2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Discrete-time Markovian Jump Linear Systems with Partly Unknown Transition Probabilities: H_∞ Filtering Problem

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***Abstract*— The problem of H_∞ filtering for a class of discrete-time Markovian jump linear systems (MJLS) with partly unknown transition probabilities is investigated in the paper. The considered systems are more general, which cover the MJLS with completely known and completely unknown transition probabilities as two special cases. A mode-dependent full-order filter is constructed and the bounded real lemma (BRL) for the resulting filtering error system is derived via LMI formulation. Then, an improved version of the BRL is further given by introducing additional slack matrix variables to eliminate the cross coupling between system matrices and Lyapunov matrices among different operation modes. Finally, the existence criterion of the desired filter is obtained such that the corresponding filtering error system is stochastically stable with a guaranteed H_∞ performance index. A numerical example is presented to illustrate the effectiveness and potential of the developed theoretical results.**

I. INTRODUCTION

As a class of stochastic hybrid systems, Markovian jump systems have been extensively studied in past decades, see for example, [1], [5], [9]. By stochastic hybrid feature, we mean that the considered systems contain continuous and discrete dynamics, which are described respectively by classical differential (or difference) equations and Markov stochastic process (or Markov chain). As a crucial factor, the transition probabilities in the jumping process determine the system behavior, and many issues on Markovian jump system have been investigated assuming the complete knowledge of the transition probabilities. A recent extension is to consider the systems with uncertain transition probabilities, in which the robust methodologies are adopted to cope with the norm-bounded or polytopic types of uncertainties in the transition probabilities matrix, see for example, [2], [8]. However, in these references, the structure and "nominal" terms of the uncertain transition probabilities are still assumed to be known *a priori*.

The ideal assumptions on the transition probabilities facilitate the treatment of considered problems, but the applicability of the obtained results is inevitably limited. A typical example could be found in Networked control systems (NCS). It is well-known that the time-varying delays induced by communication channels can be modeled as Markov chains, and accordingly the resulting closed-loop system can be studied by means of jump linear systems theory, see for example, [3], [12]. However, the variation of delays in all kinds of communication networks (especially Internet) can be vague and random, all or part of the elements in the expected transition probabilities matrix are probably hard or expensive to obtain. Consequently, the resulting NCS modeled by jump systems with completely known transition probabilities is actually questionable. Therefore, either in theory or in practice, it is necessary and significant to further consider more general jump systems with partly unknown transition probabilities.

On another research front line, state estimation is an important research issue in control field and has found many practical applications. Many useful results on estimation and filtering for all kinds of dynamic systems have been reported, and H_∞ filtering has been recognized to be one of the most popular approaches to deal with external noise sources with unknown statistics [6], [9], [10], [11]. Considering Markovian jump systems with completely known or completely unknown transition probabilities, the mode-dependent and mode-independent filter design approaches have been developed, respectively, see for example, [1], [2], [4], [7]. However, it seems more practicable and challenging to design filters, especially mode-dependent filters, for the underlying systems with partly unknown transition probabilities, which inspires us for this study.

In this paper, the H_∞ filtering problem for a class of discrete-time Markovian jump linear system (MJLS) with partly unknown transition probabilities is investigated. The considered systems are more general than the systems with completely known or completely unknown transition probabilities, which can be viewed as two special cases of the ones tackled here. A mode-dependent full-order filter is constructed and the bounded real lemma (BRL) for the resulting filtering error system is derived in terms of LMI. Also, an improved version of the BRL is given by introducing additional slack matrix variables to eliminate the cross coupling between system matrices and Lyapunov matrices among different operation modes. Furthermore, the existence condition of the desired filter is obtained such that the corresponding filtering error system is stochastically stable and has a guaranteed H_∞ performance index. A numerical example is presented to illustrate the effectiveness and potential of the developed theoretical results.

Notation: The notation used in this paper is fairly standard. The superscript "T" stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space, the notation $|\cdot|$ refers to the Euclidean vector norm. $l_2[0,\infty)$ is the space of square summable infinite sequence and for $w = \{w(k)\} \in l_2[0,\infty)$, its norm is given by $\|w\|_2 =$

This work was supported by NSERC-Canada, Grant OPG0035444

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$\sqrt{\sum_{k=0}^{\infty}|w(k)|^2}$. For notation $(\Omega,\mathcal{F},\mathcal{P})$, Ω represents the sample space, $\mathcal{F}$ is the σ-algebra of subsets of the sample space and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. $E[\cdot]$ stands for the mathematical expectation and for sequence $e=\{e(k)\}\in l_2((\Omega,\mathcal{F},\mathcal{P}),[0,\infty))$, its norm is given by $\|e\|_{E_2}=\sqrt{E\left[\sum_{k=0}^{\infty}|e(k)|^2\right]}$. In addition, in symmetric block matrices or long matrix expressions, we use * as an ellipsis for the terms that are introduced by symmetry and $diag\{\cdots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. The notation $P>0$ (≥ 0) means P is real symmetric positive (semi-positive) definite. I and 0 represent respectively, identity matrix and zero matrix.

II. Problem Formulation and Preliminaries

Fix the probability space $(\Omega,\mathcal{F},\mathcal{P})$ and consider the following class of discrete-time Markovian jump linear systems:

$$\begin{aligned} x(k+1) &= A(r_k)x(k)+B(r_k)w(k)\\ y(k) &= C(r_k)x(k)+D(r_k)w(k)\\ z(k) &= H(r_k)x(k)+L(r_k)w(k) \end{aligned} \quad (1)$$

where $x(k)\in\mathbb{R}^n$ is the state vector, $w(k)\in\mathbb{R}^l$ is the disturbance input which belongs to $l_2[0,\infty)$, $y(k)\in\mathbb{R}^m$ is the measurement output and $z(k)\in\mathbb{R}^v$ is the objective signal to be attenuated. $\{r_k,\ k\geq 0\}$ is a discrete-time homogeneous Markov chain, which takes values in a finite set $\mathcal{I}\triangleq\{1,...,N\}$ with a transition probabilities matrix $\Lambda=\{\pi_{ij}\}$ namely, for $r_k=i, r_{k+1}=j$, one has

$$\Pr(r_{k+1}=j|r_k=i)=\pi_{ij}$$

where $\pi_{ij}\geq 0\ \forall\ i,\ j\in\mathcal{I}$, and $\sum_{j=1}^{N}\pi_{ij}=1$. The set $\mathcal{I}$ contains N modes of system (1) and for $r_k=i\in\mathcal{I}$, the system matrices of the ith mode are denoted by A_i, B_i, C_i, D_i, H_i and L_i, which are considered here to be real known with appropriate dimensions.

In addition, the transition probabilities of the jumping process $\{r_k,\ k\geq 0\}$ in this paper are assumed to be partly accessed, i.e., some elements in matrix $\mathbf{\Lambda}$ are unknown. For instance, for system (1) with 5 operation modes, the transition probabilities matrix may be as:

$$\begin{bmatrix} \pi_{11} & ? & \pi_{13} & ? & \pi_{15}\\ ? & ? & ? & \pi_{24} & \pi_{25}\\ \pi_{31} & \pi_{32} & \pi_{33} & ? & ?\\ ? & ? & \pi_{43} & \pi_{44} & ?\\ ? & \pi_{52} & ? & \pi_{54} & ? \end{bmatrix}$$

where "?" represents the unaccessible elements. For notation clarity, $\forall i\in\mathcal{I}$, we denote that

$$\mathcal{I}_{\mathcal{K}}^i\triangleq\{j:\pi_{ij}\text{ is known}\},\quad \mathcal{I}_{\mathcal{UK}}^i\triangleq\{j:\pi_{ij}\text{ is unknown}\}, \quad (2)$$

Also, we denote $\pi_{\mathcal{K}}^i\triangleq\sum_{j\in\mathcal{I}_k^i}\pi_{ij}$ throughout the paper.

Remark 1: The accessibility of the jumping process $\{r_k,\ k\geq 0\}$ in the existing literature is commonly assumed to be completely accessible ($\mathcal{I}_{\mathcal{UK}}^i=\emptyset$, $\mathcal{I}_{\mathcal{K}}^i=\mathcal{I}$) or completely unaccessible ($\mathcal{I}_{\mathcal{K}}^i=\emptyset$, $\mathcal{I}_{\mathcal{UK}}^i=\mathcal{I}$). Note that the transition probabilities with polytopic or norm-bounded uncertainties can still be viewed as accessible in the sense of this paper. Therefore, our transition probabilities matrix considered in the sequel is a more natural assumption to the Markovian jump systems and hence covers the previous two cases.

Here, we are interested in designing a mode-dependent full-order filter of the form:

$$\begin{aligned} x_F(k+1) &= A_F(r_k)x_F(k)+B_F(r_k)y(k)\\ z_F(k) &= C_F(r_k)x_F(k)+D_F(r_k)y(k) \end{aligned} \quad (3)$$

where $A_F(r_k), B_F(r_k), C_F(r_k)$ and $D_F(r_k)$, $\forall r_k\in\mathcal{I}$ are filter gains to be determined. The filter with the above structure is assumed to jump synchronously with the modes in system (1), which is hereby mode-dependent.

Augmenting the model of (1) to include the states of the filter, we obtain the following dynamics:

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}(r_k)\tilde{x}(k)+\tilde{B}(r_k)w(k)\\ e(k) &= \tilde{C}(r_k)\tilde{x}(k)+\tilde{D}(r_k)w(k) \end{aligned} \quad (4)$$

where,

$$\begin{aligned} \tilde{x}(k) &= \begin{bmatrix} x(k)\\ x_F(k)\end{bmatrix},\ e(k)=z(k)-z_F(k),\\ \tilde{A}(r_k) &= \begin{bmatrix} A(r_k) & 0\\ B_F(r_k)C(r_k) & A_F(r_k)\end{bmatrix},\\ \tilde{B}(r_k) &= \begin{bmatrix} B(r_k)\\ B_F(r_k)D(r_k)\end{bmatrix},\\ \tilde{C}(r_k) &= \begin{bmatrix} H(r_k)-D_F(r_k)C(r_k) & -C_F(r_k)\end{bmatrix},\\ \tilde{D}(r_k) &= L(r_k)-D_F(r_k)D(r_k). \end{aligned}$$

Obviously, the resulting system (4) is also a Markovian jump linear system with partly unknown transition probabilities (2). Now, to present the main objective of this paper more precisely, we also introduce the following definitions for the filtering error system (4), which are essential for the later development.

Definition 1: System (4) is said to be stochastically stable if for $w(k)\equiv 0$ and every initial condition $\tilde{x}_0\in\mathbb{R}^n$ and $r_0\in\mathcal{I}$, the following holds:

$$E\left\{\sum_{k=0}^{\infty}\|\tilde{x}(k)\|^2|\tilde{x}_0,r_0\right\}<\infty$$

Definition 2: Given a scalar $\gamma>0$, system (4) is said to be stochastically stable and has an H_∞ noise attenuation performance index γ if it is stochastically stable and under zero initial condition, $\|e\|_{E_2}<\gamma\|w\|_2$ holds for all nonzero $w(k)\in l_2[0,\infty)$.

Thus, the objective of this paper is to design a mode-dependent full-order filter with the form (3) such that the filtering error system (4) is stochastically stable and has a guaranteed H_∞ noise attenuation performance.

III. Main Results

A. H_∞ Filtering Analysis:

Let us first discuss H_∞ filtering analysis for the filtering error system (4) under given filter gains in (3). The following lemma presents a bounded H_∞ performance criterion (i.e., the so-called bounded real lemma (BRL)) for system (4) with the partly unknown transition probabilities (2).

Lemma 1: Consider system (4) with partly unknown transition probabilities (2) and let $\gamma > 0$ be a given constant. If there exist matrix $P_i > 0$, $\forall i \in \mathcal{I}$ such that

$$\begin{bmatrix} -\mathcal{P}_{\mathcal{K}}^i & 0 & \mathcal{P}_{\mathcal{K}}^i\tilde{A}_i & \mathcal{P}_{\mathcal{K}}^i\tilde{B}_i \\ * & -\pi_{\mathcal{K}}^i I & \pi_{\mathcal{K}}^i\tilde{C}_i & \pi_{\mathcal{K}}^i\tilde{D}_i \\ * & * & -\pi_{\mathcal{K}}^i P_i & 0 \\ * & * & * & -\pi_{\mathcal{K}}^i\gamma^2 I \end{bmatrix} < 0, \forall j \in \mathcal{I}_{\mathcal{K}}^i \tag{5}$$

$$\begin{bmatrix} -P_j & 0 & P_j\tilde{A}_i & P_j\tilde{B}_i \\ * & -I & \tilde{C}_i & \tilde{D}_i \\ * & * & -P_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \forall j \in \mathcal{I}_{\mathcal{UK}}^i \tag{6}$$

where $\mathcal{P}_{\mathcal{K}}^i \triangleq \sum_{j\in\mathcal{I}_{\mathcal{K}}^i}\pi_{ij}P_j$, then the filtering error system (4) is stochastically stable with an H_∞ performance index γ.

Proof: Construct a stochastic Lyapunov function as

$$V(\tilde{x}_k, k) = \tilde{x}_k^T P_i \tilde{x}_k, \forall r_k = i \in \mathcal{I} \tag{7}$$

where P_i satisfy (5) and (6). Then, for $r_k = i, r_{k+1} = j$, one has

$$\begin{aligned}
& E\left[\Delta V(\tilde{x}_k, k)\right] \\
\triangleq\ & E\left[V(\tilde{x}_{k+1}, k+1|\tilde{x}_k, r_k) - V(\tilde{x}_k, k)\right] \\
=\ & \tilde{x}_{k+1}^T \sum\nolimits_{j\in\mathcal{I}}\pi_{ij}P_j\tilde{x}_{k+1} \\
& -\tilde{x}_k^T\left[\sum\nolimits_{j\in\mathcal{I}_{\mathcal{K}}^i}\pi_{ij} + \sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^i}\pi_{ij}\right]P_i\tilde{x}_k \\
=\ & \tilde{x}_{k+1}^T\left[\sum\nolimits_{j\in\mathcal{I}_{\mathcal{K}}^i}\pi_{ij}P_j + \sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^i}\pi_{ij}P_j\right]\tilde{x}_{k+1} \\
& -\tilde{x}_k^T\left[\sum\nolimits_{j\in\mathcal{I}_{\mathcal{K}}^i}\pi_{ij}P_i + \sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^i}\pi_{ij}P_i\right]\tilde{x}_k \\
=\ & \tilde{x}_{k+1}^T\left[\mathcal{P}_{\mathcal{K}}^i + \sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^i}\pi_{ij}P_j\right]\tilde{x}_{k+1} \\
& -\tilde{x}_k^T\left[\pi_{\mathcal{K}}^i P_i + \sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^i}\pi_{ij}P_i\right]\tilde{x}_k \\
=\ & \tilde{x}_{k+1}^T\mathcal{P}_{\mathcal{K}}^i\tilde{x}_{k+1} - \pi_{\mathcal{K}}^i\tilde{x}_k^T P_i\tilde{x}_k \\
& +\sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^i}\pi_{ij}\left[\tilde{x}_{k+1}^T P_j\tilde{x}_{k+1} - \tilde{x}_k^T P_i\tilde{x}_k\right] \\
=\ & \tilde{x}_k^T\left[A_i^T\mathcal{P}_{\mathcal{K}}^i A_i - \pi_{\mathcal{K}}^i P_i\right]\tilde{x}_k \\
& +\sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^i}\pi_{ij}\tilde{x}_k^T\left[A_i^T P_j A_i - P_i\right]\tilde{x}_k
\end{aligned} \tag{8}$$

On the other hand, if (5) and (6) hold, we know from some basic matrix manipulations that

$$\begin{aligned}
\begin{bmatrix} -\mathcal{P}_{\mathcal{K}}^i & \mathcal{P}_{\mathcal{K}}^i\tilde{A}_i \\ * & -\pi_{\mathcal{K}}^i P_i \end{bmatrix} &< 0,\ j \in \mathcal{I}_{\mathcal{K}}^i, \\
\begin{bmatrix} -P_j & P_j\tilde{A}_i \\ * & -P_i \end{bmatrix} &< 0,\ j \in \mathcal{I}_{\mathcal{UK}}^i,
\end{aligned}$$

Furthermore, by Schur complement, we have

$$\tilde{A}_i^T\mathcal{P}_{\mathcal{K}}^i\tilde{A}_i - \pi_{\mathcal{K}}^i P_i < 0,\ j \in \mathcal{I}_{\mathcal{K}}^i, \tag{9}$$

$$\tilde{A}_i^T P_j\tilde{A}_i - P_i < 0,\ j \in \mathcal{I}_{\mathcal{UK}}^i \tag{10}$$

Therefore, if (9) and (10) hold, we know from (8) that

$$\begin{aligned}
& E\left[\Delta V\right] \\
\leq\ & -\lambda_{\min}\left[-\left(A_i^T\mathcal{P}_{\mathcal{K}}^i A_i - \pi_{\mathcal{K}}^i P_i\right)\right]\tilde{x}_k^T\tilde{x}_k \\
& -\lambda_{\min}\left[-\left(A_i^T P_j A_i - P_i\right)\right]\tilde{x}_k^T\tilde{x}_k \\
\leq\ & -(\beta_1 + \beta_2)\,\tilde{x}_k^T\tilde{x}_k = -(\beta_1 + \beta_2)\,||\tilde{x}_k||^2
\end{aligned} \tag{11}$$

where $\beta_1 = \inf\left\{\lambda_{\min}\left[-\left(A_i^T\mathcal{P}_{\mathcal{K}}^i A_i - \pi_{\mathcal{K}}^i P_i\right)\right], i \in I\right\}$ and $\beta_2 = \inf\left\{\lambda_{\min}\left[-\left(A_i^T P_j A_i - P_i\right)\right], i \in I\right\}$. From (11), setting $\beta = \beta_1 + \beta_2$, we obtain that for any $T \geq 1$,

$$\begin{aligned}
& E\left\{\sum\nolimits_{k=0}^{T}||\tilde{x}_k||^2\right\} \\
\leq\ & \frac{1}{\beta}\left\{E\left[V(\tilde{x}_0, 0)\right] - E\left[V(\tilde{x}_{T+1}, T+1)\right]\right\} \\
\leq\ & \frac{1}{\beta}E\left[V(\tilde{x}_0, 0)\right],
\end{aligned}$$

which implies that

$$E\left\{\sum\nolimits_{k=0}^{T}||\tilde{x}_k||^2\right\} \leq \frac{1}{\beta}E\left[V(\tilde{x}_0, 0)\right] < \infty.$$

Thus, the system is stochastically stable from Definition 1.

Now, to establish the H_∞ performance for the system, consider the following performance index:

$$J \triangleq E\left\{\sum\nolimits_{k=0}^{\infty}\left[e^T(k)e(k) - \gamma^2 w^T(k)w(k)\right]\right\}$$

under zero initial condition, $V(\tilde{x}(k), r_k)\mid_{k=0} = 0$, and we have

$$\begin{aligned}
J \leq\ & E\left\{\sum\nolimits_{k=0}^{\infty}\left[e^T(k)e(k) - \gamma^2 w^T(k)w(k) + \Delta V\right]\right\} \\
=\ & \sum\nolimits_{k=0}^{\infty}\zeta^T(k)\Phi_i\zeta(k)
\end{aligned}$$

where $\zeta(k) \triangleq \begin{bmatrix} \tilde{x}^T(k) & w^T(k) \end{bmatrix}^T$ and

$$\begin{aligned}
\Phi_i &\triangleq \begin{bmatrix} [\tilde{A}_i^T\bar{\mathcal{P}}_i\tilde{A}_i - P_i + \tilde{C}_i^T\tilde{C}_i] & \tilde{A}_i^T\bar{\mathcal{P}}_i\tilde{B}_i + \tilde{C}_i^T\tilde{D}_i \\ * & [-\gamma^2 I + \tilde{B}_i^T\bar{\mathcal{P}}_i\tilde{B}_i + \tilde{D}_i^T\tilde{D}_i] \end{bmatrix} \\
\bar{\mathcal{P}}_i &\triangleq \sum\nolimits_{j\in\mathcal{I}_{\mathcal{K}}^i}\pi_{ij}P_j + \sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^i}\pi_{ij}P_j \\
&= \mathcal{P}_{\mathcal{K}}^i + \sum\nolimits_{j\in\mathcal{I}_{\mathcal{UK}}^i}\pi_{ij}P_j
\end{aligned}$$

Note that $\Phi_i < 0$ is equivalent to:

$$\begin{bmatrix} \tilde{A}_i^T\bar{\mathcal{P}}_i\tilde{A}_i - P_i & \tilde{A}_i^T\bar{\mathcal{P}}_i\tilde{B}_i \\ * & -\gamma^2 I + \tilde{B}_i^T\bar{\mathcal{P}}_i\tilde{B}_i \end{bmatrix} - \begin{bmatrix} \tilde{C}_i^T \\ \tilde{D}_i^T \end{bmatrix}(-I^{-1})\begin{bmatrix} \tilde{C}_i & \tilde{D}_i \end{bmatrix} < 0.$$

By Schur complement, one has

$$\begin{bmatrix} -I & \tilde{C}_i & \tilde{D}_i \\ * & \tilde{A}_i^T\bar{\mathcal{P}}_i\tilde{A}_i - P_i & \tilde{A}_i^T\bar{\mathcal{P}}_i\tilde{B}_i \\ * & * & -\gamma^2 I + \tilde{B}_i^T\bar{\mathcal{P}}_i\tilde{B}_i \end{bmatrix} < 0.$$

Likewise, the above inequality is equivalent to:

$$\begin{bmatrix} -I & \tilde{C}_i & \tilde{D}_i \\ * & -P_i & 0 \\ * & * & -\gamma^2 I \end{bmatrix} - \begin{bmatrix} 0 \\ \tilde{A}_i^T \bar{\mathcal{P}}_i \\ \tilde{B}_i^T \bar{\mathcal{P}}_i \end{bmatrix} (-\bar{\mathcal{P}}_i^{-1}) \begin{bmatrix} 0 & \bar{\mathcal{P}}_i \tilde{A}_i & \bar{\mathcal{P}}_i \tilde{B}_i \end{bmatrix} < 0.$$

By Schur complement again, we have

$$\Xi_i \triangleq \begin{bmatrix} -\bar{\mathcal{P}}_i & 0 & \bar{\mathcal{P}}_i \tilde{A}_i & \bar{\mathcal{P}}_i \tilde{B}_i \\ * & -I & \tilde{C}_i & \tilde{D}_i \\ * & * & -P_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0. \qquad (12)$$

Note that (12) can be rewritten as

$$\begin{aligned} \Xi_i &= \begin{bmatrix} -\sum_{j\in\mathcal{I}_{\mathcal{K}}^i} \pi_{ij} P_j & 0 & \sum_{j\in\mathcal{I}_{\mathcal{K}}^i} \pi_{ij} P_j \tilde{A}_i & \sum_{j\in\mathcal{I}_{\mathcal{K}}^i} \pi_{ij} P_j \tilde{B}_i \\ * & -\left(\sum_{j\in\mathcal{I}_{\mathcal{K}}^i} \pi_{ij}\right) I & \sum_{j\in\mathcal{I}_{\mathcal{K}}^i} \pi_{ij} \tilde{C}_i & \sum_{j\in\mathcal{I}_{\mathcal{K}}^i} \pi_{ij} \tilde{D}_i \\ * & * & -\sum_{j\in\mathcal{I}_{\mathcal{K}}^i} \pi_{ij} P_i & 0 \\ * & * & * & -\left(\sum_{j\in\mathcal{I}_{\mathcal{K}}^i} \pi_{ij}\right) \gamma^2 I \end{bmatrix} \\ &\quad + \sum_{j\in\mathcal{I}_{\mathcal{UK}}^i} \pi_{ij} \begin{bmatrix} -P_j & 0 & P_j \tilde{A}_i & P_j \tilde{B}_i \\ * & -I & \tilde{C}_i & \tilde{D}_i \\ * & * & -P_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} \\ &= \begin{bmatrix} -\mathcal{P}_{\mathcal{K}}^i & 0 & \mathcal{P}_{\mathcal{K}}^i \tilde{A}_i & \mathcal{P}_{\mathcal{K}}^i \tilde{B}_i \\ * & -\pi_{\mathcal{K}}^i I & \pi_{\mathcal{K}}^i \tilde{C}_i & \pi_i \tilde{D}_i \\ * & * & -\pi_{\mathcal{K}}^i P_i & 0 \\ * & * & * & -\pi_{\mathcal{K}}^i \gamma^2 I \end{bmatrix} \\ &\quad + \sum_{j\in\mathcal{I}_{\mathcal{UK}}^i} \pi_{ij} \begin{bmatrix} -P_j & 0 & P_j \tilde{A}_i & P_j \tilde{B}_i \\ * & -I & \tilde{C}_i & \tilde{D}_i \\ * & * & -P_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} \end{aligned}$$

Therefore, inequalities (5) and (6) guarantee $\Xi_i < 0$, i.e., $J < 0$ which means that $\|e\|_{E_2} < \gamma \|w\|_2$, this completes the proof. □

Remark 2: Note that it is hard to use Lemma 1 to design the desired filter due to the cross coupling of matrix product terms among different system operation modes, as shown in (5) and (6). To overcome this difficulty, the technique using slack matrix developed in [11] can be adopted here to obtain the following improved BRL for system (4).

Lemma 2: Consider system (4) with partly unknown transition probabilities (2) and let $\gamma > 0$ be a given constant. If there exist matrix $P_i > 0$, and R_i, $\forall i \in \mathcal{I}$ such that

$$\begin{bmatrix} \boldsymbol{\Upsilon}_j - R_i - R_i^T & 0 & R_i \tilde{A}_i & R_i \tilde{B}_i \\ * & -I & \tilde{C}_i & \tilde{D}_i \\ * & * & -P_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \qquad (13)$$

where

$$\begin{cases} \boldsymbol{\Upsilon}_j \triangleq \frac{1}{\pi_{\mathcal{K}}^i} \mathcal{P}_{\mathcal{K}}^i, & \forall j \in \mathcal{I}_{\mathcal{K}}^i \\ \boldsymbol{\Upsilon}_j \triangleq P_j, & \forall j \in \mathcal{I}_{\mathcal{UK}}^i \end{cases} \qquad (14)$$

and $\mathcal{P}_{\mathcal{K}}^i$ is denoted in Lemma 1, then the filtering error system (4) is stochastically stable with an H_∞ performance index γ.

Proof: First of all, by Lemma 1, we conclude that system (4) is stochastically stable with an H_∞ performance index γ if the inequalities (5) and (6) hold. Notice that (5) can be rewritten as:

$$\begin{bmatrix} -\frac{1}{\pi_{\mathcal{K}}^i} \mathcal{P}_{\mathcal{K}}^i & 0 & \frac{1}{\pi_{\mathcal{K}}^i} \mathcal{P}_{\mathcal{K}}^i A_i & \frac{1}{\pi_{\mathcal{K}}^i} \mathcal{P}_{\mathcal{K}}^i B_i \\ * & -I & C_i & D_i \\ * & * & -P_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0. \qquad (15)$$

From the other side, for an arbitrary matrix $R_i, \forall i \in \mathcal{I}$, we have the following facts:

$$\begin{aligned} \left(\frac{1}{\pi_{\mathcal{K}}^i} \mathcal{P}_{\mathcal{K}}^i - R_i\right)^T \left(\frac{1}{\pi_{\mathcal{K}}^i} \mathcal{P}_{\mathcal{K}}^i\right)^{-1} \left(\frac{1}{\pi_{\mathcal{K}}^i} \mathcal{P}_{\mathcal{K}}^i - R_i\right) &\geq 0, \\ (P_j - R_i)^T P_j^{-1} (P_j - R_i) &\geq 0, \end{aligned}$$

then by using (14), one has

$$\boldsymbol{\Upsilon}_j - R_i - R_i^T \geq -R_i^T \boldsymbol{\Upsilon}_j^{-1} R_i.$$

Furthermore, from (13), we can obtain that

$$\begin{bmatrix} -R_i^T \boldsymbol{\Upsilon}_j^{-1} R_i & 0 & R_i \tilde{A}_i & R_i \tilde{B}_i \\ * & -I & \tilde{C}_i & \tilde{D}_i \\ * & * & -P_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0$$

Performing now a congruence transformation using $diag\{R_i^{-1}\boldsymbol{\Upsilon}_j, I, I, I\}$ yields (15) and (6) for $j \in \mathcal{I}_{\mathcal{K}}^i$ and $j \in \mathcal{I}_{\mathcal{UK}}^i$, respectively (note that R_i is invertible if it satisfies (13)). This completes the proof. □

Remark 3: Note that in Lemmas 1 and 2, the stochastic stability for the underlying system is actually guaranteed by the two aspects, i.e., efficiently utilizing the partly known transition probabilities (see (9)) together with the requirements that $V_j(\tilde{x}_{k+1}, k+1) - V_i(\tilde{x}_k, k) < 0, \forall j \in \mathcal{I}_{\mathcal{UK}}^i$ on the latent Lyapunov function $V_i(\tilde{x}_k, k) = \tilde{x}_k^T P_i \tilde{x}_k, \forall i \in \mathcal{I}$ (see (10), where if $j \neq i$, the time k will be the mode switching times).

B. H_∞ Filter Design:

The following Theorem presents sufficient conditions for the existence of an admissible mode-dependent H_∞ filter with the form (3).

Theorem 3: Consider system (1) with partly unknown transition probabilities (2) and let $\gamma > 0$ be a given constant. If there exist matrices $P_{1i} > 0$, and $P_{3i} > 0, \forall i \in \mathcal{I}$, and matrices P_{3i}, X_i, Y_i, Z_i, A_{fi}, B_{fi}, C_{fi}, D_{fi}, $\forall i \in \mathcal{I}$, such that

$$\left[\begin{array}{cccccc} \mathbf{\Upsilon}_{1j}-X_i-X_i^T & \mathbf{\Upsilon}_{2j}-Y_i-Z_i^T & 0 & X_iA_i+B_{fi}C_i & A_{fi} & X_iB_i+B_{fi}D_i \\ * & \mathbf{\Upsilon}_{3j}-Y_i-Y_i^T & 0 & Z_iA_i+B_{fi}C_i & A_{fi} & Z_iB_i+B_{fi}D_i \\ * & * & -I & H_i-D_{fi}C_i & -C_{fi} & L_i-D_{fi}D_i \\ * & * & * & -P_{1i} & -P_{2i} & 0 \\ * & * & * & * & -P_{3i} & 0 \\ * & * & * & * & * & -\gamma^2 I \end{array}\right] < 0 \tag{16}$$

where

$$\begin{cases} \mathbf{\Upsilon}_{1j} \triangleq \frac{1}{\pi_{\mathcal{K}}^i}\mathcal{P}_{\mathcal{K}}^{1i} \triangleq \frac{1}{\pi_{\mathcal{K}}^i}\sum_{j\in\mathcal{I}_{\mathcal{K}}^i}\pi_{ij}P_{1j} \\ \mathbf{\Upsilon}_{2j} \triangleq \frac{1}{\pi_{\mathcal{K}}^i}\mathcal{P}_{\mathcal{K}}^{2i} = \frac{1}{\pi_{\mathcal{K}}^i}\sum_{j\in\mathcal{I}_{\mathcal{K}}^i}\pi_{ij}P_{2j} \\ \mathbf{\Upsilon}_{3j} \triangleq \frac{1}{\pi_{\mathcal{K}}^i}\mathcal{P}_{\mathcal{K}}^{3i} = \frac{1}{\pi_{\mathcal{K}}^i}\sum_{j\in\mathcal{I}_{\mathcal{K}}^i}\pi_{ij}P_{3j} \end{cases}, \ \forall\ j\in\mathcal{I}_{\mathcal{K}}^i \tag{17}$$

$$\begin{cases} \mathbf{\Upsilon}_{1j} \triangleq P_{1j} \\ \mathbf{\Upsilon}_{2j} \triangleq P_{2j} \\ \mathbf{\Upsilon}_{3j} \triangleq P_{3j} \end{cases}, \qquad \forall\ j\in\mathcal{I}_{\mathcal{UK}}^i \tag{18}$$

Then, there exists a mode-dependent full-order filter such that the resulting filtering error system (4) is stochastically stable with an H_∞ performance under the Markovian Chain with partly unknown transition probabilities (2). Moreover, if the LMIs (16) have a feasible solution, the gains of an admissible filter in the form (3) are given by

$$\begin{aligned} A_{Fi} &= Y_i^{-1}A_{fi},\ B_{Fi}=Y_i^{-1}B_{fi}, \\ C_{Fi} &= C_{fi},\ D_{Fi}=D_{fi},\ i\in\mathcal{I}. \end{aligned} \tag{19}$$

Proof: Consider filtering error system (4) and assume the matrices P_i, R_i in Lemma 2 to have the following forms:

$$P_i \triangleq \left[\begin{array}{cc} P_{1i} & P_{2i} \\ * & P_{3i} \end{array}\right],\ R_i \triangleq \left[\begin{array}{cc} X_i & Y_i \\ Z_i & Y_i \end{array}\right]$$

then we have

$$\begin{aligned} \mathcal{P}_{\mathcal{K}}^i &\triangleq \sum\nolimits_{j\in\mathcal{I}_{\mathcal{K}}^i}\pi_{ij}P_j = \sum\nolimits_{j\in\mathcal{I}_{\mathcal{K}}^i}\pi_{ij}\left[\begin{array}{cc} P_{1j} & P_{2j} \\ * & P_{3j} \end{array}\right] \\ &\triangleq \left[\begin{array}{cc} \mathcal{P}_{\mathcal{K}}^{1i} & \mathcal{P}_{\mathcal{K}}^{2i} \\ * & \mathcal{P}_{\mathcal{K}}^{3i} \end{array}\right] \end{aligned}$$

Further define matrix variables

$$A_{fi}=Y_iA_{Fi},\ B_{fi}=Y_iB_{Fi},\ C_{fi}=C_{Fi},\ D_{fi}=D_{Fi}$$

$$\mathbf{\Upsilon}_j \triangleq \left[\begin{array}{cc} \mathbf{\Upsilon}_{1j} & \mathbf{\Upsilon}_{2j} \\ * & \mathbf{\Upsilon}_{3j} \end{array}\right]$$

where $\mathbf{\Upsilon}_{1j}$, $\mathbf{\Upsilon}_{2j}$ and $\mathbf{\Upsilon}_{3j}$ are denoted in (17) and (18) for $j\in\mathcal{I}_{\mathcal{K}}^i$ and $j\in\mathcal{I}_{\mathcal{UK}}^i$, respectively, one can readily obtain (16) replacing $\tilde{A}_i,\tilde{B}_i,\tilde{C}_i,\tilde{D}_i,\mathbf{\Upsilon}_j,P_i$ and R_i into (13), namely, if (16) hold, the filtering error system (4) will be stochastically stable with an H_∞ performance under the Markovian Chain with partly unknown transition probabilities (2). Meanwhile, if a solution of (16) exists, the parameters of admissible filter are given by (19). This completes the proof. □

Remark 4: By setting $\delta=\gamma^2$ and minimizing δ subject to (16), we can obtain the optimal H_∞ noise attenuation performance index γ ($\gamma=\sqrt{\delta}$) and the corresponding filter gains as well. Also, it can be deduced from (16) that, given different degree of unknown elements in the transition probabilities matrix, the optimal γ achieved for system (4) and the corresponding filter gains solved for system (2) should be different, which we will illustrate via a numerical example in next section.

IV. Numerical Example

Consider the MJLS (1) with four operation modes and the following data:

$$\begin{aligned} A_1 &= \left[\begin{array}{cc} 0 & -0.41 \\ 0.81 & 0.81 \end{array}\right], A_2=\left[\begin{array}{cc} 0 & -0.27 \\ 0.81 & 1.13 \end{array}\right], \\ A_3 &= \left[\begin{array}{cc} 0 & -0.81 \\ 0.81 & 0.97 \end{array}\right], A_4=\left[\begin{array}{cc} 0 & -0.19 \\ 0.81 & 0.89 \end{array}\right], \\ B_1 &= B_2=B_3=B_4=\left[\begin{array}{cc} 0.5 & 0 \\ 0 & 0 \end{array}\right], \\ C_1 &= C_2=C_3=C_4=\left[\begin{array}{cc} 1 & 0 \end{array}\right], \\ D_1 &= D_2=D_3=D_4=\left[\begin{array}{cc} 0 & 1 \end{array}\right], \\ H_1 &= H_2=H_3=H_4=D_1, \\ L_1 &= L_2=L_3=L_4=\left[\begin{array}{cc} 0 & 0 \end{array}\right]. \end{aligned}$$

The four cases for the transition probabilities matrix will be considered in this example as shown in Table I. Our purpose here is to design a mode-dependent full-order H_∞ filter in the form of (3) such that the resulting filtering error system is stochastically stable and has a guaranteed H_∞ performance. By solving (16), the optimal H_∞ performance indices are obtained for the four different transition probabilities cases. The corresponding computation results are listed in Table II.

From Table II, it is easily seen that the more transition probabilities knowledge the system has, the smaller performance index the system can achieve. Therefore, by means of our ideas and approaches, a tradeoff can be easily built in practice between the complexity to obtain transition probabilities and the system performance benefits.

The desired filter corresponding to the optimal H_∞ performance index can be also solved using (16), for brevity, the gains are omitted here. Applying the obtained filters and giving two possible time sequences of the mode jumps, we obtain the error response of the resulting filtering error systems in Figures 1-2 for given initial condition $x=[-1.2\ 0.6\ 0\ 0]^T$ and noise signal

$$w(k)=\left[\begin{array}{c} 0.7\exp(-0.1k)\sin(0.001\pi k) \\ 0.5\exp(-0.1k)\sin(0.01\pi k) \end{array}\right]$$

It is clearly observed from the simulation curves that for the above energy bounded disturbance $w(k)$, the filtering error system is stable against different partly unknown transition probabilities, which implies that our designed filter is feasible and effective.

TABLE I

DIFFERENT TRANSITION PROBABILITIES MATRICES

Completely known

	1	2	3	4
1	0.3	0.2	0.1	0.4
2	0.3	0.2	0.3	0.2
3	0.1	0.1	0.5	0.3
4	0.2	0.2	0.1	0.5

Partly unknown (case I)

	1	2	3	4
1	0.3	0.2	0.1	0.4
2	?	?	0.3	0.2
3	0.1	0.1	0.5	0.3
4	0.2	?	?	?

Partly unknown (case II)

	1	2	3	4
1	0.3	0.2	0.1	0.4
2	?	?	0.3	0.2
3	?	0.1	?	0.3
4	0.2	?	?	?

Completely unknown

	1	2	3	4
1	?	?	?	?
2	?	?	?	?
3	?	?	?	?
4	?	?	?	?

TABLE II

MINIMUM γ^* FOR DIFFERENT TRANSITION PROBABILITIES CASES.

Transition probabilities	Completely known	Partly unknown (Case I)	Partly unknown (Case II)	Completely unknown
γ^*	1.8556	3.8215	4.2793	4.4624

V. CONCLUSIONS

The H_∞ filtering problem for the discrete-time MJLS with partly unknown transition probabilities is investigated in this paper. The systems under consideration are more general than the MJLS with completely known or completely unknown transition probabilities as two special cases. The LMI-based BRL for the underlying filtering error system is derived and its improved version is further given by means of additional slack matrix variables to eliminate the cross coupling between the Lyapunov positive matrices and system matrices. Despite the partly unknown elements in the transition probabilities matrix, the mode-dependent full-order filter is designed and the existence conditions of the desired filter are obtained such that the resulting filtering error system is stochastically stable and has a guaranteed H_∞ performance index. A numerical example is given to illustrate the effectiveness and potential of the developed theoretical results.

VI. ACKNOWLEDGMENTS

The authors gratefully acknowledge the contribution of NSERC-Canada.

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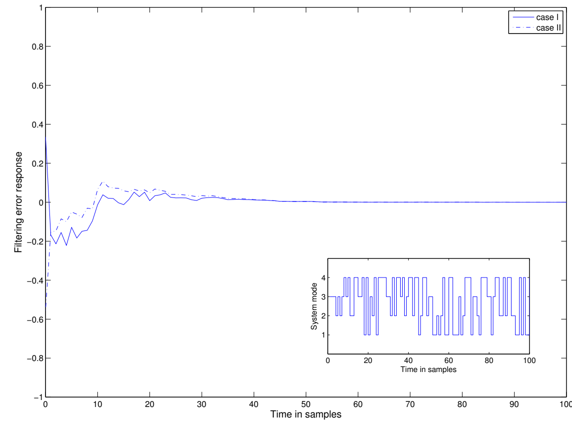

Fig. 1. Filtering Error Response for Mode Evolution r_k^1

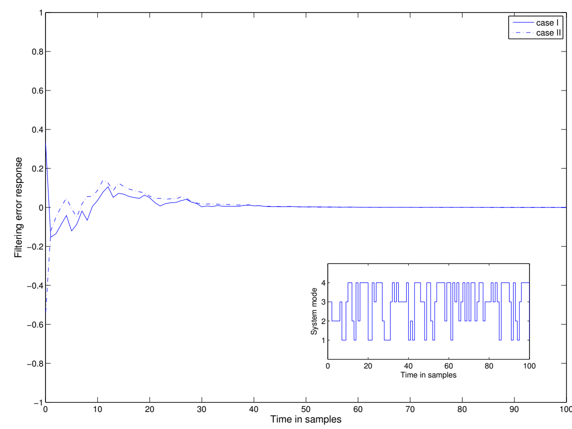

Fig. 2. Filtering Error Response for Mode Evolution r_k^2

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