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Optimal Filtering and Control for First Degree Polynomial Systems: Risk-Sensitive Method

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***Abstract*— The algorithms for the optimal filter and control have been obtained for systems with polynomial first degree drift term in the state and observations equations. Two cases are presented: systems with disturbances in L^2 and systems with Brownian motion and parameter ε multiplying both in the state and observation equations. The algorithms of the optimal risk-sensitive filter are obtained in each case and their performance is verified and compared with the algorithms of the optimal Kalman-Bucy filter through an example. The solution to the optimal control risk-sensitive problem for stochastic system, and log-exp-quadratic cost function to be minimized is obtained. This algorithms are obtained using value function as solution of PDE HJB. These algorithms are compared with the traditional control algorithms through numerical example. The optimal risk-sensitive filter and control show better performance for large values of the parameter ε.**

I. INTRODUCTION

Since the linear optimal filter was obtained by Kalman and Bucy (60's), numerous works are based on it. I could mention some as [2], [3],[12], [21], of the variety of all those. More than thirty years ago, Mortensen [16] introduced a deterministic filter model which provides an alternative to stochastic filtering theory. In this model, errors in the state dynamics and the observations are modeled as deterministic "disturbance functions," and a mean-square disturbance error criterion is to be minimized. In this case, special conditions for the existence, continuity and boundedness of $f(x(t))$ in the state equation, which is considered nonlinear, and for the linear function $h(x(t))$ in the observation equation, are given. A concept of the deterministic estimator, which is introduced more recently by McEneaney [13], is reviewed and applied to system with disturbances in L^2, where $f(x)$ has a nonlinear form in the dynamics of the system and linear observations. The problem statement uses a dynamical model of the form

$$\dot{X}(t) = f(X(t)) + w(t) \tag{1}$$

where X is the state, $f(x(t))$ represents the nominal dynamics, $w(t) \in L^2$ is a deterministic process. This model is in contrast to the the diffusion model

$$dX(t) \quad = \quad f(X(t))dt + \sqrt{\varepsilon}dW(t) \tag{2}$$

where W is a Brownian motion. The equations (1), (2) are introduced in [4] and [15] where $f(X(t))$ is a nonlinear function. This paper presents an application of the algorithms obtained in [4] and [15] for singular form of $f(x(t))$. The goal of this work is to obtain the optimal filter and control risk-sensitive equations for these models, when $f(x(t))$ and $h(x(t))$ take a polynomial of first degree form in the state and observation equation, respectively. The performance of the risk-sensitive optimal filter and control (stochastic case) algorithms is checked doing a comparison to the algorithms of the optimal Kalman-Bucy filter and traditional control through an example, for large values of ε. A long tradition of the optimal control design for nonlinear systems (see, for example, [1], [7], [11]) has been developed. Since the optimal linear control problem has been solved in 60's [9], [6], the basis of the optimal control theory is Dynamic Programming equation or Hamilton-Jacobi-Bellman equation [6], and the maximum principle of Pontryagin [17]. Following the theory of control and estimation, other method used in stochastic systems, is the finite time horizon case. Whittle [20] regarded it, using small noise asymptotic. When the process being controlled is governed by stochastic differential equation, the Whittle's formula for the optimal large-derivations rate was obtained using partial differential equation viscosity solution method in [5], [4], [15]. Runolfsson [18], used Ponsker-Varadham-type large-derivations ideas to obtain a corresponding stochastic differential game for which the game payoff is an ergodic (expected average cost per unit time) criterion. In this method is considerate the risk-averse stochastic problem and its solution is obtained taking in account a value function which is a viscosity solution to the dynamic programming equation (H-J-B)[14]. An advantage of risk-sensitive criteria is the robustness of the obtained solution with respect to noise level. Indeed, since the solution to the classical LQ problem is independent of noise level, it occurs to be too sensitive to parameter variations in noise intensity. On the other hand, the risk-sensitive problem assumes explicit presence of the small parameters in the criteria. This leads to a more robust solution, which correctly responds to parameter variations and results in close criterion values for both, large and small, parameter values. A future work is to obtain the risk-sensitive filtering and control algorithms when $f(x(t))$ takes other polynomial forms, as quadratic or cubic, and do a comparison with the polynomial filtering and control algorithms previously obtained. The performance of the obtained risk-sensitive regulator and filter for stochastic first degree polynomial systems is verified in a numerical example against the conventional linear-quadratic regulator and Kalman-Bucy filter, through comparing criteria values for both regulators and the estimator error, in both filters, for large values of ε, respectively. This work is organized as follows: The problem statement for system with disturbances

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in L^2 and for systems with Brownian motion is presented in Section II.A and II.B. In Section III is given the solution. The general form of the optimal risk-sensitive stochastic non linear control problem is given in Section IV.In Section V is presented the optimal control r-s solution. In Section VI an numerical example is solved applying the risk-sensitive optimal filter algorithms and the algorithms of Kalman-Bucy optimal filter; risk-sensitive and traditional control. Section VII are the conclusions.

II. Filtering problem statement

A. Deterministic case

For the first case, the state to be estimated $x(t)$ has differential equation (1) where $w(t)$ is the state disturbance and $x(0) = x_o$. The observation equation is given by:

$$y(t) = h(x(t)) + v(t) \tag{3}$$

where $v(t) \in \mathrm{L}^2$ is the observation disturbance. Where $x(t) \in R^n, w(t) \in R^m, y(t), v(t) \in R^p, f, h \in R^n$ with f_x, h_x bounded is assumed throughout. Here h_x is the matrix of partial derivatives of h and in the same form for Z_x. Taking $f(x(t)) = A(t) + A_1(t)x(t)$, $h(x(t)) = E(t) + E_1(t)x(t)$, with $A(t) \in R^n, A_1(t), \in M_{n\times n}, E(t) \in R^p, E_1(t) \in M_{n\times p}$ where $M_{i\times j}$ denotes the field of matrices of dimension $i \times j$. The following system equations is obtained:

$$\begin{aligned} \dot{X}(t) &= A(t) + A_1(t)x(t) + w(t), \\ y(t) &= E(t) + E_1(t)x(t) + v(t). \end{aligned} \tag{4}$$

Taking in account the state equation (1) and the observation equation (3), replacing the observation trajectory $y.$ by an accumulated observation trajectory: $Y(t) = \int_0^t y_s ds\cdot$ In [4], you can see that taking in account the accumulated observations, the function $J(T,x;w)$ has the form:

$$\begin{aligned} &\mathscr{J}(T,x;w.) = -\{\phi(x_0) + \int_0^T [\frac{1}{2}|w(t)|^2 + \\ &\frac{1}{2}|h(x(t))|^2 + (Y(t)\cdot h(x(t)))_x \cdot (f(x(t)) + w(t))]dt\}, \end{aligned} \tag{5}$$

$Z(T,x) = sup_w \mathscr{J}(T,x;w)$, then, the value function associated is given by:

$$W(T,x) = Y(T)\cdot h(x) + Z(T,x), \tag{6}$$

it is shown [4] that $Z(T,x)$ is continuous, and that Z is a viscosity solution of the dynamic programming PDE:

$$\begin{aligned} &-Z_T - f\cdot Z_x - \frac{1}{2}|h|^2 - (Y(T)\cdot h)_x \cdot f + \frac{1}{2}|(Z_x + \\ &(Y(T)\cdot h)_x)\sigma|^2 = 0, \quad Z(0,x) = -\phi(x). \end{aligned} \tag{7}$$

As was proposed in [13], and taking in account [4], in this case, $W(T,x)$ takes the form $W(T,x) = \frac{1}{2}(X - C(T))^T Q(T)(X - C(T)) + \rho_T + \frac{1}{2}\int_0^T /y(t)/^2 dt$, where $C(T)$ denotes the estimate vector, Q_T is a quadratic, positive definite symmetric matrix and ρ is a parameter with values in final time T. The filtering problem is to find the best estimate of the state $x(t)$, which minimizes the quadratic criterion (5), where $Z(T,x)$(6), is a viscosity solution of (7).

B. Filtering Stochastic case

Consider the following stochastic model (2), in which $X(t)$ denotes the state process. $X(t)$ satisfies (2), and the equation for $Y(t)$is given by:

$$dY_t = (E(t) + E_1(t)x(t))dt + \sqrt{\varepsilon}d\tilde{B}(t), \quad Y_0 = 0, \tag{8}$$

where ε is a parameter and B and $\tilde{B}$ are independent Brownian motions in themselves and both are independent of the initial state X_0. X_0 has probability density $k_\varepsilon exp(-\varepsilon^{-1}\phi(x_0))$ for some constant k_ε. The rest of the paper are verify assumptions (A1)-(A5) (from [4]). Besides, it is assumed that

$$\begin{aligned} q^\varepsilon(0,x) &= exp(-\varepsilon^{-1}\phi(x)) \\ q^\varepsilon(T,x) &= p^\varepsilon(T,x)exp[\varepsilon^{-1}Y(T)\cdot h(x)] \end{aligned} \tag{9}$$

where $p^\varepsilon(T,x)$ is called pathwise unnormalized filter density. Taking log transform: $Z^\varepsilon(T,x) = \varepsilon log p^\varepsilon(T,x)$, which satisfies the nonlinear parabolic PDE

$$\frac{\partial Z^\varepsilon}{\partial T} = \frac{\varepsilon}{2}tr(Z^\varepsilon_{xx}) + A^\varepsilon \cdot Z^\varepsilon + \frac{1}{2}Z^\varepsilon_x \cdot Z^\varepsilon_x + B^\varepsilon, \tag{10}$$

with initial data $Z^\varepsilon_x(0,x) = -\phi(x)$. The risk-sensitive optimal filter problem consists in found the estimate $C(T)^\varepsilon$, of the state $x(t)$ through verification that

$$\begin{aligned} Z^\varepsilon(T,x) &= \frac{1}{2}(x - C(T)^\varepsilon)^T Q(T)^\varepsilon (x - C(T)^\varepsilon) + \rho^\varepsilon_T \\ &\quad -Y(T)\cdot h(x(t)) \end{aligned} \tag{11}$$

is a viscosity solution of (10), and Q^ε_T is Riccati matrix equation ($Q(T)^\varepsilon$is symmetric matrix). In [4] it is proved, that the equation (10)(stochastic case) converges to the equation (7)(deterministic case) as ε goes to zero. Substituting $f(x(t)), h(x(t))$ in (2) as in deterministic case (4), the next stochastic equations system is obtained:

$$\begin{aligned} dX(t) &= A(t) + A_1(t)X(t) + \sqrt{\varepsilon}dB(t) \\ dY(t) &= E(t) + E_1(t)X(t) + \sqrt{\varepsilon}d\tilde{B}(t), \end{aligned} \tag{12}$$

where $A(t), A_1(t), E(t), E_1(t)$ are the same as in (4). The filtering problem is the same as Section A.

III. Filtering Solution

Taking in account the system of state and observation equations (4), the partial derivatives of (6) are obtained. Upon substituting into (7) and collecting x terms, the next filter equation is obtained, where $C(T)$ denotes the estimator of $x(t)$, which is the solution of the next differential equation:

$$\begin{aligned} dC(T) &= (A(t) + A_1(t)C(T))dt - Q(T)^{-1}E_1(t)(dY(t) - \\ &\quad (E_1(t)^T C(T) + E(t))dt), C_0 = c_o. \end{aligned} \tag{13}$$

where the symmetric matrix $Q(T)$ is obtained collecting x^2 terms and is the solution of the next Riccati matrix equation:

$$\begin{aligned} \dot{Q}(T) &= -A_1 Q(T) - Q(T)A_1^T + \\ &\quad (Q(T))^2 - E_1(t)^T E_1(t), \quad Q_0 = q_o. \end{aligned} \tag{14}$$

Here $Q(T)^\varepsilon$ is a symmetric matrix and the initial condition $Q^\varepsilon_0 = q^\varepsilon_o$ is gotten from the equilibrium condition $\dot{Q}(T)^\varepsilon = 0$. Where if $\overline{Q}$ is one solution. Then, should be $q^\varepsilon_0 \leq \overline{Q}$, for which $\ddot{Q}^\varepsilon(\overline{Q}) < 0$, where $\overline{Q}$ is one of two equilibrium points. Taking $Z^\varepsilon(T,x)$ and following the steps of the deterministic case, it is easy to verify that the equation for the optimal risk-sensitive stochastic estimator is the same obtained in the deterministic case.

IV. Control problem statement

The next stochastic risk-sensitive control problem has dynamics:

$$dX(t)^{\varepsilon} = f(t,X(t)^{\varepsilon},u(t))dt + \sqrt{\frac{\varepsilon}{2\gamma^2}}dB(t), \quad (15)$$

$X(s)^{\varepsilon} = x$, and log-exp-quadratic cost criterion

$$I(s,x,u) = \varepsilon log E_{s,x} exp\{\frac{1}{\varepsilon}[\int_s^T L(t,X(t)^{\varepsilon},u)dt + \psi(X(T))]\}. \quad (16)$$

Where $f(t,X(t),u(t))$ is a nonlinear function which represents the nominal dynamics with control $u(t)$ taking values in $U \in \mathbf{R}^l$ and $\{B, F\}$ is an m-dimensional Brownian motion on the probability space (Ω,F,P) where F_0 contains all the P-negligible elements of F. ε will be a measure of the risk-sensitivity and scales the diffusion term in (15) above so that the cost below will remain bounded (for each x as a function of ε), $0 \le s \le T < \infty, T$ is a fixed terminal time, $L(t,X(t)^{\varepsilon},u(t))$ is the quadratic running cost, and $\psi(X(T))$ is the quadratic terminal cost. We define the following cost functions:

$$A^{\varepsilon}(s,x,u,\omega) = \int_s^T L(t,X(t),u)dt + \psi(X(T))$$

$$J^{\varepsilon}(s,x,u) = E_{s,x} exp[\frac{1}{\varepsilon}A^{\varepsilon}(s,x,u,\omega)]. \quad (17)$$

So that

$$I^{\varepsilon}(s,x,u) = \varepsilon log J^{\varepsilon}(s,x,u) = \varepsilon log E_{s,x} exp[\frac{1}{\varepsilon}A^{\varepsilon}(s,x,u,\omega)].$$

Taking in account that the controller $u(t)$ is minimizing, and $w \in \mathbf{R}^n$ is a maximizing control, the following value functions are considered:

$$V^{\varepsilon}(s,x) = inf_{u \in A_{s,v}} I^{\varepsilon}(s,x,u) \quad (18)$$

where $A_{s,v}$ is the set of progressively measurable controls with values in U.

$$\varphi^{\varepsilon}(s,x) = inf_{u \in A_{s,v}} J^{\varepsilon}(s,x,u) \quad (19)$$

The proof can be found in [15] under certain conditions, where $f(t,X(t),u(t))$ is a nonlinear function, V^{ε} is a viscosity solution of the dynamical programming equation

$$\begin{aligned} 0 &= V^{\varepsilon}_s + \frac{\varepsilon}{2\gamma^2}\sum V^{\varepsilon}_{x_i x_j} + min_{u \in U}\{f(t,X(t)^{\varepsilon}, \\ &\quad u(t))\nabla_x V^{\varepsilon} + L(t,X(t),u(t)) + \frac{1}{2\gamma^2}\nabla V^{\varepsilon T}\nabla V^{\varepsilon}\} \\ V^{\varepsilon}(x(t),T) &= \psi(X). \end{aligned} \quad (20)$$

The next lemma shows that when $f(t,X(t)^{\varepsilon},u(t)) = A(t) + A_1(t)X(t)^{\varepsilon} + u(t)$, V^{ε} is a viscosity solution of the dynamical programming equation(20). Taking $V^{\varepsilon} = \varepsilon log \varphi^{\varepsilon}$, and substituting in (20) it is obtained the equation for φ^{ε} :

$$\begin{aligned} 0 &= \varphi^{\varepsilon}_s + \frac{\varepsilon}{2\gamma^2}\sum \varphi^{\varepsilon}_{x_i x_j} + min_{u \in U}\{f(t,X(t)^{\varepsilon}, \\ &\quad u(t))\nabla_x \varphi^{\varepsilon} + L(t,X(t),u(t))\phi^{\varepsilon}\}, \varphi^{\varepsilon}(x(t),T) = \psi(X) \end{aligned} \quad (21)$$

The optimal control problem is to show that V^{ε} is a viscosity solution to the dynamic programming equation (20) when $f(t,X(t)^{\varepsilon},u(t))$ is polynomial of first grade, to find the optimal control which minimize the quadratic criterion J and find the optimal trajectory x^*, substituting u^* in to the state equation (15). The conditions for f,L,φ,U proposed in [15] are true when $f(t,X(t)^{\varepsilon},u(t))$ takes the form proposed previously. As in [15], "cut off" problem is important, because the possibility unbounded functions f,L and ψ are replaced by bounded counterparts f^k, L^k and ψ^k in (20) and (21). The next lemma provides of proof that $V^{\varepsilon,k}$ is the unique, bounded, classical solution to (20), taking in account that $f(t,X(t),u(t))$ polynomial of first degree, the proof for $f(t,X(t),u(t))$ non linear can see in [15].

Lemma The solution to (20) is the value function $V^{\varepsilon,k}$ and the solution to (21) is the value function $\varphi^{\varepsilon,k}$. An admissible feedback solution exists which yields the minimum. Furthermore, $V^{\varepsilon,k}$ is the unique, bounded, classical solution to (20).

Proof: Let φ a solution of (21). The first part is to show $\varphi(s,x) \le J^{\varepsilon,k}(s,x,u)$, for all $u \in A_{s,v}$ and $(s,x) \in Q(T) = [0,T] \times \mathbf{R}^n$. For a fixed F_t- progressively measurable control, the solution to the stochastic differential equation, x^k (which would be denoted as x throughout the remainder of this proof) is a continuous semi-martingale, with in fact, square-integrable martingale part. Thus, since $\varphi \in C^{1,2}$, and applying Itô's rule to yield:

$$\varphi^k(t,x(t)) = \varphi^k(s,x) + \int_s^t (\frac{\partial \varphi^k(s,x(r))}{\partial s} + \frac{\partial \varphi(s,x(r))}{\partial x(r)} \times f^k(x(r)) + \frac{\varepsilon}{4\gamma^2}\frac{\partial^2 \varphi}{\partial x(r)^2})dr + \sqrt{\frac{\varepsilon}{2\gamma^2}}\int_s^t \frac{\partial \varphi^k(s,x(r))}{\partial s}dB(r), \quad (22)$$

if

$$\frac{\partial \varphi^k(s,x(r))}{\partial s} = \varphi^k_r, \frac{\partial \varphi^k(s,x(r))}{\partial x(r)} = \bigtriangledown \varphi^k, \frac{\partial^2 \varphi^k(s,x(r))}{\partial x_i(r)\partial x_j(r)} = \triangle \varphi^k,$$

then

$$\varphi^k(t,x(t)) = \varphi^k(s,x) + \int_s^t (\varphi^k_r(r,x(r)) + \bigtriangledown \varphi^k(r,x(r))f^k(r, x(r),u(r)) + \frac{\varepsilon}{4\gamma^2}\triangle \varphi^k(r,x(r)))dr + \sqrt{\frac{\varepsilon}{2\gamma^2}}\int_s^t \bigtriangledown \varphi^k(r,x(r))dB(r),$$

where

$$\int \bigtriangledown \varphi(r,x(r))dB(r) = \sum \int \varphi^k_{x_i}(r,x(r))dB(r)^{(i)}. \quad (23)$$

Since the lemma 2.3.1 in [15] states that the solution to (20) is bounded, we have from Ladyženskaja et al. [10], theorem 5.3.1 that $|\bigtriangledown V|$ is also bounded. Consequently the solution to (21) satisfies $|\varphi|, |\bigtriangledown \varphi|$ bounded also. Then exists a such that $|\bigtriangledown \varphi| \le a$ for all $(s,x) \in Q_T$ which implies $E\int_s^T |\bigtriangledown \varphi|^2 dt \le a^2 T$, and consequently $\int_s^t \bigtriangledown \varphi(r,X(r)) \cdot dB_r$ is a square-integrable martingale. Thus $\varphi(t,X(t))$ is also a continuous semi-martingale. Since L^k is bounded, $\varepsilon_t \equiv exp[\frac{1}{\varepsilon}\int_s^t L^k(r,X(r),u(r))dr]$ is a continuous semi-martingale with zero martingale part. Therefore we may apply the stochastic integration by parts formula to the product $\varepsilon_t \varphi(t,x(t))$ to yield:

$$\varepsilon_t \varphi^k(t,X(t)) - \varphi^k(s,x) = \int_s^t \varepsilon_r[\frac{\partial \varphi^k}{\partial r} + \frac{\partial \varphi^k}{\partial x} \cdot f^k(x(t)) + \frac{\varepsilon}{4\gamma^2}\frac{\partial^2 \varphi}{\partial x^2}]dr + \int_s^t \varphi^k(r,x(r))\frac{1}{\varepsilon}L^k(r,X(r),u(r))\varepsilon_r dr + \sqrt{\frac{\varepsilon}{2\gamma^2}}\int_s^t \varepsilon_r \frac{\partial \varphi^k}{\partial x}dB(t). \quad (24)$$

Using PDE (21) to eliminate the first two terms on the right, we have:

$$\varepsilon_t \varphi^k(t,X(t)) - \varphi^k(s,x) \ge \sqrt{\frac{\varepsilon}{2\gamma^2}}\int_s^t \varepsilon_r \frac{\partial \varphi^k}{\partial x}dB(t).$$

Since L^k and $\bigtriangledown\varphi^k$ are bounded, so is $\varepsilon_r \bigtriangledown \varphi(r,X(r))$. Thus, by the same argument as above, $\int_s^t \varepsilon_r \bigtriangledown \varphi^k(r,x(r))dB_r$ is a square-integrable martingale. Therefore, taking $t=T$,

$$\varphi^k(s,x) \leq E_{s,x}[\varepsilon_T \varphi(T,X(T))],$$

and substituting ε and the terminal condition in (21), we have:

$$E_{s,x}[\varepsilon_T \varphi(T,X(T))] = E_{s,x}exp\{\frac{1}{\varepsilon}[\int_s^T L^k(t,X(t),u(t))dt + \psi(X(T))]\} = J^{\varepsilon,k}(s,x,u). \quad (25)$$

Now suppose there exists $u^* \in A_{s,v}$ such that

$$u^* \in argmin_{u\in U}[f^k(t,X(t)^*,v)\bigtriangledown\varphi(t,X(t)^*) + \frac{1}{\varepsilon}L^k(t,X(t),u(t))\varphi(t,X^*)], \forall t\in[s,t].$$

Then the equality in the above is rigth, and consequently,

$$\varphi(s,x) = J^{\varepsilon,k}(s,x,u^*).$$

It is easily seen from [6], Appendix B, that exists a Borel measurable function $g(t,x)$ such that:

$$g(t,x) \in argmin_{u\in U}[f^k(t,x(t),u)\varphi^k_{x_i}(x,t) + \frac{\varphi^k(t,x)}{\varepsilon}L^k(t,X(t),u(t))], \forall(t,x)\in Q_T. \quad (26)$$

Consider the SDE:

$$dX(t) = f^k(t,X(t),g(t,X(t)))dt + \frac{\varepsilon}{2\gamma^2}dB(t). \quad (27)$$

By Veretennikov [19], Theorem 1, it has a unique strong solution for any reference probability system, ν. Letting $u^* = g(t,x(t))$ for the strong solution yields $u^* \in A_{s,\nu}$. Therefore

$$\varphi(s,x) = min_{u\in A_{s,\nu}} J^{\varepsilon,k}(s,x,u) = \varphi^{\varepsilon,k}(s,x)$$

To prove the uniqueness claim, suppose there exists another bounded, classical solution, $\tilde{\varphi}$. Then, by the same proof as above, it is equal to the value function $\varphi^{\varepsilon,k}$. The result for φ is proved. The result for V follows similarly.$\diamondsuit$.

V. Control solution

Taking in account that $f(t,X(t)^\varepsilon,u(t)) = A(t) + A_1(t)X(t) + b(t)u(t)$ and substituting in (15), the next state equation is obtained:

$$dX(t)^\varepsilon = (A(t) + A_1(t)X(t) + b(t)u(t))dt + \sqrt{\frac{\varepsilon}{2\gamma^2}}dB(t) \quad (28)$$

$X_s^\varepsilon = x$, where $X(t),u(t),A(t) \in \mathbf{R}^n, A_1(t) \in M_{nxn}$, where M_{nxn} denotes the field of matrices of dimension *nxn*, and $dB(t)$ is as in (15), $u(t)$ is the control input. Let $L(t,X(t)^\varepsilon,u) = X_t^T G X_t + u_t^T R u_t$, the exponential-quadratic cost criterion has the form:

$$I(s,x,u) = \varepsilon log E_{s,x}\{exp[\frac{1}{\varepsilon}[\int_s^T (X_t^T G X_t + u_t^T R u_t)dt + X_T^T \psi X_T]]/Y_t\} \quad (29)$$

G,ψ are non-negative symmetric matrices, R is a positive definite symmetric matrix. As is proposed in [13], in this case, the value function

$$V^\varepsilon(s,X) = \frac{1}{2}(X(t)-C(s))^T P(s)(X(t)-C(s)) + r(s) \quad (30)$$

$(C(s),P(s),r(s)$ are functions of $s\in[0,T], C(s)\in\mathbf{R}^n, P(s)$ is a symmetric matrix of dimension nxn and $r(s)$ is a scalar function) as a viscosity solution of the dynamic programming equation

$$0 = V_s + \frac{\varepsilon}{2\gamma^2}\sum V_{x_i x_j} + min_{u\in U}\{(A + A_1(t)X(t) + u(t))\nabla_x V + X(t)^2 + u(t)^2 + \frac{1}{2\gamma^2}\nabla V^T \nabla V\} \quad (31)$$

$V(x(t),T) = \psi(X)$ where V_s, V_x are the partial derivatives of V respect to s,x respectively and ∇V is the gradient of V. Then the partial derivatives of V^ε are obtained and substituting these in (31), when $f(t,x(t),u(t))$ is polynomial of first degree, following the steps as in the filter, the r-s control solution is obtained:

$$\begin{aligned}
\dot{P}(t) &= P(t)^T(\frac{b(t)R^{-1}b(t)^T}{2} - \frac{1}{\gamma^2})P(t) \quad (32)\\
&\quad -A_1^T(t)P(t) - P(t)A_1(t) - 2G\\
\dot{C}(t) &= A_1^T(t)C(t) + 2C(t)^T P(t)^{-1} + A(t)\\
\dot{r}(s) &= -C(t)^T G C(t) - \frac{\varepsilon}{2\gamma^2}\sum_{i=1}^n P_{ij}
\end{aligned}$$

with terminal conditions: $P(T) = \psi, C(T) = 0$ and from (31), the optimal control law which minimizes the quadratic criterion is given by:

$$u^*(t) = -\frac{1}{2}P(t)b^T(t)R^{-1}(X - C(t)) \quad (33)$$

VI. Applications

A. Risk-sensitive optimal filter

For the dynamical system (12), if $f(x(t)) = 1 - 0.1x(t)$, $h(x(t)) = 1 + x(t)$, the following stochastic state and observation equations are obtained:

$$\begin{aligned}
dx(t) &= (1 - 0.1x(t))dt + \sqrt{\varepsilon}dB(t), \quad (34)\\
dy(t) &= (1 + x(t))dt + \sqrt{\varepsilon}d\bar{B}(t)
\end{aligned}$$

where $x(t)\in R, dB(t), d\bar{B}(t)$ are independent Brownian motions, $\varepsilon = 1000000$. Proposing (11) as a viscosity solution of (10), getting the derivatives $Z_x^\varepsilon, Z_{xx}^\varepsilon, \frac{\partial Z^\varepsilon}{\partial T}$ of (11) and substituting in (10), the next equations are obtained for the estimate $C(T)^\varepsilon$ and for the symmetric matrix Q_T, which are equivalent to substituting the corresponding values in (13) and (14):

$$\begin{aligned}
\dot{Q}(T) &= 0.2Q(T) + Q^2(T) - 1 \quad (35)\\
dC(T)^\varepsilon &= (1 - (0.1)C(T)^\varepsilon)dt - \frac{1}{Q_T}(dY(T) - C(T)^\varepsilon dt)
\end{aligned}$$

The last equations (35) are simulated using *MatLab*7. The initial conditions for the simulation are $x_0 = y_0 = 0$, $Q_0^\varepsilon = -0.0001$, $C(T)^\varepsilon = 1000$, $T = 10seg$. The graph of the absolute values of the difference between state $x(t)$, and the estimate $C(T)^\varepsilon$, that is: $error = |x(t) - C(T)^\varepsilon|$, is shown in Figure 1.

Time	R-s estimator error	K-B estimator error
$0.2seg$	0	944.1
$2seg$	16.11	244
$4seg$	8.8	40
$6.250seg$	0	3.262
$6.7552seg$	1.694	0

TABLE I

COMPARISON OF ERRORS R-S AND K-B

B. Kalman-Bucy optimal filter equations.

Applying the Kalman-Bucy optimal filter algorithms [8] to the state equations (34), the equations for the estimate vector $m(t)$ and symmetric covariance matrix $P(t)$ are obtained:

$$\begin{aligned} dm(t) &= (-0.1m(t)+1)dt + \frac{P}{\varepsilon}(dY-(m(t)+1)dt) \\ \dot{P}(t) &= -0.2P(t)+\varepsilon-\frac{P^2(t)}{\varepsilon} \end{aligned}$$

This system of equations is simulated with the initial conditions: $m(0)=1000,\ P(0)=10000$. The graph of the absolute value of the difference between state $x(t)$, and the estimate $m(t)$, that is: $error = |x(t)-m(t)|$, can be seen in Figure 2. Table 1 presents some values of the risk-sensitive error and Kalman-Bucy error, you can see that the risk-sensitive error converges to zero in less time that the Kalman-Bucy error. Besides you can see that the values of K-B estimator error are monotonous, which is hopped, because ε is into the estimation equation, which increase their value when ε increase, while the r-s estimator works for large values of ε, and the values of the r-s estimation error are not monotonous.

C. Optimal r-s stochastic control

Give the next linear stochastic state equation:

$$dx(t) = (1+0.1x(t)+u(t))dt + \sqrt{\frac{\varepsilon}{2\gamma^2}}dB(t) \qquad (36)$$

$$L(t,x(t),u(t)) = x(t)^2+u(t)^2; \psi(x(t)) = x(t)^2$$

where $A(t)=1, A_1(t)=0.1, \sigma=1\ \ \varepsilon=0.01, \gamma=2$. The values of ε,γ are obtained of (38). The quadratic cost criterion takes the form:

$$J(s,x(t),u(t)) = \varepsilon log E_{s,x} exp(\frac{1}{\varepsilon}\int_s^T (x(t)^2+u(t)^2)dt + x(T)^2) \quad (37)$$

Substituting the values of A, A_1 into the equations (32), and (33), are obtained the next equations in reverse time:

$$\begin{aligned} \frac{dP}{dt} &= -0.2P(s)-2+P(s)^2(\frac{1}{2}-\frac{1}{\gamma^2}) \\ \frac{dC(s)}{dt} &= 1+(0.1)C(s)+2\frac{C(s)}{P(s)} \qquad (38) \\ u* &= -\frac{1}{2}P(s)(x-C(s)). \end{aligned}$$

The system (38), is stable if $|\gamma|\geq 1.40$. The final conditions in $T=5seg$ are: $P(5)=1, C(5)=0$, the initial condition for $x(0)=0; \gamma=2$. Solving this system of equations (38), the values of the optimal control law u^*, the optimal trajectory: $\dot{x}^* = (1+(0.1)x(t)-1/2P(s)(x-C(s)))+\sqrt{(\varepsilon/2\gamma^2)}dB(t)$, are obtained, substituting the optimal control u^* in to the state equation (36). The value of the criterion quadratic to be minimized J at time T is obtained. The graph of the state $x(t)$, the optimal control $u(t)$, the criterion J can be seen in the Figure 4. The value of J was approximated using Monte Carlo method.

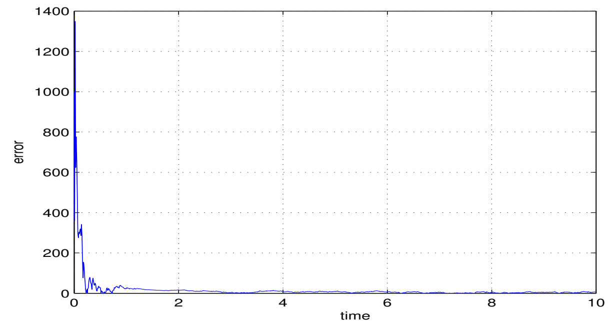

Fig. 1. Graphs of the absolute values of the difference between $x(t)$ and the linear r-s estimate $C(T)$.

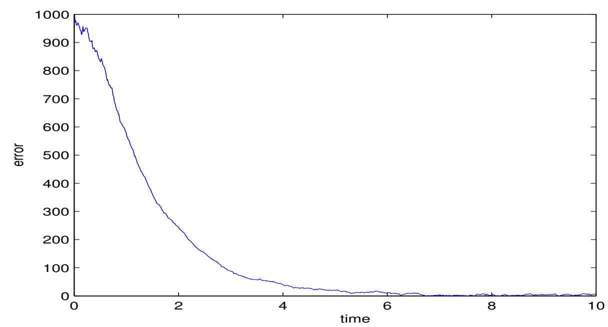

Fig. 2. Graphs of the absolute values of the difference between $x(t)$ and the linear Kalman-Bucy estimate m .

D. Optimal Linear Quadratic Control

Taking in account the state equation (36), the traditional optimal non homogeneous control [6] is obtained: $u(t)^* = R^{-1}b^T(t)(Q(t)x(t)+p(t))$, where $Q(t)$ is the solution of the gain equation:

$$\dot{Q(t)} = -0.2Q(t)+1-Q^2(t)$$

and $p(t)$ is the solution of: $\dot{p}(t) = -Q-0.1p(t)-Q(t)p(t)$ with final conditions: $Q(5)=-2, p(5)=0$ The optimal trajectory takes the form: $dx(t) = (1+(0.1)x(t)+(Q(t)x(t)+p(t)))dt+\sqrt{(\varepsilon/2\gamma^2)}dB(t)$. The quadratic criterion to be minimized is the same in both controls. The graphics of the state, optimal control and criterion, for $\varepsilon=1000$ can see in Figure 3.

VII. CONCLUSIONS

The equations of the risk-sensitive optimal filter and control when the drift term is polynomial of first grade in state and observation equations (for the filtering case) and quadratic criterion (for the control case) are obtained. These algorithms are valid when the disturbances in the

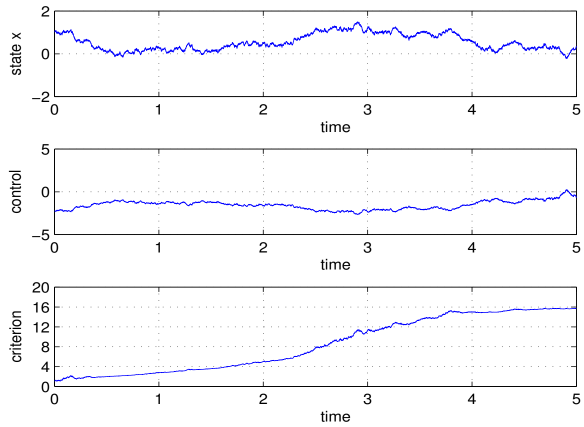

Fig. 3. Graphs of the optimal state variable $x(t)$, the optimal control u^* and the criterion J for the L-Q control.

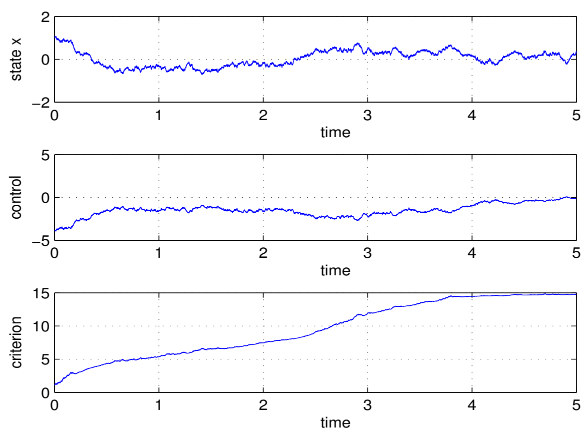

Fig. 4. Graphs of the state variable $x(t)$, the optimal control u, and the criterion J for the risk-sensitive control.

ε	J(r-s control)	J(trad. control)
0.1	9.875	7.9152
1	9.8982	8
10	10	8.278
100	10.5616	9.4371
1000	14.8716	15.774
10000	53.7456	62.5252
100000	429.1228	477.4969

TABLE II

VALUES OF J, FOR SOME ε VALUES, WITH ALGORITHMS RISK-SENSITIVE AND L-Q CONTROL

dynamical system are in L^2, and when Brownian motion is present in the dynamical system. A numerical example is solved in the stochastic case with multiplier ε on the weak derivative of Brownian motion and with parameter γ in the control problem. The optimal filter risk-sensitive algorithms and Kalman-Bucy optimal filter are obtained, and compared. When ε grows, the estimate risk-sensitive converges in less time to the real value than the Kalman-Bucy estimate, as shown in Figure 1 and 2. The optimal control risk-sensitive algorithms and traditional optimal control are obtained, and compared, using the log-exp-criterion quadratic of risk-sensitive method. When ε takes small values, the performance of traditional control is verify, when ε grow, the performance of risk-sensitive control is verify (values of J are lowest).

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