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Delay-dependent Robust Stability and Stabilization for Uncertain Singular System with Time-varying Delay

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Abstract— **The problems of delay-dependent robust stability and stabilization for uncertain singular systems with time-varying delay are addressed in this paper. The uncertainty is assumed to be norm bounded. By establishing an *integral inequality* based on quadratic terms, a new delay-dependent robust stability criterion is derived and expressed in terms of linear matrix inequality(LMI). Based on the criterion, the problem of robust stabilization is solved via state feedback controller, which guarantees that the resultant closed-loop system is regular, impulse-free and stable for all admissible uncertainties. An explicit expression for the desired controller is also given. Numerical examples are given to demonstrate the applicability and the less conservatism of the proposed method.**

I. INTRODUCTION

Time-delays are frequently encountered in many fields of science and engineering, including communication network, manufacturing systems, biology, economy and other areas [1], [2]. During the last two decades, the problem of stability analysis and control of time-delay systems has been the subject of considerable research efforts. Many significant results have been reported in the literature, see for example, [3]–[10], and references therein. On the other hand, singular systems, which are known as descriptor systems, implicit systems, generalized state-space systems or semi-state systems, have received much attention since singular model can preserve the structure of practical systems and can better describe a large class of physical systems than regular ones [11], [12]. Therefore, the study of robust stability and stabilizability problem for uncertain singular time-delay system is of theoretical and practical importance.

It should be pointed out that when the robust stability problem for singular systems is investigated, the regularity and absence of impulses (for continuous systems) and causality (for discrete systems) are required to be considered simultaneously [13]–[15]. Hence, the robust stability problem for singular time-delay systems is much more complicated than that for state-space ones. The existing results can be classified into two type: delay-independent stabilization and delay-dependent stabilization. Generally, the delay-independent case is more conservation than the delay-dependent case, especially when the time delay is comparatively small. For a system with small delay, a model transformation technique or bounding cross terms technique is often used to reduce the conservatism. But the model transformation may introduce additional dynamics [16], [17]. Using bounding technique requires that some matrix variables should be limited to a certain structure to obtain controller synthesis conditions in terms of LMIs [18] and [19]. This limitation introduces some conservatism. Some delay-dependent stability criteria for singular time-delay systems were presented in [21], [25]–[35], but in [25], [27], [29], it was required to assume that the considered system is regular and impulse free. Moreover, these conditions in [21], [25]–[35] were established under the assumption that the delay was constant, when the delay is time-varying, they are inapplicable. In [20], delay-dependent robust H_∞ controller is designed for uncertain descriptor systems with time-varying discrete and distributed delays, but the given results are based on a set of nonconvex matrix inequalities, not on strict linear matrix inequalities. To the best of our knowledge, the class of uncertain singular systems with time-varying delay has not yet been fully investigated. Particularly delay-dependent sufficient conditions of robust stability are few even not existing in the literature.

In this paper, the problems of robust stability and stabilization are considered for a class of singular systems with time-varying delay and norm-bounded uncertainties. With the introduction of a new *integral inequality*, which avoids using both model transformation and bounding technique for cross terms, a strict LMI sufficient criterion for singular time-varying delay systems is obtained. The robust stability and stabilization problems are also solved and an explicit expression of the desired state feedback control law is given, which can be obtained by solving the feasibility problem of a strict LMI.

Notations: Through this paper, the superscripts ”T” and ”-1” stand for the transpose of a matrix and the inverse of a matrix; $\mathbb{R}^n$ denotes n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$ is the set of all real matrices with m rows and n columns; $P > 0$ means that P is positive definite; I is the identity matrix with appropriate dimensions; $\lambda_{max}(P)$ and $\lambda_{min}(P)$ denote the maximal and minimal eigenvalues of the matrix P respectively; $\|x\|$ refers to the Euclidean norm of the vector x, that is $\|x\| = \sqrt{x^T x}$; For a symmetric matrix, $*$ denotes the matrix entries implied by symmetry.

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II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the uncertain singular system with time-varying delay described by

$$\begin{cases} E\dot{x}(t) = (A+\Delta A)x(t) + (A_d+\Delta A_d)x(t-d(t)) \\ \qquad\qquad + (B+\Delta B)u(t) \\ x(t) = \phi(t), t = [-\max(d(t)), 0] \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector. E, A, A_d and B are constant matrices of appropriate dimensions, where E may be singular and we assume that rank$E = r \leq n$. $\Delta A, \Delta A_d$ and ΔB are unknown and possibly time-varying matrices representing norm-bounded parameter uncertainties and are assumed to be of the following form,

$$\begin{bmatrix} \Delta A & \Delta A_d & \Delta B \end{bmatrix} = MF(t)\begin{bmatrix} N_a & N_d & N_b \end{bmatrix} \tag{2}$$

where M, N_a, N_d, N_b are known constant matrices of appropriate dimensions, and $F(t)$ is an unknown matrix function satisfying $F^T(t)F(t) \leq I$. $\phi(t)$ is a compatible vector valued initial function. $d(t)$ is time-varying delay with known bound in system (1) such that

$$0 < d(t) \leq d_m,\ \dot{d}(t) \leq d^* < \infty \tag{3}$$

Clearly, $d^* = 0$ means that the time-delay $d(t)$ is time-invariant.

The nominal unforced singular system of (1) can be written as

$$E\dot{x}(t) = Ax(t) + A_d x(t-d(t)) \tag{4}$$

Definition 1: [11]–[13]
1) The pair (E, A) is said to be regular if $det(sE - A)$ is not identically zero.
2) The pair (E, A) is said to be impulse-free if $deg(det(sE-A)) = \text{rank}\ E$.

Definition 2: [13]
1) The singular system (4) is said to be regular and impulse free if the pair (E, A) is regular and impulse free.
2) The singular system (4) is said to stable if for any $\epsilon > 0$, there exists a scalar $\delta(\epsilon) > 0$ such that for any compatible initial conditions $\phi(t)$ satisfying $\sup\limits_{-d(t)\leq t\leq 0} \|\phi(t)\| \leq \delta(\epsilon)$, the solution $x(t)$ of the system (4) satisfies $\|x(t)\| \leq \epsilon$ for $t \geq 0$. Furthermore, $\lim\limits_{t\to\infty} x(t) = 0$.

Definition 3: The uncertain singular time-varying delay system (1) is said to be robustly stable if the system (1) with $u(t) \equiv 0$ is regular, impulse-free and stable for all admissible uncertainties (2) and (3).

Definition 4: The uncertain singular time-varying delay system (1) is said to be robustly stabilizable if there exists a linear state feedback control law $u(t) = Kx(t), K \in \mathbb{R}^{m\times n}$ such that its closed-loop system is robustly stable in the sense of Definition 3. In this case, $u(t) = Kx(t)$ is called as a robust state feedback controller.

The aim of this paper is to design a feedback gain K such that $u(t) = Kx(t)$ is a robust state feedback controller.

We conclude this section by presenting several preliminary results, which will be used in the proof of our main results.

Lemma 1: [22] Given matrices Γ, Λ and symmetric matrix Ω, we have $\Omega + \Gamma F\Lambda + \Lambda^{\mathrm{T}}F^{\mathrm{T}}\Gamma^{\mathrm{T}} < 0$ for any $F^{\mathrm{T}}F \leq I$, if and only if there exists a scalar $\epsilon > 0$ such that $\Omega + \epsilon^{-1}\Gamma\Gamma^{\mathrm{T}} + \epsilon\Lambda^{\mathrm{T}}\Lambda < 0$.

Lemma 2: [23] Consider the function $\varphi:\ R^+ \to R$, if $\dot{\varphi}$ is bounded on $[0, \infty)$, that is, there exists a scalar $\alpha > 0$ such that $\mid \dot{\varphi}(t) \mid \leq \alpha$ for all $t \in [0, \infty)$, then $\varphi(t)$ is uniformly continuous on $[0, \infty)$.

Lemma 3: (Barbalat's Lemma) [23] Consider the function $\varphi:\ R^+ \to R$, if φ is uniformly continuous and $\int_0^\infty \varphi(s)ds < \infty$, then $\lim\limits_{t\to\infty} \varphi(t) = 0$.

III. MAIN RESULTS

In this section, the problems of robust stability and stabilization based on LMI approach for uncertain singular system (1) are discussed. First, we present the delay-dependent stability condition for nominal unforced singular system of (1) based on a new integral inequality.

A. Delay-dependent Stability Analysis for Nominal Singular System

For the nominal system (4), we introduce two vectors as follows

$$\xi(t) = \begin{bmatrix} x^T(t) & x^T(t-d(t)) \end{bmatrix}^T, y(t) = E\dot{x}(t).$$

The following lemma gives the relationship between the vectors $\xi(t)$ and $\dot{x}(t)$, which will play a key role in the delay-dependent stability analysis.

Lemma 4: (Integral Inequality) For any constant matrices $N_1 \in \mathbb{R}^{n\times n}$, $N_2 \in \mathbb{R}^{n\times n}$, a positive-definitive symmetric matrix $Z \in \mathbb{R}^{n\times n}$, and a time-varying delay $d(t)$, then

$$\begin{aligned} -\int_{t-d(t)}^{t} \dot{x}^T(s)E^T ZE\dot{x}(s)ds \leq & \xi^T(t)\{\Pi \\ & + d(t)Y^T Z^{-1}Y\}\xi(t) \end{aligned} \tag{5}$$

where

$$\begin{aligned} \Pi &= \begin{bmatrix} N_1^T E + E^T N_1 & E^T N_2 - N_1^T E \\ * & -N_2^T E - E^T N_2 \end{bmatrix} \\ Y &= \begin{bmatrix} N_1 & N_2 \end{bmatrix} \end{aligned} \tag{6}$$

Proof: Let $C = \begin{bmatrix} Z^{1/2} & Z^{-1/2}Y \\ 0 & 0 \end{bmatrix}$, then

$$\begin{bmatrix} Z & Y \\ Y^T & Y^T Z^{-1}Y \end{bmatrix} = C^T C \geq 0.$$

It follows

$$\begin{aligned} \int_{t-d(t)}^{t} \begin{bmatrix} E\dot{x}(s) \\ \xi(t) \end{bmatrix}^T \begin{bmatrix} Z & Y \\ Y^T & Y^T Z^{-1}Y \end{bmatrix} \begin{bmatrix} E\dot{x}(s) \\ \xi(t) \end{bmatrix} ds \\ \geq 0 \end{aligned} \tag{7}$$

Notice that

$$\int_{t-d(t)}^{t} 2\xi^T(t)Y^T E\dot{x}(s)ds = 2\xi^T(t)Y^T \begin{bmatrix} E & -E \end{bmatrix} \xi(t)$$

Rearranging (7) yields (5). ■

Based on Lemma 4, the following theorem presents a stability condition of the nominal system (4).

Theorem 1: The nominal singular system (4) with time-varying delay is regular, impulse-free, stable if there exist positive-definite symmetric matrices P, Q, Z and matrices S, S_d, N_1, N_2 of appropriate dimensions such that

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & d_m N_1^T & d_m A^T Z \\ * & \Xi_{22} & d_m N_2^T & d_m A_d^T Z \\ * & * & -d_m Z & 0 \\ * & * & * & -d_m Z \end{bmatrix} < 0, \quad (8)$$

where

$$\begin{aligned} \Xi_{11} =& A^T PE + SR^T A + E^T PA + A^T RS^T \\ &+ N_1^T E + E^T N_1 + Q, \\ \Xi_{12} =& A^T RS_d^T + SR^T A_d + E^T PA_d + E^T N_2 - N_1^T E, \\ \Xi_{22} =& -(1-d^*)Q + A_d^T RS_d^T + S_d R^T A_d \\ &- N_2^T E - E^T N_2, \end{aligned}$$

and $R \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $E^T R = 0$

Proof: Since rank$E = r \le n$, there must exist two invertible matrices G and $H \in \mathbb{R}^{n\times n}$ such that

$$\bar{E} = GEH = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad (9)$$

Then, R can be parameterized as $R = G^T \begin{bmatrix} 0 \\ \bar{\Phi} \end{bmatrix}$, where $\bar{\Phi} \in \mathbb{R}^{(n-r)\times(n-r)}$ is any nonsingular matrix.

Similar to (9), we define

$$\begin{aligned} \bar{A} &= GAH = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \\ \bar{P} &= G^{-T} P G^{-1} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix} \\ \bar{N}_1 &= G^{-T} N_1 H = \begin{bmatrix} \bar{N}_{1,11} & \bar{N}_{1,12} \\ \bar{N}_{1,21} & \bar{N}_{1,22} \end{bmatrix} \\ \bar{S} &= H^T S = \begin{bmatrix} \bar{S}_{11} \\ \bar{S}_{21} \end{bmatrix} \\ \bar{R} &= G^{-T} R = \begin{bmatrix} 0 \\ \bar{\Phi} \end{bmatrix} \end{aligned}$$

Since $\Xi_{11} < 0$ and $Q > 0$, we can formulate the following inequality easily,

$$\begin{aligned} \Psi = & A^T PE + SR^T A + E^T PA + A^T RS^T \\ &+ N_1^T E + E^T N_1 < 0 \end{aligned}$$

Pre- and post-multiplying $\Psi < 0$ by H^T and H, respectively, yields

$$\begin{aligned} \bar{\Psi} =& H^T \Psi H \\ =& \bar{A}^T \bar{P} \bar{E} + \bar{S} \bar{R}^T \bar{A} + \bar{E}^T \bar{P} \bar{A} + \bar{A}^T \bar{R} \bar{S}^T \\ &+ \bar{N}_1^T \bar{E} + \bar{E}^T \bar{N}_1 \\ =& \begin{bmatrix} \bar{\Psi}_{11} & \bar{\Psi}_{12} \\ * & \bar{A}_{22}^T \bar{\Phi} \bar{S}_{21}^T + \bar{S}_{21} \bar{\Phi}^T \bar{A}_{22} \end{bmatrix} \\ <& 0 \end{aligned} \quad (10)$$

Since $\bar{\Psi}_{11}$ and $\bar{\Psi}_{12}$ are irrelevant to the results of the following discussion, the real expression of these two variables are omitted here. From (10), it is easy to see that

$$\bar{A}_{22}^T \bar{\Phi} \bar{S}_{21}^T + \bar{S}_{21} \bar{\Phi}^T \bar{A}_{22} < 0 \quad (11)$$

and thus $\bar{A}_{22}$ is nonsingular. Otherwise, supposing $\bar{A}_{22}$ is singular, there must exist a non-zero vector $\zeta \in \mathbb{R}^{n-r}$, which ensures $\bar{A}_{22}\zeta = 0$. And then we can conclude that $\zeta^T(\bar{A}_{22}^T \bar{\Phi} \bar{S}_{21}^T + \bar{S}_{21} \bar{\Phi}^T \bar{A}_{22})\zeta = 0$, and this contradicts (11). So $\bar{A}_{22}$ is nonsingular. Then, the pair of (E, A) is regular and impulse-free, which implies from Definition 2 that the system (4) is regular and impulse-free. In the following, we will prove that the system (4) is also stable.

Considering the (4), we define the functional

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)), \quad (12)$$

where

$$\begin{aligned} V_1(x(t)) &= x^T(t) E^T P E x(t), \\ V_2(x(t)) &= \int_{t-d(t)}^{t} x^T(s) Q x(s) ds, \\ V_3(x(t)) &= \int_{-d_m}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) E^T Z E \dot{x}(s) ds d\theta. \end{aligned}$$

Differentiating $V(x(t))$ with respect to t, we have

$$\begin{aligned} \dot{V}_1 &= \dot{x}^T(t) E^T P E x(t) + x^T(t) E^T P E \dot{x}(t), \\ \dot{V}_2 &= x^T(t) Q x(t) - (1 - \dot{d}(t)) x^T(t-d(t)) Q x(t-d(t)), \\ \dot{V}_3 &= d_m \dot{x}^T(t) E^T Z E \dot{x}(t) - \int_{t-d_m}^{t} \dot{x}^T(s) E^T Z E \dot{x}(s) ds \\ &\le d_m \dot{x}^T(t) E^T Z E \dot{x}(t) - \int_{t-d(t)}^{t} \dot{x}^T(s) E^T Z E \dot{x}(s) ds. \end{aligned}$$

Furthermore, noting $E^T R = 0$, we can deduce

$$0 = 2\dot{x}^T(t) E^T R (S^T x(t) + S_d^T x(t - d(t))) \quad (13)$$

Then it follows from (13) and Lemma 4 that

$$\dot{V}(x(t)) \le \xi^T(t) \Xi \xi(t)$$

It is easy to see that (8) guarantees $\dot{V}(x(t)) < 0$ and

$$\begin{aligned} \lambda_1 \|x(t)\|^2 - V(x(0)) &\le x^T(t) E^T P E x(t) - V(x(0)) \\ &\le V(x(t)) - V(x(0)) \\ &= \int_0^t \dot{V}(x(s)) ds \\ &\le -\lambda_2 \int_0^t \|x(s)\|^2 ds \\ &< 0 \end{aligned} \quad (14)$$

where $\lambda_1 = \lambda_{min}(E^T P E) > 0, \lambda_2 = -\lambda_{max}(\Xi) > 0$.

Taking into account (14), we can deduce that

$$\lambda_1 \|x(t)\|^2 + \lambda_2 \int_0^t \|x(s)\|^2 ds \le V(x(0))$$

Therefore

$$0 < \|x(t)\|^2 \le \frac{1}{\lambda_1} V(x(0)),$$

$$0 < \int_0^t \|x(s)\|^2 ds \le \frac{1}{\lambda_2} V(x(0)).$$

Thus, $\|x(t)\|$ and $\int_0^t \|x(s)\|^2 ds$ are bounded. Using same method, we have that $\|\dot{x}(t)\|$ is bounded. By Lemma 2, we obtain $\|\dot{x}(t)\|^2$ is uniformly continuous. Therefore, noting that $\int_0^t \|x(s)\|^2 ds$ is bounded, and using Lemma 3, we get

$$\lim_{t\to\infty} x(t) = 0,$$

according to Definition 2, the singular delay system (4) is stable. This completes the proof. ■

Remark 1: When $E = I$, it follows from $E^T R = 0$ that $R = 0$. Therefore, it is easy to show that Theorem 1 coincides with the Theorem 1 in [10] not considering the H_∞ performance by slight modifications.

Remark 2: Employing model transformation and bounding technique for cross terms, we also can consider the problem of stability for the singular system (4), then the assumption that $\dot{d}(t) \le d^* < 1$ on the time-varying delay $d(t)$ is required. However, from the proof of Theorem 1, one can clearly see that neither neither model transformation nor bounding technique for cross terms is involved. Therefore, the stability criteria are expected to be less conservative. Moveover, the restriction $d^* < 1$ is **removed**, which means that a **fast** time-varying delay is allowed.

B. Robust State Feedback Controller Design

In this sequel, we give a strict LMI design algorithm for the system (1). For notational simplicity, we first consider the system (1) with $\Delta A = \Delta A_d = \Delta B = 0$, which, with the control law $u(t) = Kx(t)$, results in the following closed-loop system,

$$E\dot{x}(t) = (A + BK)x(t) + A_d x(t - d(t)) \quad (15)$$

For this system, we have the following theorem.

Theorem 2: The singular system (15) with time-varying delay is robustly stabilizable if there exist positive-definite symmetric matrices P, Q, Z and matrices S, N_1, N_2, X, L of appropriate dimensions such that

$$\Upsilon = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & d_m N_1^T & 0 \\ * & \Upsilon_{22} & X^T A_d^T & 0 & d_m Z \\ * & * & \Upsilon_{33} & d_m N_2^T & 0 \\ * & * & * & -d_m Z & 0 \\ * & * & * & * & -d_m Z \end{bmatrix} < 0, \quad (16)$$

where

$$\begin{aligned} \Upsilon_{11} =& AX + X^T A^T + BL + L^T B^T \\ &+ N_1^T E^T + E N_1 + Q, \\ \Upsilon_{12} =& EP + SR^T - X^T + AX + BL, \\ \Upsilon_{13} =& X^T A_d^T + E N_2 - N_1^T E^T \\ \Upsilon_{22} =& -X - X^T, \\ \Upsilon_{33} =& -(1 - d^*)Q - E N_2 - N_2^T E^T \end{aligned}$$

and $R \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $ER = 0$. Furthermore, a suitable state feedback control law is given by $u(t) = LX^{-1}x(t)$.

Proof: Following the same philosophy as that in [24], we represent the system (15) as the following form,

$$\bar{E}\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{A}_d \bar{x}(t - d(t)), \quad (17)$$

where

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \bar{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},$$

$$\bar{A} = \begin{bmatrix} 0 & I \\ A + BK & -I \end{bmatrix}, \bar{A}_d = \begin{bmatrix} 0 & 0 \\ A_d & 0 \end{bmatrix}.$$

Then, by the result of Theorem 1, we have that the system (15) is robust stable, if (8) holds, where E,A,A_d,P,Q,Z,R,S,S_d,N_1,N_2 are replaced by $\bar{E}$,$\bar{A}$,$\bar{A}_d$, $\bar{P}$,$\bar{Q}$,$\bar{Z}$,$\bar{R}$,$\bar{S}$,$\bar{S}_d$,$\bar{N}_1$,$\bar{N}_2$ respectively. Especially, we select

$$\bar{P} = \begin{bmatrix} P & 0 \\ 0 & \beta I \end{bmatrix}, \bar{Q} = \begin{bmatrix} Q & 0 \\ 0 & \beta I \end{bmatrix}, \bar{Z} = \begin{bmatrix} Z & 0 \\ 0 & \beta I \end{bmatrix},$$
$$\bar{R} = \begin{bmatrix} R & 0 \\ 0 & X \end{bmatrix}, \bar{S} = \begin{bmatrix} S & I \\ 0 & I \end{bmatrix}, \bar{N}_1 = \begin{bmatrix} N_1 & 0 \\ 0 & \beta I \end{bmatrix},$$
$$\bar{N}_2 = \begin{bmatrix} N_2 & 0 \\ 0 & \beta I \end{bmatrix}, \bar{S}_d = 0$$

where $P \in \mathbb{R}^{n\times n}, Q \in \mathbb{R}^{n\times n}, Z \in \mathbb{R}^{n\times n}$ are positive-definite symmetric matrices, $R \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $E^T R = 0$, $X \in \mathbb{R}^{n\times n}$ is any nonsingular matrix, $S \in \mathbb{R}^{n\times(n-r)}, N_1 \in \mathbb{R}^{n\times n}, N_2 \in \mathbb{R}^{n\times n}$ is any matrices. It is easy to see that $\bar{R}$ is with full column rank and satisfies $\bar{E}^T \bar{R} = 0$. Then, the following condition can be obtained by using Schur complement and letting $\beta \longrightarrow 0$,

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & d_m N_1^T & 0 \\ * & \Lambda_{22} & X^T A_d & 0 & d_m Z \\ * & * & \Lambda_{33} & d_m N_2^T & 0 \\ * & * & * & -d_m Z & 0 \\ * & * & * & * & -d_m Z \end{bmatrix} < 0 \quad (18)$$

where

$$\begin{aligned} \Lambda_{11} =& (A + BK)^T X + X^T (A + BK) \\ &+ N_1^T E + E^T N_1 + Q \\ \Lambda_{12} =& E^T P + SR^T - X^T + (A + BK)^T X \\ \Lambda_{13} =& X^T A_d + E^T N_2 - N_1^T E \\ \Lambda_{22} =& -X - X^T \\ \Lambda_{33} =& -(1 - d^*)Q - E^T N_2 - N_2^T E \end{aligned}$$

Now, consider the following singular time-varying delay system,

$$E^T \dot{\varsigma}(t) = (A + BK)^T \varsigma(t) + A_d^T \varsigma(t - d(t)) \quad (19)$$

where $\varsigma(t) \in \mathbb{R}^n$ is the state vector.

Note that $\det(sE - (A+BK))$ =$\det(sE^T - (A+BK)^T)$, then the pair $(E, A+BK)$ is regular, impulse-free and stable if and only if the pair $(E^T, (A+BK)^T)$ is regular, impulse-free and stable and thus, the system (15) is regular, impulse-free and stable if and only if the system (19) is regular, impulse-free and stable.

Therefore, as long as the regularity, free of impulse and stability are concerned, we can consider the system (19) instead of (15). Then, LMI (16) can be obtained by replacing $E, (A+BK), A_d$ in (19) by $E^T, (A+BK)^T, A_d^T$ respectively and introducing a matrix $L = KX$. According to Definition 3 and Definition 4, the singular time-varying delay system (15) is robustly stabilizable. ■

The robust stabilizability result for uncertain singular time-varying system (1) is presented in the following theorem.

Theorem 3: Consider the uncertain singular system (1) with time-varying delay, if there exist positive-definite symmetric matrices P, Q, Z, matrices S, N_1, N_2, X, L of appropriate dimensions and scalars $\epsilon_1 > 0, \epsilon_2 > 0$ such that

$$\begin{bmatrix} \Theta_{11} & \Upsilon_{12} & \Upsilon_{13} & d_m N_1^T & 0 & \Theta_{16} & \Theta_{17} \\ * & \Upsilon_{22} & \Theta_{23} & 0 & d_m Z & \Theta_{26} & \Theta_{27} \\ * & * & \Theta_{33} & d_m N_2^T & 0 & 0 & 0 \\ * & * & * & -d_m Z & 0 & 0 & 0 \\ * & * & * & * & -d_m Z & 0 & 0 \\ * & * & * & * & * & -\epsilon_1 I & 0 \\ * & * & * & * & * & * & -\epsilon_2 I \end{bmatrix} < 0, \tag{20}$$

then, we can construct a robust state feedback control law $u(t) = LX^{-1}x(t)$ such that the resultant closed-loop system is robustly stable for all admissible uncertainties, where

$$\Theta_{11} = \Upsilon_{11} + \epsilon_1 M M^T,\ \Theta_{23} = X^T A_d^T,$$
$$\Theta_{33} = \Upsilon_{33} + \epsilon_2 M M^T,\ \Theta_{16} = \Theta_{26} = (N_a X + N_b L)^T,$$
$$\Theta_{17} = \Theta_{27} = X^T N_d^T,$$

$R \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $ER = 0$ and Υ_{11},Υ_{12},Υ_{13},Υ_{22},Υ_{33} follow the same definition as those in (16).

Proof: Replacing A by $A + MF(k)N_a$,A_d by $A_d + MF(k)N_d$ and B by $B + MF(k)N_b$ in (16) respectively result in the following condition,

$$\Upsilon + \Gamma_1 F(t)\Phi_1 + \Phi_1^T F^T(t)\Gamma_1^T + \Gamma_2 F(t)\Phi_2 + \Phi_2^T F^T(t)\Gamma_2^T < 0 \tag{21}$$

where

$$\Gamma_1 = \begin{bmatrix} M^T & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$
$$\Phi_1 = \begin{bmatrix} N_a X + N_b L & N_a X + N_b L & 0 & 0 & 0 \end{bmatrix}.$$
$$\Gamma_2 = \begin{bmatrix} 0 & 0 & M^T & 0 & 0 \end{bmatrix}^T,$$
$$\Phi_2 = \begin{bmatrix} N_d X & N_d X & 0 & 0 & 0 \end{bmatrix}.$$

By Lemma 1, it follows that (21) holds for any $F(t)$ satisfying $F^T(t)F(t) \leq I$ if there exists scalars $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\Upsilon + \varepsilon_1^{-1}\Gamma_1\Gamma_1^T + \varepsilon_1\Phi_1^T\Phi_1 + \varepsilon_2^{-1}\Gamma_2\Gamma_2^T + \varepsilon_2\Phi_2^T\Phi_2 < 0, \tag{22}$$

which is equal to (20) in the sense of Schur complement. ■

IV. NUMERICAL EXAMPLE

In this section, some examples are provided to demonstrate the effectiveness and the less conservatism of the proposed design algorithm.

Example 1: (Stability Analysis) Consider the following singular delay system [28]

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1.1 & 1 \\ 0 & 0.5 \end{bmatrix} x(t-d)$$

In this example, we choose $R = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Since the time-delay is time invariant, by setting $d^* = 0$, the upper bounds on the time delay from Theorem 1 are shown in Table I. For Comparison, the table also lists the upper bounds obtained from the criteria in [25]–[35]. It can be seen that our method is less conservative.

But when the time-delay $d = d(t)$ is varying delay, the criteria in [25]–[35] fail to make any decision for this case. According to Theorem 1, The upper bound on the time delay is shown in Table II for different d^*.

Example 2: (Stability Synthesis) Consider the uncertain time-varying delay singular system (1) with parameters as follows,

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 0.2 & 0.1 & 1 \\ -1 & 0 & 1 \\ 0.5 & 0 & 1 \end{bmatrix}$$
$$A_d = \begin{bmatrix} -1.5 & 0.5 & -0.8 \\ 1 & 1 & 0.5 \\ 0.7 & 0.5 & 1 \end{bmatrix},\ M = \begin{bmatrix} 0.4 \\ 0.3 \\ 0.1 \end{bmatrix},\ N_b = 0.2,$$
$$N_a = \begin{bmatrix} 0.2 & 0.4 & 0.5 \end{bmatrix},\ N_d = \begin{bmatrix} 0.3 & 0.1 & 0.5 \end{bmatrix}$$

In this example, we choose $R = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. According to Theorem 3, Table III shows the allowed maximum d_m and the corresponding state feedback gain K for the different d^*.

V. CONCLUSION

In this paper, the delay-dependent robust stability and stabilization for uncertain singular systems with time-varying delay are studied. With the introduction of a new *integral inequality*, which avoids using both model transformation and bounding technique for cross terms, a new delay-dependent robust stability condition is derived and expressed in terms of linear matrix inequality(LMI). Meanwhile, a control law design algorithm is also given, which guarantees that the resultant closed-loop system is regular, impulse-free and stable for all admissible uncertainties and time-varying delay. Finally, two numerical examples are given to show the effectiveness of the proposed approach.

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TABLE I: Comparison of delay-dependent stability conditions of Example 1

Methods	[25], [27], [29]	[26], [30]	[31]	[32]	[33], [34]	[28]	[35]	Theorem 1
Maximum d_m allowed	-	0.5567	0.8708	0.9091	0.9680	1.0423	1.0660	1.0660

TABLE II: Delay-dependent stability conditions of Example 1 with different d^*

d^*	0	0.3	0.5	1.1
Maximum d_m allowed	1.0660	1.013	0.9816	0.1524

TABLE III: Delay-dependent stability and stabilization conditions of Example 2 with different d^*

d^*	0	0.5
Maximum d_m allowed	0.8261	0.6493
K	$\begin{bmatrix} -32.2287 & -27.5604 & -49.6299 \end{bmatrix}$	$\begin{bmatrix} -15.3519 & -16.0582 & -25.4512 \end{bmatrix}$

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