (ideally) instantaneous impact phase. During the swing phase, the point of contact between ground and the first link (the "stance leg") is assumed to be without slipping. Since knee-lock impacts introduce another level of complexity to the hybrid model, we assume that foot-ground impacts are the only discrete events (which does not preclude knees without impacts [6]). Moreover, foot impacts are assumed to be instantaneous and perfectly plastic. We begin by introducing the necessary hybrid system terminology and then describe the model of interest for this paper.

Hybrid Systems. Hybrid systems are systems that display both continuous and discrete behavior. Simple hybrid systems with one continuous phase are often modeled as "systems with impulse effects" (see [6], [8]), which are a subset of more general hybrid systems (the different notation can be seen in [2]). This section introduces the basic terminology of hybrid systems, from the former perspective, to model a biped without knee-lock impacts.

Definition 2: A *hybrid control system* has the form

$$\mathscr{HC} : \begin{cases} \dot{x} = f(x) + g(x)u & x \in D\backslash G \\ x^+ = R(x^-) & x^- \in G \end{cases},$$

where $G \subset D$ is called the *guard* and $R : G \to D$ is a smooth map called the *reset map* (or *impact equations*). In this context, state $x = (q^T, \dot{q}^T)^T$ is in domain $D \subseteq TQ$ and control input u is in admissible control space $U \subseteq \mathbb{R}^n$.

A *hybrid system* is a hybrid control system without an explicit control input u, i.e., it has the form

$$\mathscr{H} : \begin{cases} \dot{x} = f(x) & x \in D\backslash G \\ x^+ = R(x^-) & x^- \in G \end{cases}.$$

A *hybrid flow* is a solution to a hybrid system $\mathscr{H}$.

Since bipedal walking gaits correspond to periodic orbits of hybrid systems, we offer the definition of a periodic hybrid flow. For example, hipped bipedal walking gaits correspond to 2-periodic orbits due to natural side-to-side swaying and turning motions over two steps. Letting $x(t)$ be a hybrid flow of $\mathscr{H}$, it is k-periodic if $x(t) = x(t + \sum_{i=1}^{k} T_i)$, for all $t \geq 0$, where T_i is the fixed time between the $(i-1)^{th}$ and i^{th} discrete events. A *hybrid k-periodic orbit* $\mathcal{O} \subset D$ is defined by $\mathcal{O} = \{x(t)|t \geq 0\}$ for some k-periodic hybrid flow $x(t)$. Moreover, this hybrid k-periodic orbit is *locally exponentially stable* if there exist constants $K > 0$, $\alpha > 0$ and $\delta > 0$ such that for all hybrid flows $x(t)$ with $d(x(0),\mathcal{O}) < \delta$, we have $d(x(t),\mathcal{O}) \leq Ke^{-\alpha t}d(x(0),\mathcal{O})$, for all $t \geq 0$. Here, the distance between a point x and a set Y is defined as usual: $d(x,Y) = \inf_{y \in Y} \|x - y\|$.

In order to determine the stability of a k-periodic orbit, we study the corresponding Poincaré map as described in [2]. In particular, taking G to be the Poincaré section, one obtains the Poincaré map, $P : G \to G$, which is a partial map defined by $P(z) = x(\tau(z))$, where $x(t)$ is the solution to $\dot{x} = f(x)$ with $x(0) = R(z)$ and $\tau(z)$ is the *time-to-impact* function (see [8]). If $z^* \in G$ is a k-fixed point of P with certain assumptions (cf. [2]), a k-periodic orbit $\mathcal{O}$ containing z^* is locally exponentially stable iff P^k is locally exponentially stable as a discrete-time system. Although it is not possible to analytically derive the Poincaré map here, it can be numerically approximated through simulation so as to compute its linearization's eigenvalues. This allows us to directly analyze the stability of hybrid periodic orbits (see [4] for more on this numerical approximation technique).

Four-Degree-of-Freedom Biped Model. The model of interest is a 4-d.o.f. bipedal robot with a hip and splayed legs, as seen in Fig. 1 with an additional yaw-d.o.f., which is a three-dimensional version of the planar "compass-gait"

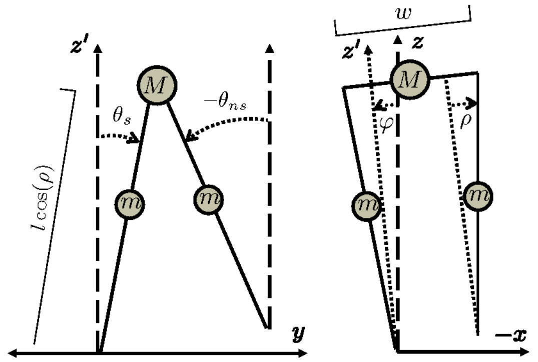

Fig. 1. The sagittal and lateral planes of a three-dimensional bipedal robot.

biped (as in the sagittal plane of Fig. 1). Given this robot, we will explicitly construct the hybrid control system $\mathscr{HC}_{4\mathrm{D}}$.

The configuration space for the 4-d.o.f. biped can be represented by $Q_{4\mathrm{D}} = SO(3) \times \mathbb{S}^1$ (cf. [12]). In particular, 3×3-matrix $R_s \in SO(3)$ is the orientation of the stance leg and $\theta_{ns} \in \mathbb{S}^1$ is the relative shape the nonstance/swing leg. However, before we obtain the equations of motion, we must parameterize the configuration space $Q_{4\mathrm{D}}$. An element of $SO(3)$ can be minimally represented by an ordered set of three ZYX Euler angles $(\omega, \varphi, \theta_s) \in \mathbb{T}^3$, which correspond to the yaw, roll, and pitch angles of the stance leg (and are the robot's first three d.o.f.). For the sake of distinguishing the sagittal-plane configuration, we take $Q_{4\mathrm{D}} = \mathbb{T}^2 \times \mathbb{T}^2 \cong \mathbb{T}^3 \times \mathbb{S}^1$, with coordinates $q = (\omega, \varphi, \theta^T)^T$, where ω is the yaw (or heading), φ is the roll (or lean) from vertical, and $\theta = (\theta_\mathrm{s}, \theta_\mathrm{ns})^T$ is the vector of sagittal-plane variables as in the 2D compass-gait model. The hip width w, leg length l, and leg splay angle ρ are held constant.

The domain and guard are constructed from a unilateral constraint function representing the height of the swing foot above the level ground:

$$H_{4\mathrm{D}}(q) = l(\cos(\theta_\mathrm{s}) - \cos(\theta_\mathrm{ns}))\cos(\varphi) - (w + 2l\sin(\rho))\sin(\varphi).$$

In particular, the domain $D_{4\mathrm{D}} \subset TQ_{4\mathrm{D}}$ is given by requiring that this height be nonnegative, i.e., $H_{4\mathrm{D}}(q) \geq 0$. The guard $G_{4\mathrm{D}}$ is the subset of the domain corresponding to the set of configurations in which the height of the swing foot is zero and infinitesimally decreasing. That is,

$$G_{4\mathrm{D}} = \Big\{(q^T, \dot{q}^T)^T \in D_{4\mathrm{D}} : H_{4\mathrm{D}}(q) = 0, \ \left(\frac{\partial H_{4\mathrm{D}}(q)}{\partial q}\right)^T \dot{q} < 0\Big\}.$$

Using Mathematica and the procedures outlined in [6], we compute the reset map

$$R_{4\mathrm{D}}(q, \dot{q}) = \begin{pmatrix} S_{4D}q \\ P_{4D}(q)\dot{q} \end{pmatrix},$$

where S_{4D}, $P_{4D}(q) \in \mathbb{R}^{4\times 4}$; the complexity of this map prevents its elements from being included but more detail can be found in [5]. Also, note that the signs of w and ρ are flipped during impact to model the change in stance leg (technically the hybrid model should then have two sets of continuous/discrete phases, but we forgo this for simplicity).

Finally, the dynamics for $\mathscr{HC}_{4\mathrm{D}}$ are derived with the previously described method. This system's Lagrangian is

$$L_{4\mathrm{D}}(q, \dot{q}) = \frac{1}{2}\dot{q}^T M_{4\mathrm{D}}(q)\dot{q} - V_{4\mathrm{D}}(q),$$

with recursively-cyclic 4×4 inertia matrix

$$M_{4\mathrm{D}}(q) = \begin{pmatrix} m_\omega(\varphi, \theta) & — & M_{\omega,\varphi,\theta}(\varphi, \theta) \\ | & m_\varphi(\theta) & M_{\varphi,\theta}(\theta) \\ M^T_{\omega,\varphi,\theta}(\varphi, \theta) & M^T_{\varphi,\theta}(\theta) & M_\theta(\theta) \end{pmatrix}$$

and potential energy

$$V_{4\mathrm{D}}(\varphi, \theta) = V_\theta(\theta)\cos(\varphi) - \frac{g}{2}(2m + M)(w + 2l\sin(\rho))\sin(\varphi), \quad (18)$$

which contains the planar subsystem potential energy

$$V_\theta(\theta) = \frac{gl}{2}(m(\cos(\rho) - 2)\cos(\theta_\mathrm{ns}) + (2m + (m + 2M)\cos(\rho))\cos(\theta_\mathrm{s})). \quad (19)$$

Using the controlled Euler-Lagrange equations, the dynamics for the fully-actuated walker are then given by

$$M_{4\mathrm{D}}(q)\ddot{q} + C_{4\mathrm{D}}(q, \dot{q})\dot{q} + N_{4\mathrm{D}}(q) = B_{4\mathrm{D}}u, \quad (20)$$

where $C_{4\mathrm{D}}$ and $N_{4\mathrm{D}}$ are defined as usual, and 4×4-matrix

$$B_{4\mathrm{D}} = \begin{pmatrix} I_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & B_\theta \end{pmatrix}$$

is assumed invertible. These equations (which can be found in [5]) yield the control system $(f_{4\mathrm{D}}, g_{4\mathrm{D}})$ with input u. We assume that we can only directly affect the angular accelerations within actuator saturation, so the set of admissible controls is $U_{4\mathrm{D}} = \{u \in \mathbb{R}^4 : |u_i| \leq U_{max}, \forall i \in \{1, 4\}\}$. We now construct a reduction-based control law for this robot.

IV. Control Law Construction

In the case of controlled multistage reduction, the controller is designed to recursively break cyclic symmetries in the special almost-cyclic manner. The inner loop of the control law shapes the energy of our robot to the 2-almost-cyclic form, and the nested outer loop plays two roles:

1) Implements passivity-based control on the reduced 2D-subsystem to construct robust flat-ground gaits.
2) Stabilizes to a surface defined by constraint (17) so that Theorem 1 holds.

We begin by describing the Lagrangian-shaping inner loop.

Lagrangian Shaping Controller. The goal of this controller is to shape both the kinetic and potential energies of $L_{4\mathrm{D}}$ so as to render it 2-almost-cyclic for controlled reduction to the biped's planar subsystem. Given configuration vector $q = (\omega, \varphi, \theta^T)^T$, the 4-d.o.f. potential $V_{4\mathrm{D}}$ of (18) is clearly not cyclic in the second variable φ, and thus we must make it so by imposing a "controlled symmetry" with respect to the second degree-of-freedom's rotation group $\mathbb{S}^1$. This is most naturally accomplished with potential shaping to replace $V_{4\mathrm{D}}$

$$u := \text{sat}\left(B_{4\text{D}}^{-1}\left(C_{4\text{D}}(q,\dot{q})\dot{q} + N_{4\text{D}}(q) + M_{4\text{D}}(\varphi,\theta)M_{\lambda_1^2}(\varphi,\theta)^{-1}\left(-C_{\lambda_1^2}(q,\dot{q})\dot{q} - N_{\lambda_1^2}(q) + B_{4\text{D}}v\right)\right), U_{max}\right)$$
$$= B_{4\text{D}}^{-1}\left(C_{4\text{D}}(q,\dot{q})\dot{q} + N_{4\text{D}}(q) + M_{4\text{D}}(\varphi,\theta)M_{\lambda_1^2}(\varphi,\theta)^{-1}\left(-C_{\lambda_1^2}(q,\dot{q})\dot{q} - N_{\lambda_1^2}(q) + B_{4\text{D}}\text{sat}(v,\tilde{U}_{max})\right)\right) \quad (21)$$

---

with V_θ of (19), the planar walker's cyclic potential energy (constructed from a scaled height due to splay angle ρ). We will incorporate this shaping into the control law.

Consider the generalized almost-cyclic Lagrangian of (3a) for $j = 1$, $k = 2$, $n = 4$:

$$L_{\lambda_1^2}(q,\dot{q}) = \frac{1}{2}\dot{q}^T M_{\lambda_1^2}(q_2^4)\dot{q} - W_{\lambda_1^2}(q,\dot{q}_2^4) - V_{\lambda_1^2}(q),$$

where $M_{\lambda_1^2}$, $W_{\lambda_1^2}$ and $V_{\lambda_1^2}$ are defined by substituting $M_{4\text{D}}$ for M and V_θ for V_{fct} in (4)-(5). It follows that the stage-2 functional Routhian associated with 2-almost-cyclic Lagrangian $L_{\lambda_1^2}$ is the Lagrangian of the scaled planar walker:

$$L_{2\text{D}}(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T M_\theta(\theta)\dot{\theta} + V_\theta(\theta),$$

which yields the reduced control system $(f_{2\text{D}}, g_{2\text{D}})$ with v_θ.

Given this desired reduction, we define a feedback control law that transforms L to $L_{\lambda_1^2}$ and allows auxiliary control of the shaped system. We saturate this control law absolutely at constant $U_{max} = 20$ Nm to demonstrate its practicality, under the assumption that the Lagrangian-shaping controller of the inner loop is always within the saturation limits. Therefore, let u be defined by (21), where as in (8), $C_{\lambda_1^2}$ is the shaped Coriolis matrix and $N_{\lambda_1^2} = \frac{\partial}{\partial q}V_{\lambda_1^2}$ is the vector of shaped potential torques. The vector $v = (v_\omega, v_\varphi, v_\theta^T)^T$ of auxiliary control inputs, to be defined later, is saturated by the time-varying limit $\tilde{U}_{max}$ that is assumed uniformly bounded from zero. Finally, using momentum maps $\lambda_1(\omega) = -\alpha_1(\omega - \bar{\omega})$ and $\lambda_2(\varphi) = -\alpha_2\varphi$, for $\alpha_1, \alpha_2 > 0$, we have directional control to constant angle $\bar{\omega}$ for the yaw/heading d.o.f. and correction to vertical for the roll/lean d.o.f.

Applying this law to the control system $(f_{4\text{D}}, g_{4\text{D}})$, we have the shaped 2-almost-cyclic dynamical system

$$M_{\lambda_1^2}(\varphi,\theta)\ddot{q} + C_{\lambda_1^2}(q,\dot{q})\dot{q} + N_{\lambda_1^2}(q) = B_{4\text{D}}\text{sat}(v,\tilde{U}_{max}),$$

which is associated with the new control system $(f_{\lambda_1^2}, g_{\lambda_1^2})$ as in (9) with bounded control input v to be defined next.

Subsystem Passivity-Based Controller. Since we can decouple the robot's last two degrees-of-freedom (the reduced subsystem), we can control it as a planar 2-d.o.f. biped with well-known passivity-based techniques in v_θ. The first of these techniques is that of slope-changing "controlled symmetries," which will allow our biped to walk on flat ground given planar walking cycles down slopes [10], [12].

In three dimensions, the orientation of the ground (the slope) can be represented by a rotation of the world frame, i.e., an element of $SO(3)$. Thus, any change of slope is characterized by a group action of $SO(3)$ on our bipedal walker's configuration space $Q_{4\text{D}}$:

$$\Phi_A(R_s,\theta_{ns}) = (A \cdot R_s, \theta_{ns}), \;\; A \in SO(3).$$

The behavior of a bipedal walker is strongly dependant on the ground slope. Spong and Bullo prove in [12] that both the kinetic energy and impact events are invariant under the slope changing action Φ, but this is not the case for the potential energy. We can, however, control the robot's potential to the desired world orientation and thus impose symmetry on the system, i.e., a *controlled symmetry*. With such a symmetry, any stable limit cycle down a slope can be mapped to a stable limit cycle on an arbitrary slope.

For the sagittal-plane compass-gait biped, stable passive/uncontrolled limit cycles exist down shallow slopes between about 3° and 5°, as shown in [4]. This is the range of slope angles for which the potential energy introduced by gravity over each stride is matched by the energy dissipated at foot impact with ground. For this planar robot,

$$\Phi_A(\theta) = \theta + \beta = (\theta_s + \beta, \theta_{ns} + \beta)^T,$$

where $\beta = \sigma - \delta$ is the angle of rotation parameterizing $A \in SO(2)$, σ is the slope angle yielding the desired passive limit cycle (such as $\pi/50$), and δ is the actual ground slope angle for controlled walking. Since our 4-d.o.f. biped's forward walking motion is dominated by its sagittal plane, we want to implement controlled symmetries on this 2D subsystem to construct gaits on flat ground ($\delta = 0$), and the control that achieves the desired trajectory mapping is

$$v_\theta^\beta = B_\theta^{-1}\frac{\partial}{\partial\theta}\left(V_\theta(\theta) - V_\theta(\theta+\beta)\right), \quad \beta = \frac{\pi}{50}.$$

We assume[1] that v_θ^β is within the saturation limit $\tilde{U}_{max}$.

As for the full-order system, each impact event can amplify limit cycle perturbations in several dimensions, especially when the biped is turning. In order to increase limit cycle robustness, we implement passivity-based constant-energy tracking from [11] on the critical 2D subsystem (this energy is nearly constant in the hipped case).

To begin, we define the Lyapunov-like storage function

$$S = \frac{1}{2}(E_{2D} - E_{2D}^{ref})^2 \geq 0,$$

where E_{2D}^{ref} is a constant reference energy and E_{2D} is the 2D-subsystem energy after controlled symmetries:

$$\begin{aligned} E_{2D} &= K_\theta + V_\theta^\beta \\ &= K_\theta + V_\theta + (-V_\theta + V_\theta^\beta), \end{aligned} \quad (22)$$

with $K_\theta = \frac{1}{2}\dot{\theta}^T M_\theta(\theta)\dot{\theta}$ and $V_\theta^\beta = V_\theta(\theta+\beta)$.

Due to the passivity property of robots, we have

$$\dot{E_{2D}} = \dot{\theta}^T\left(B_\theta v_\theta - \frac{\partial V_\theta}{\partial\theta} + \frac{\partial V_\theta^\beta}{\partial\theta}\right)$$

[1]All assumptions on the control input and its limits are confirmed in simulation by observing the required torques.

along trajectories of the shaped system. And, using passivity-based control on the 2D subsystem:

$$v_\theta := v_\theta^\beta + \tilde{v}_\theta = B_\theta^{-1}\left(\frac{\partial V_\theta}{\partial \theta} - \frac{\partial V_\theta^\beta}{\partial \theta}\right) + \tilde{v}_\theta, \tag{23}$$

it follows that $\dot{E_{2D}} = \dot{\theta}^T \tilde{v}_\theta$. Then, taking the derivative of the storage function yields $\dot{S} = (E_{2D} - E_{2D}^{ref})\dot{\theta}^T \tilde{v}_\theta$.

If we wisely choose the auxiliary input $\tilde{v}_\theta$ for energy tracking, such as feedback law

$$\tilde{v}_\theta \quad = \quad -B_\theta^{-1} p(E_{2D} - E_{2D}^{ref})\dot{\theta} \tag{24}$$

with $p > 0$, then we have the negative semidefinite

$$\dot{S} \quad = \quad -2p||\dot{\theta}||^2 S \leq 0.$$

It is proven in [11] that under reasonable conditions (including saturation), this implies exponential convergence of a planar biped's total energy to the reference energy between step impacts. If the reference is chosen to be the constant energy corresponding to a stable limit cycle (assuming that the limit cycle has nearly constant energy), then this passivity-based controller should expand the limit cycle's basin of attraction.

This subsystem control law v_θ of (23) is incorporated into the full-order shaped system ($f_{\lambda_1^2}$,$g_{\lambda_1^2}$) by defining the new control system ($\hat{f}_{\lambda_1^2}$,$\hat{g}_{\lambda_1^2}$) with input $v_1^2 = (v_\omega, v_\varphi)^T$ as in (10). Similarly, the 2-reduced, v_θ-controlled vector field $\hat{f}_{2\mathrm{D}}$ is defined as in (15). We now design the control law v_1^2 to handle conditions that do not satisfy equation (16).

Zero Dynamics Controller. The decoupling effect of Theorem 1 is only valid when (16) is satisfied. Since most initial conditions will not satisfy this equation, we adopt the approach of [1] in using output linearization to stabilize to the surface defined by constraint (17).

In order to satisfy (17), we define output functions

$$h_i(q_i^4, \dot{q}_i^4) := \dot{q}_i - \frac{1}{m_{q_i}(q_{i+1}^4)}\left(\lambda_i(q_i) - M_{q_i,q_{i+1}^4}(q_{i+1}^4)\dot{q}_{i+1}^4\right), \tag{25}$$

for $i \in \{1, 2\}$. We construct this control law to drive functions h_i to zero, i.e., we force the system to the surface

$$\mathcal{Z} = \left\{\left(\begin{array}{c} q \\ \dot{q} \end{array}\right) \in TQ : h_i(q_i^4, \dot{q}_i^4) = 0, \quad \forall i \in \{1, 2\}\right\}.$$

Given the standard method for zeroing multiple output functions in a MIMO nonlinear control system (cf. [9]), we first define the matrix of Lie derivatives with respect to $\hat{g}_{\lambda_1^2}$:

$$A(q) = \left(\begin{array}{cc} \frac{\partial h_1(q,\dot{q})}{\partial q} & \frac{\partial h_2(q_2^4,\dot{q}_2^4)}{\partial q} \end{array}\right)^T \hat{g}_{\lambda_1^2}(q),$$

of which element $A_{i,j}$ is $L_{\hat{g}_{\lambda_1^2} e_j} h_i$, the Lie derivative of h_i with respect to $\hat{g}_{\lambda_1^2} e_j$, where e_j is the j^{th} standard basis vector of $\mathbb{R}^2$. Also, $A(q)$ is positive-definite, since

$$L_{\hat{g}_{\lambda_1^2} e_i} h_i(q_i^4, \dot{q}_i^4) = \frac{1}{m_{q_i}(q_{i+1}^4)},$$

and $m_{q_i}(q_{i+1}^4) > 0$ by the positive-definiteness of $M_{4\mathrm{D}}(q)$.

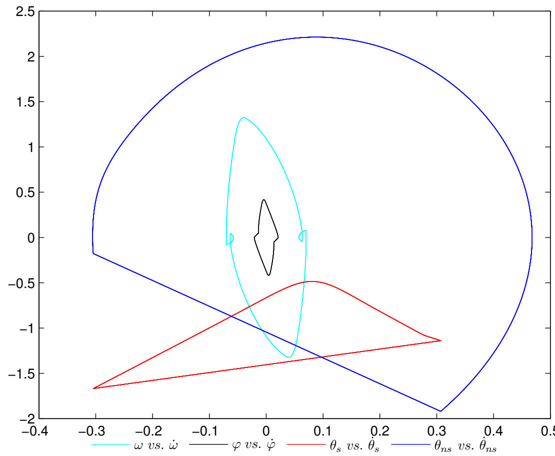

Fig. 2. Phase portrait of the 4-d.o.f. biped's straight-walking limit cycle.

We then define the following feedback control law:

$$v_1^2 \quad := \quad -A(q)^{-1}\left(\left(\begin{array}{c} L_{\hat{f}_{\lambda_1^2}} h_1(q,\dot{q}) \\ L_{\hat{f}_{\lambda_1^2}} h_2(q_2^4,\dot{q}_2^4) \end{array}\right) + \left(\begin{array}{cc} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{array}\right)^{-1}\left(\begin{array}{c} h_1(q,\dot{q}) \\ h_2(q_2^4,\dot{q}_2^4) \end{array}\right)\right), \tag{26}$$

where $L_{\hat{f}_{\lambda_1^2}} h_i$ is the Lie derivative of h_i with respect to $\hat{f}_{\lambda_1^2}$, and $1/\epsilon_i > 0$ is a new "proportional" control gain, for $i \in \{1, 2\}$. Note that v_1^2 is well-defined by the positive-definiteness of $A(q)$ and the control gain matrix. And finally, $v_1^2|_{h_i=0, \forall i \in \{1,2\}} = 0$, so this controller does not interfere with the reduction theorem when on the surface $\mathcal{Z}$.

V. Simulations and Concluding Remarks

The bipedal model of interest has hip mass $M = 10$ kg, hip width $w = 0.1$ m, mid-leg mass $m = 5$ kg, leg length $l = 1$ m, and leg splay angle $\rho = 0.0188$ rad. Given overall control law u of (21), we tighten the momentum map λ_1, by setting its gain constant to $\alpha_1 = 15$ and desired heading to $\bar{\omega} = 0$, to counteract the induced yaw caused by the hip's coupling with sagittal-plane motion. Setting the other gains to $\alpha_2 = 10$, $\epsilon_1 = \frac{1}{30}$, $\epsilon_2 = \frac{1}{15}$, $\beta = \frac{\pi}{50}$, $p = 20$, and $E_{2D}^{ref} = 154.6088$ J, we have the reduction-shaped hybrid system $\mathscr{H}_{4\mathrm{D}}^{\alpha,\epsilon,\beta,p}$, which is the closed-loop hybrid system of $\mathscr{H}\mathscr{C}_{4\mathrm{D}}$ after applying control law u.

As was the case with the hipped 3-d.o.f. walker in [2], an analytical proof of limit cycle stability does not seem possible because the momentum quantities are not conserved through impact. The velocity discontinuities at every step introduce conserved quantity errors to be corrected by zero dynamics law v_1^2, so the assumptions of Theorem 1 do not always hold and the solutions of $\hat{f}_{\lambda_1^2}$ and $\hat{f}_{2\mathrm{D}}$ cannot be analytically related. The decoupling of the 2D-subsystem limit cycle is temporarily violated at each impact, resulting in a perturbation in this limit cycle. However, we argue that for sufficiently small gains ϵ_1, ϵ_2 (and thus sufficiently fast convergence to conserved quantities surface $\mathcal{Z}$), each perturbation will be within the planar limit cycle's basin of attraction (which is expanded by passivity-based control $\tilde{v}_\theta$) and that the subsystem θ-dynamics and divided ω,φ-dynamics will essentially evolve according to Theorem 1.

Therefore, we claim that when walking straight forward on flat ground, this controlled 4-d.o.f. biped has a stable 2-periodic limit cycle, $\mathcal{O}_{4\mathrm{D}}$ of Fig. 2, which is constructed from its planar subsystem's limit cycle (shown in red and blue). The 2-fixed point of $\mathcal{O}_{4\mathrm{D}}$ at the Poincaré section is

$$\begin{aligned}\begin{pmatrix} q^* \\ \dot{q}^* \end{pmatrix} &= \mathcal{O}_{4\mathrm{D}} \cap G_{4\mathrm{D}} \\ &\approx (-0.0697, -0.0133, -0.3044, 0.3071, \\ &\qquad -0.0774, 0.0491, -1.6676, -1.9198)^T.\end{aligned}$$

We numerically calculate the eigenvalue magnitudes of the linearized Poincaré map to be within the unit circle: 0.2734, 0.1654, 0.1654, 0.0070, 0.0040, 0.0038, 0.0001, and 0.0001, thus verifying that $\mathcal{O}_{4\mathrm{D}}$ is a locally exponentially stable periodic orbit of $\mathscr{H}_{4\mathrm{D}}^{\alpha,\epsilon,\beta,p}$. It is easily observed that limit cycle convergence takes far fewer steps with the passivity-based control. We also see that the yaw and lean dynamics follow 2-periodic orbits, a natural result of functional Routhian reduction, our choice of functional momentum maps, and the velocity discontinuities of each impact event.

In order to demonstrate the directional capabilities of this controlled 4-d.o.f. walker, we instruct the biped to perform a 90° turn over several steps. This is done by starting with $\bar{\omega} = 0$, and at every other step incrementing the desired yaw angle by $\pi/10$ until $\bar{\omega} = \pi/2$. Moreover, we re-tune the reference 2D-energy $E_{2D}^{ref} = 154.6088$ J, as the desired energy level is slightly lower when turning. The walking gait is shown in Fig. 3, and we see in Fig. 4 that the passivity-based control keeps the sagittal subsystem energy at the desired level, despite the injected energy from yaw rotation. Once the biped meets desired heading $\bar{\omega} = \pi/2$, its gait converges to the straight-walking 2-periodic limit cycle of $\mathcal{O}_{4\mathrm{D}}$ with a horizontally-shifted yaw orbit.

These results suggest that a completely 3-D bipedal robot can achieve stable directional walking with a feasible reduction/passivity-based control law. Since this form of controlled reduction is presented in the general k-stage case, this method easily extends to higher-dimensional systems. However, this theory is limited to fully-actuated serial-chain manipulators, precluding application to robots with feet, torsos, or arms. Future work will generalize this to branched chains, but the underactuated case demands further attention.

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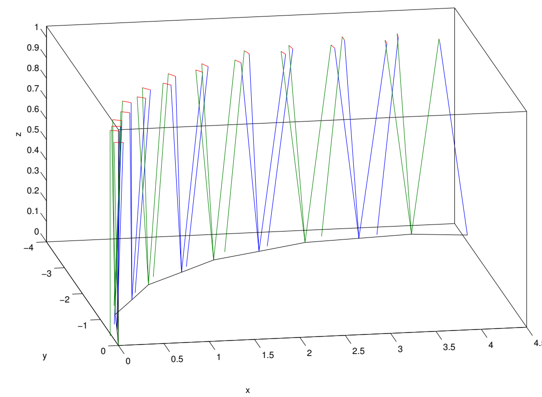

Fig. 3. A 90°-turn gait for the hipped 4-d.o.f. biped.

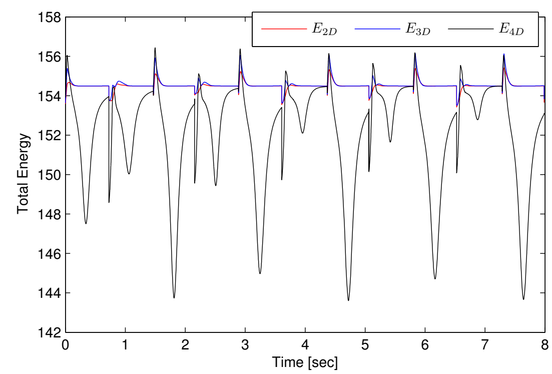

Fig. 4. Subsystem energies of the hipped 4-d.o.f. biped's 90°-turn gait.

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