2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Design of Proportional Integral Adaptive Observers

Elnaz Vahedforough and Bahram Shafai

***Abstract*— This paper generalizes the conventional Proportional Adaptive Observer (PAO) to the case of Proportional Integral Adaptive Observer (PIAO). A systematic design procedure is provided for estimating the states and parameters of a dynamic system using PIAO. The proof of convergence is given and simulation results demonstrate the validity of theoretical development.**

I. INTRODUCTION

Adaptive observers were first introduced by Carroll and Lindorff in [1], and later on modified by Kudva and Narendra in [2]. For a more detailed analysis and design of adaptive systems see [3], [4], [5]. In many applications where the parameters are unknown and states are not accessible, adaptive observer appears to be a valuable method in estimation of both parameters and states of the system. Literature reports several useful applications in which adaptive observers have successfully been employed [6] - [11]. In almost all of these applications the conventional structure of adaptive observer (PAO) is used.

A new type of observer, known as proportional Integral Observer (PIO) has been extensively used over the past several years for various purposes. The PIO has been originally introduced in connection to LTR problem, in which time and frequency recoveries have been realized independently [12]. In [13], PIO is used for fault detection and it was further analyzed in [9]. In a recent paper [14] the authors provide a new formulation for PIO design by using the integral gain to stabilize the noise free error dynamics while the proportional gain is used to decouple the disturbance. In spite of various designs available for PIO, its generalization to *adaptive* case has not sufficiently been explored. Although an ad hoc design of PI Adaptive Observer (PIAO) has been reported in [15] and [16], there is no systematic design approach available for it. The difficulty encountered in previous publications was due to the fact that a direct generalization of PAO design to PIAO was not possible and, consequently, the observer gains were adjusted until satisfactory convergence results were achieved. This paper attempts to make use of conventional Proportional Adaptive Observer (PAO) and extend it to Proportional Integral Adaptive Observer (PIAO). This step is not trivial and extra care is necessary to accomplish the design procedure. Therefore the main result of this paper is to provide a systematic design procedure for PIAO to make this generalization possible and prove stability and convergence of parameters.

E. Vahedforough is with Electrical and Computer Engineering Department, Northeastern University, Boston, MA 02115, USA evahed@ece.neu.edu

B. Shafai is with faculty of Electrical and computer Engineering Department, Northeastern University, Boston, MA 02115, USA shafai@ece.neu.edu

In Section II, we provide a modified treatment of conventional adaptive observer, which facilitates the derivation of proportional integral adaptive observer in Section III. The theoretical development of PIAO is concluded by the proof of stability in Section IV. Simulations are carried out in Section V. Two numerical examples using PAO and PIAO are provided and convergence of estimated state and parameters are compared. The superior performance of PIAO is evident from simulation results. Finally, concluding remarks are given in section VI.

II. ADAPTIVE OBSERVERS

Consider the linear time invariant system described by:

$$y(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-2}s^2 + a_{n-1}s + a_n} u(s)$$

with unknown parameters a_i and b_i, $\forall\ i = \{1, 2, \cdots, n\}$.

The state space representation of this system in observable canonical form is written as:

$$\dot{x} = \begin{pmatrix} -a_1 & 1 & 0 & 0 & 0 \\ -a_2 & 0 & 1 & 0 & 0 \\ \vdots & & & \ddots & \\ -a_{n-1} & 0 & 0 & 0 & 1 \\ -a_n & 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} u \quad (1)$$

$$y = h^T x$$
$$h^T = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where x is an n^{th} order state vector, and u, y are input and output signals respectively.

The objective is to estimate all unknown parameters and the states of the system using available input and output signals. According to definition of the adaptive observers in [2], the adaptive observer realization of the above system is given by:

$$\dot{\hat{x}} = K\hat{x} + [k - \hat{a}(t)]\, x_1 + \hat{b}u \quad (2)$$
$$\hat{y}(t) = h^T \hat{x} = \hat{x}_1$$

where $\hat{a}$ and $\hat{b}$ are the estimated parameter vectors, and K is a stable matrix written in compact form as:

$$K = \begin{pmatrix} -k_1 & 1 & 0 & 0 & 0 \\ -k_2 & 0 & 1 & 0 & 0 \\ \vdots & & & \ddots & \\ -k_{n-1} & 0 & 0 & 0 & 1 \\ -k_n & 0 & 0 & 0 & 0 \end{pmatrix}.$$

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

An alternative characterization of the above development was introduced in [18], which makes the derivation of the proportional adaptive observer transparent. We briefly discuss it here and provide a generalization of this approach for the design of PIAO in section III.

Applying $\hat{y} = \hat{x}_1$ to the PAO described above will result in the following representation:

$$\begin{cases} \dot{\hat{x}}_1 &= k_1(y-\hat{y}) + \hat{x}_2 - \hat{a}_1 y + \hat{b}_1 u \\ \dot{\hat{x}}_2 &= k_2(y-\hat{y}) + \hat{x}_3 - \hat{a}_2 y + \hat{b}_2 u \\ \vdots & \\ \dot{\hat{x}}_{n-1} &= k_{n-1}(y-\hat{y}) + \hat{x}_n - \hat{a}_{n-1} y + \hat{b}_{n-1} u \\ \dot{\hat{x}}_n &= k_n(y-\hat{y}) - \hat{a}_n y + \hat{b}_n u \end{cases} \tag{3}$$

Using the auxiliary filters:

$$\rho = \begin{pmatrix} -s^{n-1} \\ -s^{n-2} \\ \vdots \\ -1 \end{pmatrix} \frac{y(s)}{s^{n-1} + k_1 s^{n-2} + k_2 s^{n-3} + \ldots + k_{n-1}}$$

and

$$\gamma = \begin{pmatrix} s^{n-1} \\ s^{n-2} \\ \vdots \\ 1 \end{pmatrix} \frac{u(s)}{s^{n-1} + k_1 s^{n-2} + k_2 s^{n-3} + \ldots + k_{n-1}}$$

as shown in Fig. 1, the adaptive observer (3) can be described by a single expression as follows:

$$\dot{\hat{y}} = k_p(y-\hat{y}) + k_c y + \Psi + \rho^T \hat{a} + \gamma^T \hat{b} \tag{4}$$

with

$$\dot{\hat{a}} = -\Gamma \rho (y-\hat{y}) \tag{5}$$

$$\dot{\hat{b}} = -\Lambda \gamma (y-\hat{y}) \tag{6}$$

where Ψ is an $(n-1)^{th}$ order filter described by:

$$\Psi(s) = \frac{-\lambda_1 s^{n-2} - \lambda_2 s^{n-1} - \ldots - \lambda_{n-1}}{s^{n-1} + k_1 s^{n-2} + k_2 s^{n-3} + \ldots + k_{n-1}} y(s)$$

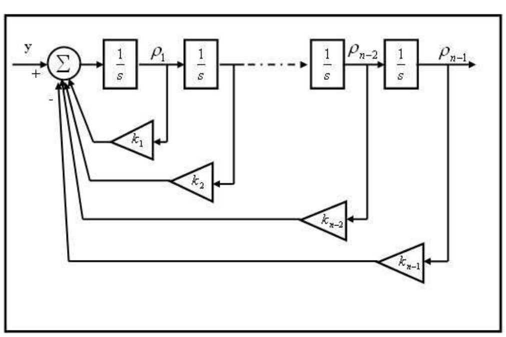

Fig. 1. The Auxiliary Filter Structure

III. PI Adaptive Observers

The adaptive observer discussed above is a strong tool in estimation of unknown parameters and state vectors. However for this type of observers (PAO) the convergence rate of the parameters is slow and very sensitive to the gain adjustment. By inclusion of an integral term to PAO, we introduce a new observer structure called PIAO and derive its mathematical representation. The extra freedom in integral gain enables to improve the convergence rate.

Considering the state space representation of (1), we define the following PI adaptive observer:

$$\begin{cases} \dot{\hat{x}}_1 &= k_1(y-\hat{y}) + \hat{x}_2 - \hat{a}_1 y + \hat{b}_1 u + l_1 \int (y-\hat{y}) \\ \dot{\hat{x}}_2 &= k_2(y-\hat{y}) + \hat{x}_3 - \hat{a}_2 y + \hat{b}_2 u + l_2 \int (y-\hat{y}) \\ \vdots & \\ \dot{\hat{x}}_{n-1} &= k_{n-1}(y-\hat{y}) + \hat{x}_n - \hat{a}_{n-1} y + \hat{b}_{n-1} u \\ & \quad + l_{n-1} \int (y-\hat{y}) \\ \dot{\hat{x}}_n &= k_n(y-\hat{y}) - \hat{a}_n y + \hat{b}_n u + l_n \int (y-\hat{y}) \end{cases} \tag{7}$$

The denominator of the overall transfer function can be expressed as:

$$D(s) = s^{n+1} + d_1 s^n + d_2 s^{n-1} + \cdots + d_n s + d_{n+1}$$

where $d_i = k_i + l_{i-1}$ with $l_0 = 0$ and $k_{n+1} = 0$. Since k_i's are known, we select l_i's such that the polynomial D(s) has at least one real root. Consequently, we construct the following n^{th} order auxiliary filters:

$$\rho = \begin{pmatrix} s^n \\ s^{n-1} \\ \vdots \\ s \end{pmatrix} \frac{-1}{P(s)} \cdot y(s) \tag{8}$$

$$\gamma = \begin{pmatrix} s^n \\ s^{n-1} \\ \vdots \\ s \end{pmatrix} \frac{1}{P(s)} \cdot u(s), \tag{9}$$

with

$$P(s) = s^n + p_1 s^{n-1} + p_2 s^{n-2} + \ldots + p_{n-1} s + p_n$$

where coefficients p_i, $\forall\ i = \{1, 2, \cdots, n\}$ are derived by extracting the real root using Euclidean Division Algorithm.

Applying the auxiliary filters ρ and γ to PIAO described by (7) will lead to following compact PIAO representation:

$$\dot{\hat{y}} = k_p(y-\hat{y}) + k_c y + \Psi + \rho^T \hat{a} + \gamma^T \hat{b} \tag{10}$$

with

$$\dot{\hat{a}} = -\Gamma \rho (y-\hat{y}) \tag{11}$$

$$\dot{\hat{b}} = -\Lambda \gamma (y-\hat{y}) \tag{12}$$

where Ψ is a n^{th} order augmented filter described by:

$$\Psi(s) = \frac{Q(s)}{P(s)} \cdot y(s) \tag{13}$$

where

$$Q(s) = -\lambda_1 s^{n-1} - \lambda_2 s^{n-2} - \ldots - \lambda_n.$$

which has the structure of an observable canonical form as shown in Fig. 2.

The schematic diagram of such a PIAO is shown in Fig. 3. In this system the input and output signals represented by $u(t)$ and $y(t)$ pass through auxiliary filters described by (8), (9). The generated signals along with error signal enter the Identification block represented by (11) and (12), and after integration, the estimated parameter vectors are constructed. Subsequently, the resulting signals $\hat{a}$ and $\hat{b}$ are multiplied by ρ and γ respectively which along with filtered output signal Ψ described by (13) defines the PI adaptive observer (10). The estimated output signal which represents the first state of the system will be generated in the last step.

IV. Proof of Stability

The objective in PI adaptive observer estimation is to devise a scheme to adjust the parameters $\hat{a}$ and $\hat{b}$ such that:

$$\lim_{t\to\infty} \hat{a}(t) \to \ a, \quad \lim_{t\to\infty} \hat{b}(t) \to \ b$$

$$\lim_{t\to\infty} \hat{x}(t) \to \ x$$

defining the estimation errors:

$$e_y = y - \hat{y}, \quad e_a = a - \hat{a}, \quad e_b = b - \hat{b}$$

We obtain the following error dynamics:

$$\dot{e}_y = -ke_y + e_\Psi + \rho^T e_a + \gamma^T e_b \tag{14}$$

$$\dot{e}_a = -\Gamma \rho e_y$$

$$\dot{e}_b = -\Lambda \gamma e_y$$

$$\dot{e}_\Psi = K e_\Psi$$

since ρ, γ, Ψ are bounded and smooth, using the Lyapunov function:

$$V = \frac{1}{2}({e_y}^2 + {e_\Psi}^2 + \frac{1}{\Gamma}{e_a}^2 + \frac{1}{\Lambda}{e_b}^2) \tag{15}$$

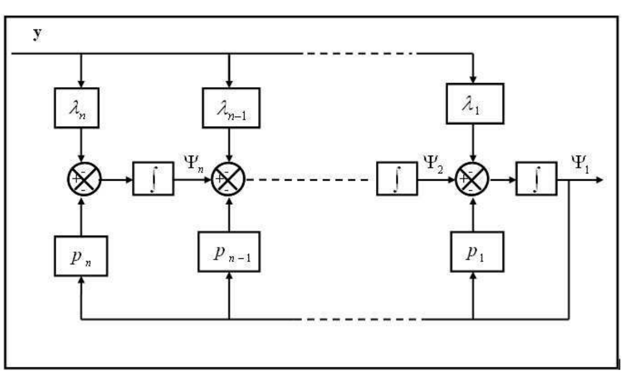

Fig. 2. The Augmented Filter Structure

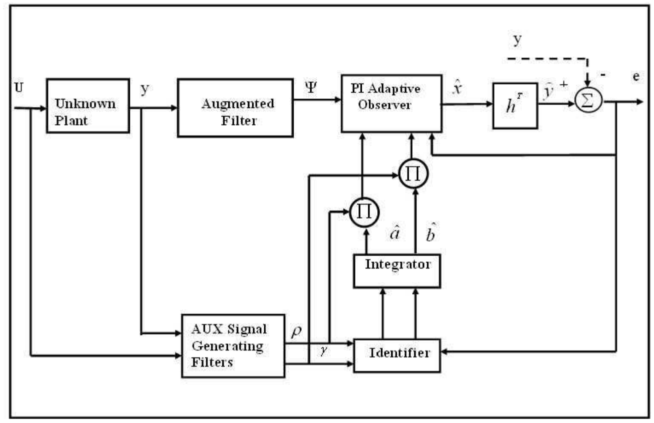

Fig. 3. The Proportional Integral Adaptive Observer Diagram

One can easily show that its time derivative

$$\dot{V} = -k{e_y}^2 + e_y e_\Psi + K{e_\Psi}^2$$

is negative semi-definite, which guarantees the existence and boundedness of solution in (14). Using the standard argument in adaptive control, the persistently exciting condition:

$$\lim_{T\to\infty} \int_t^{t+T} \rho(\tau)^T \rho(\tau) + \gamma(\tau)^T \gamma(\tau) d\tau =$$

$$\int_{-\infty}^{\infty} \left|\rho(\nu)^2\right| S_y d\nu + \int_{-\infty}^{\infty} \left|\gamma(\nu)^2\right| S_u d\nu \geq k_p I > 0$$

is stratified, where S_u and S_y are the spectral measures of the input u and the output y, respectively, and k_p is a positive scalar. Therefore ρ and γ are persistently exciting, and the error dynamic described by (14) is exponentially stable. One can equivalently show this, using Kalman-Yakubovich Lemma.

V. Simulation Results

In this section the simulation results for two plants with unknown parameters are demonstrated. The state response and the convergence behaviors of the parameters using PI adaptive observer are compared to the corresponding results of P adaptive observers.

A. Parameter Estimation For Second Order system

Simulations for the following second order plant are carried out for P and PI adaptive observers, separately.

$$\dot{x} = \begin{pmatrix} -5 & 1 \\ -10 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u \tag{16}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x.$$

The PI adaptive observer for the above system is constructed as:

$$\dot{\hat{y}} = 2(y - \hat{y}) + 4y + \Psi + \rho_1 \hat{a}_1 + \rho_2 \hat{a}_2 + \gamma_1 \hat{b}_1 + \gamma_2 \hat{b}_2$$

$$\dot{\hat{a}} = -\Gamma \rho (y - \hat{y})$$

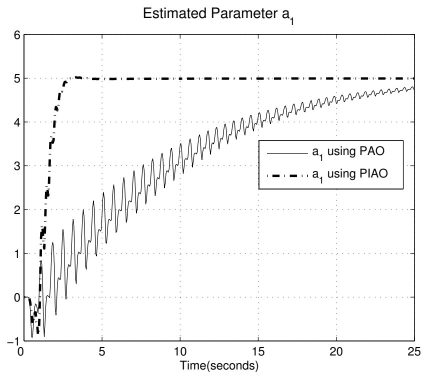

Fig. 4. Estimated $\hat{a}_1$ using adaptive and PI adaptive observers

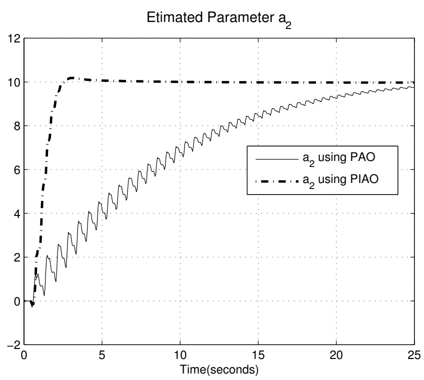

Fig. 5. Estimated $\hat{a}_2$ using adaptive and PI adaptive observers

$$\dot{\hat{b}} = -\Lambda\gamma(y - \hat{y})$$

where Γ and Λ are estimation gains given by:

$$\Gamma = \begin{pmatrix} 750 \\ 1650 \end{pmatrix}, \Lambda = \begin{pmatrix} 50 \\ 75 \end{pmatrix}$$

with ρ and γ filters:

$$\rho = \begin{pmatrix} -s^2 \\ -s \end{pmatrix} \frac{1}{s^2+4s+4} y(s),$$

$$\gamma = \begin{pmatrix} s^2 \\ s \end{pmatrix} \frac{1}{s^2+4s+4} u(s),$$

and the augmented filter:

$$\Psi(s) = \frac{-12s-16}{s^2+4s+4} y(s).$$

However for the P adaptive observer the corresponding filters of ρ and γ will be converted to first order low pass filters with a pole at -2 and the augmented filter given by $\Psi(s) = \frac{-4}{s+2} y(s)$. The results of estimated parameters are illustrated in Fig. 4, Fig. 5, Fig. 6 and Fig. 7. As it is evident from these figures, the estimation results for PI adaptive observer outperform the P adaptive observer. The state estimation error for PI adaptive observer also shown in Fig. 8.

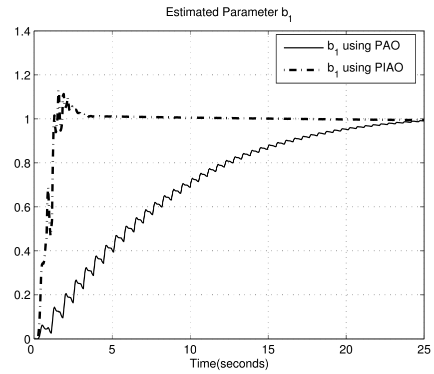

Fig. 6. Estimated $\hat{b}_1$ using adaptive and PI adaptive observers

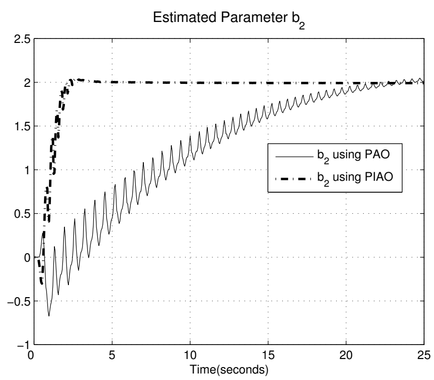

Fig. 7. Estimated $\hat{b}_2$ using adaptive and PI adaptive observers

B. Parameter Estimation For Forth Order System

We applied the PI Adaptive Observer to following forth order plant:

$$\dot{x} = \begin{pmatrix} -3 & 1 & 0 & 0 \\ -8 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 4 \\ 1 \\ 3 \\ 6 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x$$

Using the derived equations for PI adaptive observer in Section III, we have:

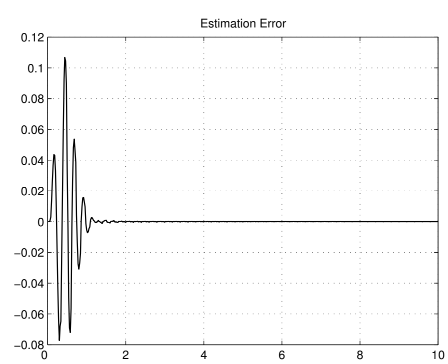

Fig. 8. The State Estimation Error

$$\dot{\hat{y}} = 2(y - \hat{y}) + 8y + \Psi + \rho^T \hat{a} + \gamma^T \hat{b}$$

$$\dot{\hat{a}} = -\Gamma \rho (y - \hat{y})$$

$$\dot{\hat{b}} = -\Lambda \gamma (y - \hat{y})$$

where the estimation gains Γ, and Λ are selected as:

$$\Gamma = \begin{pmatrix} 100 \\ 400 \\ 600 \\ 500 \end{pmatrix}, \Lambda = \begin{pmatrix} 200 \\ 75 \\ 400 \\ 700 \end{pmatrix}$$

and the auxiliary filters ρ and γ are constructed as:

$$\rho = \begin{pmatrix} -s^4 \\ -s^3 \\ -s^2 \\ -s \end{pmatrix} \frac{1}{s^4 + 8s^3 + 24s^2 + 32s + 16} y(s)$$

$$\gamma = \begin{pmatrix} s^4 \\ s^3 \\ s^2 \\ s \end{pmatrix} \frac{1}{s^4 + 8s^3 + 24s^2 + 32s + 16} u(s)$$

while augmented filter Ψ is specified:

$$\Psi(s) = \frac{-40s^3 - 160s^2 - 240s - 128}{s^4 + 8s^3 + 24s^2 + 32s + 16} y(s). \qquad (17)$$

Fig. 9 and Fig. 10, illustrate estimated parameter vectors $\hat{a}$ and $\hat{b}$ respectively. Fig. 11 and Fig. 12 show the estimated state of the system and its corresponding estimation error.

VI. Conclusions

In this paper the proportional integral adaptive observer is introduced for estimation of unknown parameters and states of a system. The design procedure is outlined and the proof of stability is provided. The simulations are carried out for second order and forth order systems. The results demonstrate the fast convergence of proposed PI adaptive observer. Although the aim of this paper was to provide a systematic design for PIAO, it is important to point out that PIAO can also be useful in connection to robust fault detection whereby the fault detection and parameter estimation can be treated separately. Finally the design of PIAO for SISO system reported in this paper can easily be generalized to MIMO case, which will be reported in a future publication.

References

[1] R. Carroll and D. Lindorff, "An adaptive observer for single-input single-output linear systems", *IEEE Trans. on Automatic Control*, vol. 18, pp. 428-435, 1973.

[2] P. Kudva and K. S. Narendra, "Synthesis of an adaptive observer using Lyapunov's direct method", *Journal of control*, vol. 18, pp. 1201-1210, 1973.

[3] K. S. Narendra, A. M. Annaswamy, "Stable Adaptive Systems", Dover Publications, 2005.

[4] P. Ioannou and J. Sun, "Robust Adaptive Control", Prentice Hall, 1996.

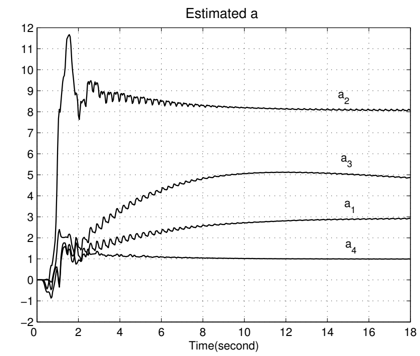

Fig. 9. The Estimated Parameter Vector $\hat{a}$

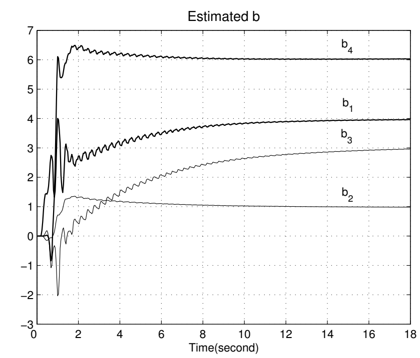

Fig. 10. The Estimated Parameter Vector $\hat{b}$

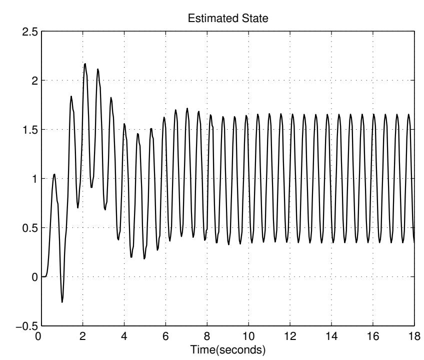

Fig. 11. The Estimated State

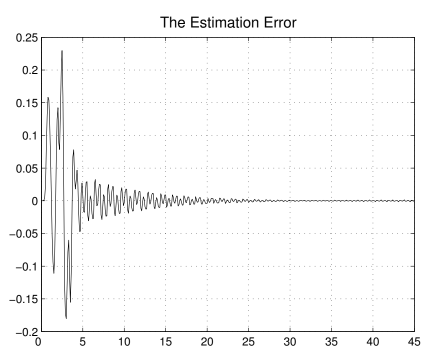

Fig. 12. The State Estimation Error

[5] S. S. Sastry and M. Bodson, ”Adaptive Control: Stability, Convergence, and Robustness”, Prentice Hall, 1994.

[6] N. Hung, N. Duy-Ky, A. Shannon, D. Owens, ”Estimation of minimal model parameters with the use of an adaptive observer for suprabasal insulin action”, *IEEE Conferance of Engineering in Medicine and Biology society*, vol. 5, pp. 2146-2148, 1997.

[7] R. Jimenez, L. Alvarez-Icaza, ”Civil structures semi-active control with limited measurements”, *IEEE Trans. of American Control Conference*, vol. 6, pp. 5467-5471, 2004.

[8] M. Arcak, H. Gorgun, L. M. Pedersen, S. Varigonda, ”An adaptive observer design for fuel cell hydrogen estimation”, *IEEE Trans. of American Control Conference*, vol. 3, pp. 2037-2042, 2003.

[9] Wu. Qing and M. Saif, ”Neural adaptive observer based fault detection and identification for satellite attitude control systems”, *IEEE Trans. of American Control Conference*, vol. 2, pp. 1054-1059, 2005.

[10] L. Alvarez-Icaza, L. Munoz sun, R. XiaotianHorowitz, ”Adaptive observer for traffic density estimation”, *IEEE Trans. of American Control Conference*, vol. 3, pp. 2705-2710, 2004.

[11] R. Ginhoux, J. A. Gangloff, M. F. de Mathelin, L. Soler, M. M. A. Sanchez, J. Marescaux, ”Beating heart tracking in robotic surgery using 500 Hz visual servoing, model predictive control and an adaptive observer”, *IEEE International Conference of Robotics and Automation*, vol. 1, pp. 274-279, 2004 .

[12] H. H. Niemann, J. Stoustrup, B. Shafai, and S. Beale, ”LTR design of proportional integral observer”, *International Journal of Robust and Nonlinear Control*, pp. 671-693, 1995.

[13] M. Saif, ”An observer based approach for fault diagnoisis”, *Proc. IASTED International Conference*, pp. 239-242, 1997.

[14] K. K. Busawon and P. Kabore, ”Disturbance attenuation using proportional integral observers”, *International Journal of Control*, pp. 618-627, 2001.

[15] B. Shafai, C.T. Pi, S. Nork and S.P. Linder,”Proportional integral adaptive observer for parameter and disturbance estimations”„ *IEEE Conference of Decision and Control*, vol. 4, pp.4694 - 4699, 2002.

[16] S. Linder and B. Shafai,” Robust PI Adaptive Observer”, *Proc. IASTED International control Conference* , pp. 151-154, 1997.

[17] Q. Zhang, ”Revisiting different adaptive observers through a unified formulation”, *IEEE Conf. Decision and Control*, vol. 5, pp. 3067-3072, 2005.

[18] R. Marino, P. Tomei, ”Global estimation of n unknown frequencies”, *Proceedings of the 39th IEEE Conference of Decision and Control*, vol.2, pp. 1143 - 1147, Dec 2000.