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Uniformly Ultimate Boundedness Control for Switched Linear Systems with Parameter Uncertainties

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Abstract— **This paper presents uniformly ultimate boundedness (UUB) control design for switched linear systems with parametric uncertainties. Only the possible bound of the uncertainty is needed. Under arbitrary switching laws, a continuous state feedback control scheme is proposed in order to guarantee uniformly ultimate boundedness of every system response within an arbitrary small neighborhood of the zero state. The design techniques are based on common Lyapunov functions and Lyapunov minimax approach.**

I. INTRODUCTION

A switched system is a particular kind of hybrid system that consists of several subsystems and a switching law determining at any time instant which subsystem is active. There are indeed many switched systems that occur naturally or by design, in the fields of control, communication, computer and signal processes. System analysis of switching dynamics, such as stability, reachability, and controllability has been studied extensively in the recent years. The reader is referred to [1], [2], [3], [4], and [5] for more information. Most of the existing work on control design for switched linear systems is developed without uncertainty. In this paper, we shall extend the scope to address the parametric uncertainty issue.

Consider a switched linear systems represented by the differential equations of the form

$$\begin{aligned}&\dot{x}(t) = A_{\sigma(t)}(\omega)x(t) + B_{\sigma(t)}(\omega)u(t),\\ &\sigma(t) : R^+ \to S = \{1, \cdots, N\},\end{aligned} \tag{1}$$

where state $x(t) \in R^n$, input $u(t) \in R^m$ and R^+ denotes non-negative real numbers. Piecewise constant function $\sigma(t)$ is the switching law indicating the active subsystem at each instant. Assume $A_i(\omega)$, $B_i(\omega)$, $i = 1, \cdots, N$, are continuous functions of $\omega \in \Omega$, where ω is an unknown and possibly fast time-varying vector, and $\Omega \subset R^q$ is a prescribed compact set. The uncertainty is nonlinear and time-varying, and only the possible bound of the set of uncertainty is known.

For this uncertain switched linear system (1), we are interested in seeking a continuous state feedback control such that the closed-loop switched system response $x(t)$ under arbitrary switching laws, enters a neighborhood of the equilibrium $x_e = 0$ in finite time and remains within it thereafter; that is, we desire system performance uniformly ultimate boundedness (UUB) or practical stability.

Definition 1.1: The uncertain switched system (1) under arbitrary switching law $\sigma(t)$ is Uniform Ultimate Bounded (UUB) with ultimate bound b if there exist positive constants b and c, for every $a \in (0, c)$, there is $T = T(a, b)$, such that

$$\|x(0)\| \le a \Rightarrow \|x(t)\| \le b, \forall t \ge T.$$

Uniform stability properties of the switched systems are intimately related to the existence of a common Lyapunov function for all individual subsystems. Various constructive approaches have been presented [4], [5], [6], [12] to find a common quadratic Lyapunov function ensuring the asymptotic stability of switched systems for any switching law. In [4] and [7], Lie algebra conditions are given, which imply the existence of a common quadratic Lyapunov function. In [12], by means of an elegant iterative procedure, a common quadratic Lyapunov function is constructed for switched linear systems with commuting Hurwitz system matrices.

In this paper, we propose to relax the conclusion [12] by utilizing the technique developed in [13]. In [13], necessary and sufficient conditions of quadratic stability of uncertain linear systems are proposed. For the uncertain switched linear systems, if the uncertainty is matched, a robust control scheme is proposed, which renders the switched system UUB, and if the uncertainty is mismatched, we show that a mismatched threshold is needed to ensure stability.

II. STABILITY ANALYSIS OF SWITCHED LINEAR SYSTEMS

Consider the nominal switched linear systems with control input $u(t) = 0$,

$$\begin{aligned}&\dot{x}(t) = A_{\sigma(t)}x(t),\\ &\sigma(t) : R^+ \to S = \{1, \cdots, N\}.\end{aligned} \tag{2}$$

For all $i \in S$, if A_i is Hurwitz, and

$$A_iA_j = A_jA_i, j \in S,$$

then a stability condition for (2) is given below [12].

Theorem 2.1: If $\{A_i : i \in S\}$ is a finite set of commuting Hurwitz matrices, then the corresponding switched linear systems (2) is globally uniform asymptotic stability.

An elegant iterative procedure also given to construct a common quadratic Lyapunov function.

Theorem 2.2: For a given positive definite matrix Q, let $P_1, P_2, \cdots, P_N > 0$ be the unique solutions of the following Lyapunov equations:

$$\begin{aligned}&A_1^TP_1 + P_1A_1 = -Q,\\ &A_i^TP_i + P_iA_i = -P_{i-1}, i = 2, \cdots, N,\end{aligned} \tag{3}$$

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with the condition of Theorem 1, the function $V(x) = x^T P_N x$ is a common Lyapunov function for the switched linear system (2).

Theorem 2.2 shows a systematic way to find a common positive definite matrix P_N in (3). Next, we propose to relax the condition by utilizing the technique developed in [13]. In [13], necessary and sufficient conditions of quadratic stability of uncertain linear systems are proposed.

First, we decompose A_i of (2) as follows:

$$A_i = \bar{A}_i + \Delta A_i, i = 1, \cdots, N, \tag{4}$$

where $\bar{A}_i$ satisfies commuting Hurwitz and ΔA_i is the extra portion.

Substituting (4) into (2), we obtain

$$\begin{aligned} \dot{x}(t) &= (\bar{A}_i + \Delta A_i)_{\sigma(t)} x(t), \\ \sigma(t) &: R^+ \to S = \{1, \cdots, N\}. \end{aligned} \tag{5}$$

From the definition of quadratic stability given in [13], we conclude that system (5) is quadratically stable if there exists a scalar α_i such that

$$x^T[(\bar{A}_i + \Delta A_i)^T P_N + P_N(\bar{A}_i + \Delta A_i)]x \le -\alpha_i \|x\|^2 \tag{6}$$

for all $x \in R^n$.

Above conclusions indicate that stability can also be determined even if uncertainties exist in the switched linear systems (1).

III. UUB CONTROL DESIGN FOR SWITCHED LINEAR SYSTEMS

Based on Theorem 2, we propose a robust control, which renders the uncertain switched linear systems globally UUB by utilizing the Lyapunov minimax approach [11].

Decompose $A_i(\omega)$ and $B_i(\omega)$ into

$$A_i(\omega) = \bar{A}_i + \Delta A_i(\omega), \tag{7}$$

$$B_i(\omega) = \bar{B} + \Delta B_i(\omega), \tag{8}$$

$i = 1, \cdots, N$, where $\bar{A}_i$ satisfies commuting Hurwitz. Therefore, there exists a common positive definite matrix P_N satisfying (3). For the uncertainties term $\Delta A_i(\omega)$ and $\Delta B_i(\omega)$, we discuss the matched and mismatched cases respectively.

A. *ROBUST CONTROL DESIGN FOR MATCHED CASES*

Parametric uncertainty of matched case means there exist continuous function $D_i : \Omega \to R^{m\times n}$ and $E_i : \Omega \to R^{m\times m}$ and a scalar $\delta > 0$ such that for all $\omega \in \Omega$, $i = 1, \cdots, N$,

$$\Delta A_i(\omega) = \bar{B} D_i(\omega), \tag{9}$$

$$\Delta B_i(\omega) = \bar{B} E_i(\omega), \tag{10}$$

$$I + \frac{1}{2}(E_i(\omega) + E_i^T(\omega)) \ge \delta I. \tag{11}$$

For any $\epsilon > 0$, let the control scheme be

$$u(t) = \begin{cases} -\frac{\mu(x,t)}{\|\mu(x,t)\|}\rho(x,t) & if \ \ \|\mu(x,t)\| > \epsilon \\ -\frac{\mu(x,t)}{\epsilon}\rho(x,t) & if \ \ \|\mu(x,t)\| \le \epsilon \end{cases}, \tag{12}$$

where

$$\mu(x,t) = \bar{B}^T P_N x \rho(x,t), \tag{13}$$

$$\rho(x,t) = \frac{1}{\delta} \max_i \max_{\omega\in\Omega} \|D_i(\omega)\| \|x\|. \tag{14}$$

Theorem 3.1: Uncertain switched linear system (1) satisfying the matched conditions (9),(10) is UUB with the state feedback control (12), and the sizes of the uniform ultimate bounded region and the uniform stability region can be made arbitrarily small by a suitable choice of ϵ.

Proof: Choose the Lyapunov function candidate to be

$$V(x) = x^T P_N x. \tag{15}$$

The derivative of $V(x)$ along the trajectory of system (1) is given by

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P_N x + x^T P_N \dot{x} \\ &= [x^T(\bar{A}_i^T + \Delta A_i^T) + u^T(\bar{B}_i^T + \Delta B_i^T)]P_N x \\ &+ x^T P_N[(\bar{A}_i + \Delta A_i)x + (\bar{B}_i + \Delta B_i)u]. \end{aligned} \tag{16}$$

Substituting (9) and (10) into (16) yields

$$\begin{aligned} \dot{V}(x) &= x^T[\bar{A}_i^T P_N + P_N\bar{A}_i + (\bar{B}D_i)^T P_N \\ &+ P_N(\bar{B}D_i)]x + u^T(I + E_i^T)\bar{B}^T P_N x \\ &+ x^T P_N \bar{B}(I + E_i)u]. \end{aligned} \tag{17}$$

Applying the control scheme given by (12), we consider two cases.

(1) if $\|\mu(x,t)\| > \epsilon$:

$$\begin{aligned} \dot{V}(x) &= x^T[\bar{A}_i^T P_N + P_N\bar{A}_i \\ &+ (\bar{B}D_i)^T P_N + P_N(\bar{B}D_i)]x \\ &- \frac{\rho}{\|\mu\|}\mu^T(I + E_i^T)\bar{B}^T P_N x - \frac{\rho}{\|\mu\|}x^T P_N\bar{B}(I + E_i)\mu] \\ &= x^T[\bar{A}_i^T P_N + P_N\bar{A}_i + (\bar{B}D_i)^T P_N + P_N(\bar{B}D_i)]x \\ &- \frac{\rho^2}{\|\mu\|}x^T P_N\bar{B}(2I + E_i^T + E_i)\bar{B}^T P_N x \\ &\le x^T(\bar{A}_i^T P_N + P_N\bar{A}_i)x + 2\delta\|x^T P_N\bar{B}\|\rho - 2\delta\|\mu\| \\ &= x^T(\bar{A}_i^T P_N + P_N\bar{A}_i)x. \end{aligned} \tag{18}$$

For the sake of brevity, let

$$\bar{A}_i^T P_N + P_N\bar{A}_i = -R_i, \tag{19}$$

where,

$$R_i > 0, i \in 1, \cdots, N.$$

Substitute (19) into (18),

$$\dot{V}(x) \le -x^T R_i x \le -\lambda_{\min}(R_i)\|x\|^2. \tag{20}$$

Let

$$\underline{\lambda} = \min_i \lambda_{\min}(R_i),$$

we obtain

$$\dot{V}(x) \le -\underline{\lambda}\|x\|^2. \tag{21}$$

(2) if $\|\mu(x,t)\| \leq \epsilon$:

$$\dot{V}(x) = x^T[\bar{A}_i{}^T P_N + P_N\bar{A}_i + (\bar{B}D_i)^T P_N + P_N(\bar{B}D_i)]x - \frac{\rho}{\epsilon}\mu^T(I+E_i^T)\bar{B}^T P_N x - \frac{\rho}{\epsilon}x^T P_N \bar{B}(I+E_i)\mu]$$

$$= x^T[\bar{A}_i{}^T P_N + P_N\bar{A}_i + (\bar{B}D_i)^T P_N + P_N(\bar{B}D_i)]x - \frac{\rho^2}{\epsilon}x^T P_N\bar{B}(2I + E_i^T + E_i)\bar{B}^T P_N x$$

$$\leq x^T(\bar{A}_i{}^T P_N + P_N\bar{A}_i)x + 2\delta\|\mu\| - 2\delta\frac{\|\mu\|^2}{\epsilon}$$

$$\leq x^T(\bar{A}_i{}^T P_N + P_N\bar{A}_i)x - 2\frac{\delta}{\epsilon}(\|\mu\|^2 - \epsilon\|\mu\|)$$

$$\leq x^T(\bar{A}_i{}^T P_N + P_N\bar{A}_i)x - 2\frac{\delta}{\epsilon}(\|\mu\| - \frac{\epsilon}{2})^2 + \frac{\delta\epsilon}{2}$$

$$\leq -\underline{\lambda}\|x\|^2 + \frac{\delta\epsilon}{2}. \tag{22}$$

Following the standard argument in [11], the controlled system is globally practically stable. The uniform bounded region is with radius

$$\underline{d}(r) = \begin{cases} kR^2 & if \ \ r \leq R, \\ kr^2 & if \ \ r > R, \end{cases} \tag{23}$$

where

$$k = \frac{\lambda_{\max}(P_N)}{\lambda_{\min}(P_N)},$$

$$R = \frac{\delta\epsilon}{2\underline{\lambda}}.$$

The uniform ultimate bounded ball is with radius $\bar{d} > kR^2$ and the maximum amount of time it takes to enter this ball (and remains there thereafter) is

$$T(\bar{d}, r) = \begin{cases} 0 & if \ \ r \leq \bar{R}, \\ \frac{\lambda_{\max}(P_N)r^2 - \lambda_{\min}(P_N)\bar{R}^2}{\underline{\lambda}\bar{R}^2 - \frac{1}{2}\delta\epsilon} & if \ \ r > \bar{R}, \end{cases}, \tag{24}$$

where

$$\bar{R} = k\bar{d}^2.$$

The uniform stability ball is with radius R. Both $\bar{d}$ and R can be made arbitrarily small by an appropriate choice of ϵ. The proof is thus completed.

B. ROBUST CONTROL DESIGN FOR MISMATCHED CASES

In case the matching conditions (9) and (10) are not met, we need to investigate the mismatched case. Let us decompose the uncertainty in the following way:

$$\Delta A_i(\omega) = \bar{B}D_i(\omega) + \Delta\tilde{A}_i(\omega), \tag{25}$$

$$\Delta B_i(\omega) = \bar{B}E_i(\omega) + \Delta\tilde{B}_i(\omega). \tag{26}$$

Let

$$\rho_A = \max_i \max_{\omega\in\Omega} \|\Delta\tilde{A}_i(\omega)\|, \tag{27}$$

$$\rho_B = \max_i \max_{\omega\in\Omega} \|\Delta\tilde{B}_i(\omega)\|, \tag{28}$$

$$\bar{\rho} = \max_i \max_{\omega\in\Omega} \|D_i(\omega\|. \tag{29}$$

Theorem 3.2: Uncertain switched linear system (1) under the mismatched conditions (27),(28) is UUB with the state feedback control (12), if

$$\gamma < \underline{\lambda},$$

where

$$\gamma = 2\lambda_{max}(P_N)(\rho_A + \frac{1}{\delta}\rho_B\bar{\rho}),$$

and the sizes of the uniform ultimate bounded region can be made arbitrarily small by a suitable choice of ϵ.

Proof: Let the Lyapunov function candidate $V(x)$ be the same as (15). The derivative of $V(x)$ along the trajectory of the controlled system of (1) is

$$\dot{V}(x) = \dot{x}^T P_N x + x^T P_N \dot{x} = [x^T(\bar{A}_i{}^T + \Delta A_i^T) + u^T(\bar{B}_i{}^T + \Delta B_i^T)]P_N x + x^T P_N[(\bar{A}_i + \Delta A_i)x + (\bar{B}_i + \Delta B_i)u] + \tilde{e}_i^T P_N x + x^T P_N\tilde{e}_i \tag{30}$$

where

$$\tilde{e}_i(x,\omega) = \Delta\tilde{A}_i(\omega)x + \Delta\tilde{B}_i(\omega)u(x). \tag{31}$$

By the proof of Theorem 3, we have

$$\dot{V}(x) \leq -\underline{\lambda}\|x\|^2 + \frac{\delta\epsilon}{2} + \tilde{e}_i^T P_N x + x^T P_N \tilde{e}_i$$

$$= -\underline{\lambda}\|x\|^2 + \frac{\delta\epsilon}{2} + [\Delta\tilde{A}_i(\omega)x + \Delta\tilde{B}_i(\omega)u(x)]^T P_N x + x^T P_N[\Delta\tilde{A}_i(\omega)x + \Delta\tilde{B}_i(\omega)u(x)]$$

$$\leq -\underline{\lambda}\|x\|^2 + \frac{\delta\epsilon}{2} + 2\lambda_{max}(P_N)(\rho_A + \frac{1}{\delta}\rho_B\bar{\rho})\|x\|^2$$

$$= -\underline{\lambda}\|x\|^2 + \frac{\delta\epsilon}{2} + \gamma\|x\|^2$$

$$= -(\underline{\lambda} - \gamma)\|x\|^2 + \frac{\delta\epsilon}{2}. \tag{32}$$

Therefore, if $\gamma < \underline{\lambda}$ holds, the controlled system of (1) is UUB by following the similar argument as in the proof of Theorem 3.1. The size of the ultimate bounded region can be determined subsequently.

The proof is thus completed.

IV. A NUMERICAL EXAMPLE

Consider a uncertain switched linear system (1) with two subsystems,

$$A_1(\omega) = \begin{pmatrix} 0 & 1 \\ -0.01 + \omega_2(t) & -1 + \omega_1(t) \end{pmatrix},$$

$$B_1(\omega) = \begin{pmatrix} 0 \\ 1.4387 + \omega_3(t) \end{pmatrix},$$

$$A_2(\omega) = \begin{pmatrix} 0 & 1 \\ -0.235 + \omega_2(t) & -1 + \omega_1(t) \end{pmatrix},$$

$$B_2(\omega) = \begin{pmatrix} 0 \\ 0.5613 + \omega_3(t) \end{pmatrix},$$

where the uncertain parameters

$$\|\omega_1(t)\| \leq 0.5, \|\omega_2(t)\| \leq 1.0, \|\omega_1(t)\| \leq 0.25,$$

for all $t \geq 0$.

Decompose $A_i(\omega)$, $B_i(\omega)$, $i = 1, 2$ as in form (7),(8),

$$\bar{A}_i = \begin{pmatrix} 0 & 1 \\ -0.01 & -1 \end{pmatrix},$$

$$\bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\Delta A_1(\omega) = \begin{pmatrix} 0 & 0 \\ \omega_2(t) & \omega_1(t) \end{pmatrix},$$

$$\Delta B_1(\omega) = \begin{pmatrix} 0 \\ 0.4387 + \omega_3(t) \end{pmatrix},$$

$$\Delta A_2(\omega) = \begin{pmatrix} 0 & 0 \\ -0.225 + \omega_2(t) & \omega_1(t) \end{pmatrix},$$

$$\Delta B_2(\omega) = \begin{pmatrix} 0 \\ -0.4387 + \omega_3(t) \end{pmatrix},$$

which satisfies the matched condition (9),(10).

We can choose $\delta = 0.4$, then get

$$\rho = 3.375\|x\|,$$

and

$$P_2 = \begin{pmatrix} 5050.5 & -50.2 \\ -50.2 & 0.8 \end{pmatrix} > 0,$$

Therefore,

$$\mu = \bar{B}^T P_N x \rho$$

$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 5050.5 & -50.2 \\ -50.2 & 0.8 \end{pmatrix} x\rho$$

$$= (-169.4x_1 + 2.7x_2)\|x\|.$$

V. CONCLUSIONS

A system way to design a robust control for uncertain switched systems is suggested. The uncertainty may or may not meet the matched condition. The resulting controlled system performance, under the matching condition, is (global) uniformly ultimate bounded. In the mismatched case, if the mismatched portion of the uncertainty is within a threshold, which is designated by $\underline{\lambda}$, as shown in Theorem 3.2, the same performance is guaranteed.

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