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Observer with Sample-and-Hold Updating for Lipschitz Nonlinear Systems with Nonuniformly Sampled Measurements

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Abstract— **This paper presents a new observer design for Lipschitz nonlinear continuous-time systems with nonuniformly sampled measurements. Based on recent results in sampled-data control of linear continuous-time systems, linear matrix inequality (LMI) conditions are established to guarantee global stability of the estimation error dynamics and to design the observer matrix. The applicability of the proposed observer is demonstrated via two examples, that are the flexible joint robotic arm and Chua's circuit.**

I. INTRODUCTION

Many observer design techniques for nonlinear continuous-time systems have been developed in the last decades, e.g. the extended Luenberger observer [9], the Lipschitz observer [8], or the circle-criterion observer [2]. All these observer design techniques are based on the assumption that the output of the nonlinear system is measured continuously, i.e. $y(t) = Cx(t)$. However in some engineering applications, e.g. in network control systems (NCSs) in which the output is transmitted over a shared digital communication network, the output is only available at discrete time instants, i.e. $y_k = Cx(t_k)$. For nonlinear continuous-time systems with sampled measurements notable observer design techniques, thereunder the Kalman filtering technique or the high-gain observer design technique, are e.g. [3–5] and the references therein.

In this paper a new observer for Lipschitz nonlinear continuous-time systems with nonuniformly sampled measurements is designed via LMI-based sampled-data control techniques. Since the measurements are only available at discrete time instants, the proposed observer is updated in a sample-and-hold fashion. Due to this the dynamics of the observer, accordingly also the dynamics of the estimation error, is described in a hybrid way. To ensure that the proposed observer globally reconstructs the system state, the stability of the hybrid nonlinear estimation error dynamics is analyzed with a discontinuous Lyapunov function proposed in [7]. From this analysis, LMI conditions are established to guarantee global stability of the hybrid nonlinear estimation error dynamics and to design the observer matrix.

The remainder of the paper is organized as follows: In Section II an observer with sample-and-hold updating for Lipschitz nonlinear continuous-time systems with nonuniformly sampled measurements is presented. In Section III the applicability of the proposed observer is demonstrated via examples. Finally, Section IV concludes the paper.

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II. MAIN RESULT

Consider the Lipschitz nonlinear continuous-time system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \rho(t, u(t)) + G\sigma(Hx(t)) \\ y_k &= Cx(t_k) \end{aligned} \tag{1}$$

with initial condition $x(t_0) = x_0$. In (1) $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^p$ the control input, $y \in \mathbb{R}^q$ the measured output, that is available at time instants t_k satisfying $0 \leq t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots$ with $\lim_{k\to\infty} t_k = \infty$ and time-varying intervals $\delta_{k+1} = t_{k+1} - t_k$. Furthermore, $A \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{q\times n}$ are constant matrices and $\rho : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^n$ is a locally Lipschitz function which depends on known arguments. Finally, it is assumed that (A, C) is observable and that $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is a Lipschitz nonlinearity with Lipschitz constant $\gamma > 0$, i.e.

$$||\sigma(v) - \sigma(w)|| \leq \gamma||v - w|| \ \ \forall v, w \in \mathbb{R}^m. \tag{2}$$

The proposed observer for system (1) is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \rho(t, u(t)) + G\sigma(H\hat{x}(t)) + K(t_k) \tag{3}$$

with $\hat{x} \in \mathbb{R}^n$, initial condition $\hat{x}(t_0) = \hat{x}_0$, and $K(t_k) = L(\hat{y}(t_k) - y(t_k))$ for $t \in [t_k, t_{k+1})$, where $\hat{y}(t_k) = C\hat{x}(t_k)$ is the estimated output and $L \in \mathbb{R}^{n\times q}$ the observer matrix. Note that the dynamics of observer (3) is of hybrid nature. The reason for this is that the state estimate $\hat{x}$ is described in continuous-time while the innovation process $\hat{y}(t_k) - y(t_k)$ is only changed at time instants t_k, i.e. the current innovation process is used until the new innovation process is available. To show that observer (3) estimates the state of system (1), the stability of the estimation error dynamics

$$\dot{e}(t) = Ae(t) + G\upsilon(t, e(t)) - K(t_k) \tag{4}$$

with $e = x - \hat{x}$ and $\upsilon(t, e) = \sigma(Hx) - \sigma(H(x - e))$ is studied. The next theorem provides sufficient conditions, expressed in terms of LMIs, to guarantee that estimation error dynamics (4) is globally exponentially stable.

Theorem 1: Suppose that nonlinearity σ of system (1) satisfies condition (2) and that the time-varying sampling intervals δ_{k+1} are bounded by $\underline{\delta} \leq \delta_{k+1} \leq \overline{\delta}$ for all $k \in \mathbb{N}_0$ with $0 \leq \underline{\delta} \leq \overline{\delta}$. Then observer (3) globally exponentially estimates the state of system (1) if there exist matrices $P = P^T > 0$ with $P \in \mathbb{R}^{n\times n}$, $\tilde{L} \in \mathbb{R}^{n\times q}, N_1 \in \mathbb{R}^{n\times n}$, $N_2 \in \mathbb{R}^{n\times n}$, $N_3 \in \mathbb{R}^{m\times n}$ such that LMI conditions (9) are satisfied. The actual observer matrix L is given by $L = P^{-1}\tilde{L}$.

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Proof: The main ideas of the following proof are based on the proof of Theorem 2 in [7] for linear sampled-data systems. First of all, estimation error dynamics (4) is rewritten as

$$\begin{aligned} \dot{e}(t) &= Ae(t) + G\upsilon(t, e(t)) + LC\epsilon(t) & \\ \dot{\epsilon}(t) &= 0 & t_k \le t < t_{k+1} \\ e(t_k^+) &= e(t_k) & t = t_k \\ \epsilon(t_k^+) &= \epsilon(t_k) & k = 0, 1, \ldots \end{aligned} \tag{5}$$

in order to consider more easily its hybrid effects in the stability analysis. The stability of (5) is studied via the function

$$\begin{aligned} V(t) &= e^T(t)Pe(t) + \int_{t-\tau(t)}^{t} (\bar{\delta} - t + s)\dot{e}^T(s)P\dot{e}(s)\, ds \\ &\quad + (\bar{\delta} - \tau(t))(e(t) - \epsilon(t))^T P(e(t) - \epsilon(t)) \end{aligned} \tag{6}$$

with $\tau(t) = t - t_k$, $t_k \le t < t_{k+1}$, for $k = 0, 1, \ldots, \infty$. The time derivative of (6) along the trajectories of estimation error dynamics (5) is

$$\begin{aligned} \dot{V}(t) &= \xi^T(t) \underbrace{\begin{bmatrix} PA + A^TP & PLC & PG \\ C^TL^TP & 0 & 0 \\ G^TP & 0 & 0 \end{bmatrix}}_{Q_1} \xi(t) \\ &\quad + \bar{\delta}\xi^T(t)Q_2\xi(t) - \int_{t-\tau(t)}^{t} \dot{e}^T(s)P\dot{e}(s)ds \\ &\quad + \xi^T(t) \underbrace{\begin{bmatrix} -P & P & 0 \\ P & -P & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{Q_3} \xi(t) \\ &\quad + (\bar{\delta} - \tau(t))\xi^T(t)Q_4\xi(t) \end{aligned} \tag{7}$$

with $\xi = [e^T \ \epsilon^T \ \upsilon^T]^T$ and Q_2, Q_4 defined in (8).

Using the fact that

$$0 \le \int_{t-\tau(t)}^{t} \begin{bmatrix} \dot{e}(s) \\ \xi(t) \end{bmatrix}^T \begin{bmatrix} P & N^T \\ N & NP^{-1}N^T \end{bmatrix} \begin{bmatrix} \dot{e}(s) \\ \xi(t) \end{bmatrix} ds \tag{10}$$

holds for any matrix $N = [N_1^T \ N_2^T \ N_3^T]^T$ and that

$$0 \le \xi^T(t) \underbrace{\begin{bmatrix} \gamma^2 H^TH & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{bmatrix}}_{Q_5} \xi(t)$$

is satisfied due to (2), one obtains

$$\dot{V}(t) \le \xi^T(t)(R_1 + \tau(t)R_2 + (\bar{\delta} - \tau(t))R_3)\xi(t) \tag{11}$$

with $R_1 = Q_1 + \bar{\delta}Q_2 + Q_3 + Q_5 + Q_6$, where Q_6 is

$$Q_6 = \begin{bmatrix} -N_1 - N_1^T & N_1 - N_2^T & -N_3^T \\ N_1^T - N_2 & N_2 + N_2^T & N_3^T \\ -N_3 & N_3 & 0 \end{bmatrix}, \tag{12}$$

$R_2 = N^TP^{-1}N$, and $R_3 = Q_4$. To ensure that $\dot{V}(t)$ is negative definite, $R_1 + \tau(t)R_2 + (\bar{\delta} - \tau(t))R_3 < 0$ has to hold for $0 \le \tau(t) \le \bar{\delta}$. To avoid the time dependency it is shown in [7] that (11) is negative definite iff the following LMI conditions, which are equivalent to (9) by setting $\tilde{L} = PL$ and applying the Schur complement to Q_2, R_2, are feasible:

$$R_1 + \bar{\delta}R_3 < 0, \quad R_1 + \bar{\delta}R_2 < 0. \tag{13}$$

Hence, global exponential stability of estimation error dynamics (4) follows from the feasibility of LMI conditions (9) since in that case V is a Lyapunov function for (5), i.e. V satisfies all conditions, including the important conditions $\dot{V}(t) < 0$ and $V(t_k) \le \lim_{t \to h} V(t_k - h)$, to guarantee stability of hybrid systems [7]. ■

$$Q_2 = \begin{bmatrix} A^T \\ C^TL^T \\ G^T \end{bmatrix} P \begin{bmatrix} A & LC & G \end{bmatrix}, \; Q_4 = \begin{bmatrix} PA + A^TP & PLC - A^TP & PG \\ C^TL^TP - PA & -PLC - L^TC^TP & -PG \\ G^TP & -G^TP & 0 \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} (1+\bar{\delta})(PA + A^TP) + \gamma^2H^TH - P - N_1 - N_1^T & (1+\bar{\delta})\tilde{L}C + P - \bar{\delta}A^TP + N_1 - N_2^T & (1+\bar{\delta})PG - N_3^T & \bar{\delta}A^TP \\ (1+\bar{\delta})C^T\tilde{L} + P - \bar{\delta}PA + N_1^T - N_2 & -\bar{\delta}(\tilde{L}C + C^T\tilde{L}^T) - P + N_2 + N_2^T & -\bar{\delta}PG + N_3^T & \bar{\delta}C^T\tilde{L}^T \\ (1+\bar{\delta})G^TP - N_3 & -\bar{\delta}G^TP + N_3 & -I & \bar{\delta}G^TP \\ \bar{\delta}PA & \bar{\delta}\tilde{L}C & \bar{\delta}PG & -\bar{\delta}P \end{bmatrix} < 0 \tag{9a}$$

$$\begin{bmatrix} PA + A^TP + \gamma^2H^TH - P - N_1 - N_1^T & \tilde{L}C + P + N_1 - N_2^T & PG - N_3^T & \bar{\delta}A^TP & \bar{\delta}N_1 \\ C^T\tilde{L}^T + P + N_1^T - N_2 & -P + N_2 + N_2^T & N_3^T & \bar{\delta}C^T\tilde{L}^T & \bar{\delta}N_2 \\ G^TP - N_3 & N_3 & -I & \bar{\delta}G^TP & \bar{\delta}N_3 \\ \bar{\delta}PA & \bar{\delta}\tilde{L}C & \bar{\delta}PG & -\bar{\delta}P & 0 \\ \bar{\delta}N_1^T & \bar{\delta}N_2^T & \bar{\delta}N_3^T & 0 & -\bar{\delta}P \end{bmatrix} < 0 \tag{9b}$$

Remark 1: If an Lipschitz observer is designed under the assumption that the system output is continuously measurable, i.e. $y(t) = Cx(t)$ instead of $y_k = Cx(t_k)$ in (1), its convergence behavior with nonuniformly sampled measurements can be analyzed via the less conservative function [7]

$$V(t) = e^T(t)Xe(t) + \int_{t-\tau(t)}^{t} (\overline{\delta} - t + s)\dot{e}^T(s)Y\dot{e}(s)\, ds + (\overline{\delta} - \tau(t))(e(t) - \epsilon(t))^T Z(e(t) - \epsilon(t)), \tag{14}$$

where X, Y, Z are symmetric positive definite matrices. Note that (6) is obtained from (14) by setting $P = X = Y = Z$.

III. Examples

In this section the applicability of the proposed observer design for Lipschitz nonlinear systems with nonuniformly sampled measurements is demonstrated via two examples.

A. Flexible Joint Robotic Arm

In the following observer (3) is applied to estimate the state of a flexible joint robotic arm [6] shown in Figure 1. The dynamics of this robotic arm is described by (1) with system state $x^T = [x_1\ x_2\ x_3\ x_4]$, system matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, H^T = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, C^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

function $\rho(t, u) = Bu$, and nonlinearity $\sigma(Hx) = 3.3 \sin x_3$ with Lipschitz constant $\gamma = 3.3$. Furthermore, it is assumed that $\overline{\delta} = 0.1$. Solving the LMI conditions of Theorem 1 with the above specified matrices and constants, one obtains the observer matrix $L^T = \begin{bmatrix} -52 & 20.7 & -1.2 & -9.7 \\ -0.5 & -6.4 & -0.3 & 2.0 \end{bmatrix}$. The simulation results with $u(t) = \sin(t)$ and time-varying sampling intervals δ_k (uniform probability distribution between $\underline{\delta} = 0.001$ and $\overline{\delta} = 0.1$) are plotted in Figure 3 and it can be seen that the system state of the flexible joint robotic arm is reconstructed.

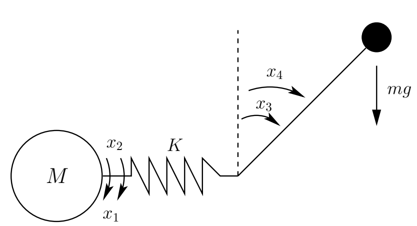

Fig. 1. Flexible joint robotic arm.

B. Chua's Circuit

Now observer (3) is applied to Chua's circuit, that is shown in Figure 2. The dynamics of this circuit, see e.g. [1], is given by (1) with $x^T = [x_1\ x_2\ x_3]$, system matrices

$$A = \begin{bmatrix} -3.2 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14.87 & 0 \end{bmatrix}, G = H^T = C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

function $\rho(t, u) = 0$, and nonlinearity $\sigma(Hx) = 2.95(|x_1 + 1| - |x_1 - 1|)$. Solving LMI conditions (9) with the above specified matrices and constant $\overline{\delta} = 0.05$, one obtains the observer matrix $L^T = \begin{bmatrix} -0.9 & -1.3 & 1.2 \end{bmatrix}$. Figure 4 shows that the system state of Chua's circuit is estimated via observer (3).

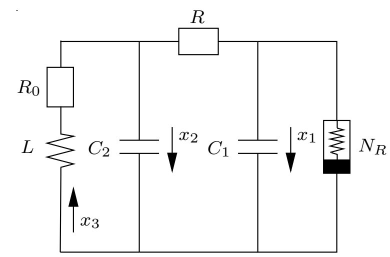

Fig. 2. Chua's circuit.

IV. Summary

In this paper an observer design for Lipschitz nonlinear continuous-time systems with nonuniformly sampled measurements has been proposed. To the authors' best knowledge, the proposed LMI-based observer design approach is new for the considered observer design problem. Finally, the applicability of the proposed observer has been successfully demonstrated via two well-known observer examples, that are the flexible joint robotic arm and Chua's circuit.

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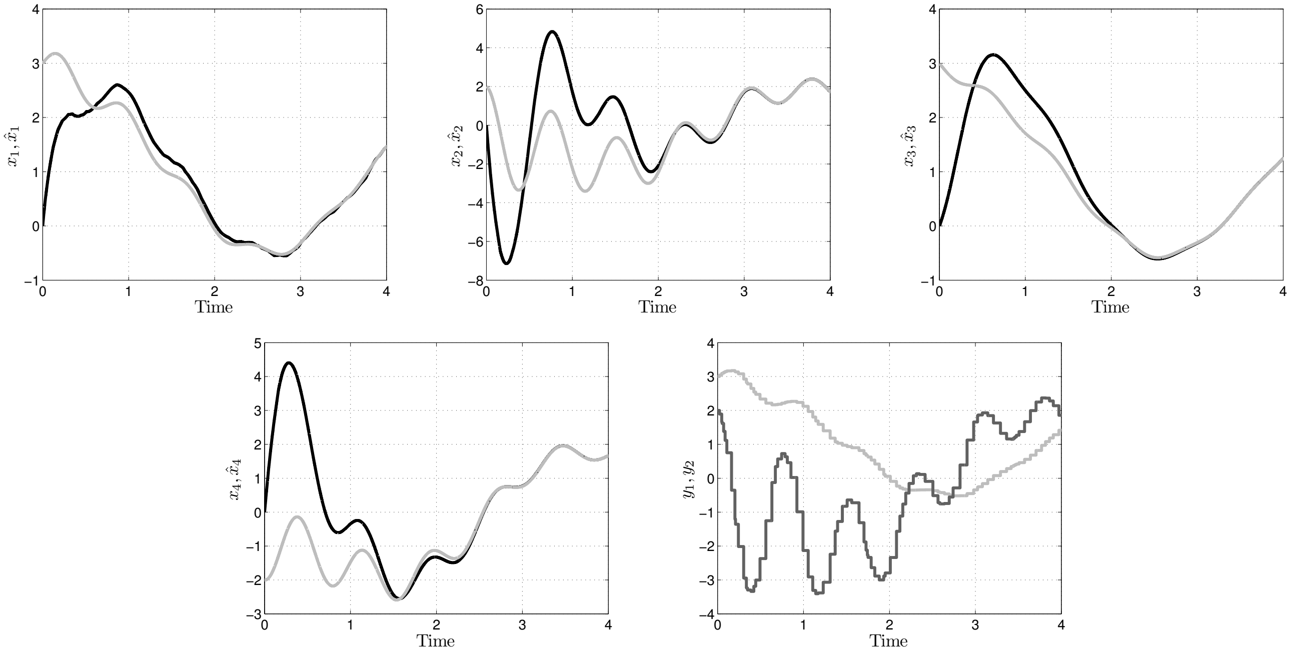

Fig. 3. Simulation results of Section III.A: The system state of the flexible joint robotic arm (gray) with input $u(t) = \sin(t)$, measured output y_1 (gray), measured output y_2 (dark gray), time-varying sampling intervals δ_k that are modeled by an uniform probability distribution between $\underline{\delta} = 0.001$ and $\overline{\delta} = 0.1$, and initial condition $x_0 = [3\ 2\ 3\ -2]^T$ is estimated via observer (3) (black) with initial condition $\hat{x}_0 = [0\ 0\ 0\ 0]^T$.

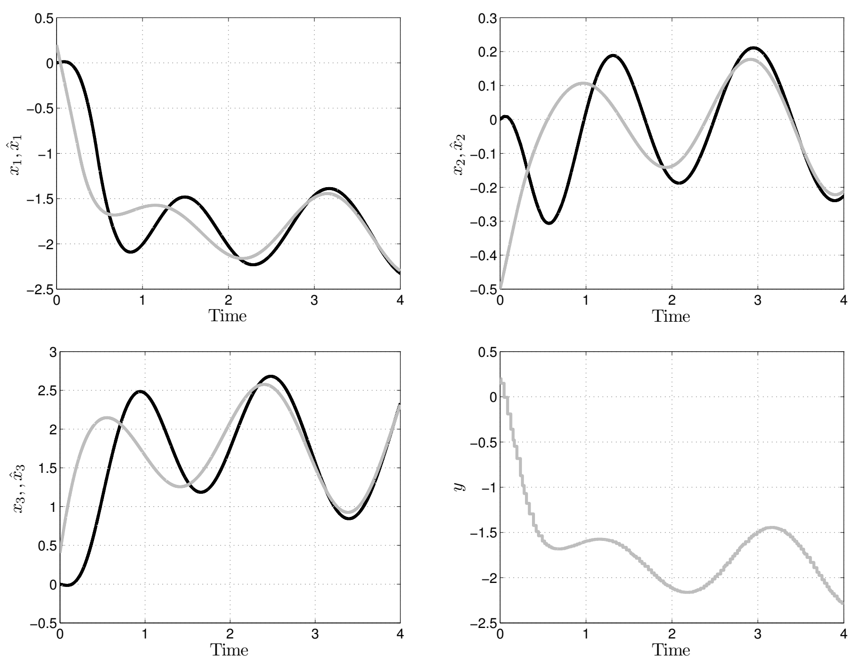

Fig. 4. Simulation results of Section III.B: The system state of Chua's circuit (gray) with measured output y, time-varying sampling intervals δ_k that are modeled by an uniform probability distribution between $\underline{\delta} = 0.001$ and $\overline{\delta} = 0.05$, and initial condition $x_0 = [0.2\ -0.5\ 0.4]^T$ is estimated via observer (3) (black) with initial condition $\hat{x}_0 = [0\ 0\ 0]^T$.