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Robust Tracking for Matched Disturbance Nonlinear Systems

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***Abstract*— In this paper we extend a robust tracking method to more general uncertainties or matched disturbances. Therefore the tracking method consisting of an inverse control part and a feedback control part will be repeated and extended in such a way that robustness bounds can be guaranteed even in presence of constant disturbances. Furthermore, besides robustness bounds we will discuss performance bounds as well. Therefore we will focus on the state penalty of the performance value in the case of stabilizing stationary operating points in order to simplify the tuning of the feedback controller. At the end, a simulation result will verify the design procedure.**

I. INTRODUCTION

Recently in [1] a novel approach for robust output tracking has been presented. The main novelty of this approach was that although there are model uncertainties and disturbances, an inverse controller is applied in order to achieve a good tracking performance. This of course requires a robustifying feedback controller which guarantees robust stability (of the tracking error differential equations) as it is discussed in [1]. The approach in [1] is restricted to disturbances which are square integrable. Hence, constant disturbances, which often occur in practical life (actuator faults, sensor offsets,...), are not considered so far. Furthermore, since the robustifying feedback controller is calculated using a central robust stabilizing controller, the performance of robustifying feedback controller results from the central controller as well. Hence, tuning the central controller includes the tuning of the robustifying feedback controller. Thus, it would be useful to estimate the performance of the robustifying feedback controller *a priori* to consider it already in the tuning phase of the stabilizing controller.

The purpose of this paper is to improve the controller design and to deal with a more general class of matched disturbances.

The inverse control part of the presented strategy is quite standard and can be realized by the use of the inverse control approach of [3], [4], [7] or [5]. For designing a robustifying feedback controller a central stabilizing controller that guarantees robustness in terms of the L_2 gain from the disturbance input to the performance output (see *e.g.* [11] for linear systems) is extended for solving the tracking problem. Instead, the tracking problem is solved by formulating a robust feedback control problem which guarantees robust stability and robust performance.

The paper is organized as follows: in the next section the main results of [1] are recalled, since these results are the base of the following considerations. After this, in section III, it is shown how the tracking controller can be extended to a more general kind of matched disturbances. Furthermore, the state penalty will be explained and a theorem for a guaranteed state penalty will be given. Finally a simulation result shows the effectiveness of the presented method. The paper will be completed with the conclusions.

II. ROBUST TRACKING FOR A CLASS OF NONLINEAR SYSTEMS

A. Problem Statement

Consider a nonlinear uncertain system that has a well defined relative degree:

$$\dot{x} = f(x) + g(x)(u + w) \tag{1a}$$
$$y = h_y(x) \tag{1b}$$
$$z = h_e(x) + d(x)u \tag{1c}$$

where $x \in R^n$ is the state, $u \in R^m$ the control input, $y \in R^p$ the output variable and $z \in R^p$ the performance variable for a stabilizing optimal L_2 or suboptimal H_∞ controller (see section [2]) and $w \in R^m$ is the disturbance input. Note that by this definition it is possible to consider input disturbances caused *e.g.* by actuator uncertainties as well as model uncertainties.

For system (1) we assume that $x = 0$ is an equilibrium point, i.e. $f(0) = 0$, and that $h_2(0) = 0$. Moreover, we also assume that $f(x)$ is a smooth vector field and that $g(x)$, $h_e(x)$ and $h_y(x)$ are smooth mappings. Without loss of generality it is assumed that $d(x)'h_e(x) = 0$. Finally, assume that $g(x)$ and $d(x)$ has full column rank. In particular, $d(x)'d(x) = R(x) > 0$, for each x.

For $w = 0$ we assume that there exists a feedforward control law that guarantees perfect tracking $y = \tilde{y}$ for any reference output $\tilde{y}$ as long as $\tilde{x} \in X \subset R$ ($\tilde{x}$ is the corresponding desired state trajectory), $\dot{\tilde{x}}$ exists, and the initial state is well known. This feedforward control law, called $k_{ff}\left(\dot{\tilde{x}}, \tilde{x}\right)$, can be calculated using different methods (see [3], [4], [7] and [5]). It is obvious that if $w \neq 0$ the feedforward control law fails and a tracking error will occur. Hence, we want to solve a problem

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which is called **R**obustifying the **F**eedforward **C**ontrol (RFC) problem.

B. Definitions and preliminaries

Definition 1 (RFC problem): Consider the nonlinear system with matched disturbances (1) and the feedforward control law $k_{ff}\left(\dot{\tilde{x}},\tilde{x}\right)$. Find an additional feedback controller $k_{fb}\left(\tilde{x},x\right)$ such that the control law

$$u = k_{ff}\left(\dot{\tilde{x}},\tilde{x}\right) + k_{fb}\left(\tilde{x},x\right) \tag{2}$$

with

$$\left.k_{fb}\left(\tilde{x},x\right)\right|_{x=\tilde{x}} = 0$$

asymptotically stabilizes the closed loop system $\forall\tilde{x}\in X$ and the state error

$$e = x - \tilde{x} \tag{3}$$

is bounded in L_2 as long as the input disturbance is bounded in L_2.

Remark 1: For validating the performance of the controller that satisfies the RFC problem, we introduce an additional performance variable $z_e = h_e\left(e\right)$ where $h_e(e)$ is the state error depending output map.

Definition 2 (Gradient difference field): The gradient difference field in x is defined as difference of the gradients of a scalar function $V\left(x\right)$ at two different points x and $x+e$ for $x,e\in R^n$.

$$\Gamma_x\left(e\right) = \left.\frac{\partial V(x)}{\partial x}\right|_{x+e} - \left.\frac{\partial V(x)}{\partial x}\right|_{x} \tag{4}$$

Lemma 1: For any point x the gradient difference field $\Gamma_x\left(e\right)$ defines the gradient of a local scalar function $V_x\left(e\right)$ (scalar function of e which is locally varying with x).

$$\Gamma_x\left(e\right) = \frac{\partial V_x(e)}{\partial e}$$

Lemma 2: If $V(x)$ is a scalar convex function in the convex set $x\in C\subseteq X$ then the local scalar function $V_x\left(e\right)$ from Lemma 1 is convex, too. Furthermore, with $V_x\left(0\right)=0$ the function is greater than or equal to zero.

For the proof of these lemmas we want to refer the reader to [1]

Remark 2: The local scalar function can be calculated using the path independent line integral

$$V_x\left(e\right) = \int_0^e \Gamma_x\left(e\right)de. \tag{5}$$

Setting $V_x\left(0\right)=0$ ensures that the scalar function is positive and if it is convex it is positive definite. Hence this function may be a candidate Lyapunov function.

Definition 3: The distribution of the error vector field along a given trajectory caused by an imperfect feedforward tracking law is called the feedforward tracking error vector field and is defined as

$$l_{\tilde{x},\dot{\tilde{x}}}\left(e\right) = f\left(\tilde{x}+e\right) + g\left(\tilde{x}+e\right)k_{ff}\left(\dot{\tilde{x}},\tilde{x}\right) - \dot{\tilde{x}}$$

Definition 4: The maximum eigenvalue of the input weighting matrix is defined as follows

$$\overline{\lambda}_R = \max_{x\in X}\lambda_{\max}\left(R\left(x\right)\right)$$

C. Solution of the RFC problem

Consider a system given by (1) and assume that a stabilizing controller with guaranteed robustness bounds exists. As shown in [8], [2], [13] and in [6] such a controller for a given system (1) is

$$u = -R\left(x\right)^{-1}g\left(x\right)'\frac{\partial V(x)}{\partial x}' \tag{6}$$

where $V\left(x\right)$ is a control Lyapunov function which guarantees robustness and which may be the solution of the Hamilton Jacobi Belmann (HJB) for the L_2 optimal control law or for the Hailton Jacobi ISsaak for the suboptimal H_∞ control law. For the latter approaches it has been shown that robustness (see [9] for linear systems and [2] for nonlinear systems) in terms of a maximum H_∞ attenuation level from the input disturbance w to the performance output variable z can be guaranteed and calculated *a priori*.

D. Sufficient conditions for solving the RFC problem

Assume that there exists a stabilizing control law (6) which is either an optimal L_2 controller or a suboptimal H_∞ controller. Thus, there exists a solution - called $V(x)$ - of the HJB or of the HJI for which it should be assumed that $V\left(x\right)$ is strictly convex and its Hessian matrix does not vanish.

$$\frac{\partial^2 V(x)}{\partial x^2} > 0, \qquad \forall x\in X. \tag{7}$$

Since $V\left(x\right)$ is a convex function we can now apply Lemma 2 and the Remark 2 to define a local scalar positive definite function $V_x\left(e\right)$ by using the gradient difference field (see Definition 2). The following theorem is then sufficient for solving the RFC problem.

Theorem 1: If there exists a positive definite scalar function $V(x)$ which obeys (7) such that the controller (6) guarantees robustness and if there exist positive constants $k>0$, $\kappa\geq 1$ and $0<\alpha<\kappa/\left(k\overline{\lambda}_R\right)$ such that the scalar function $V_{\tilde{x}}\left(e\right)$ satisfies

$$\begin{aligned}\frac{m}{\alpha}\left\|\frac{\partial V_{\tilde{x}}(e)}{\partial \tilde{x}}\right\|_2 + \frac{1}{\alpha}\frac{\partial V_{\tilde{x}}(e)}{\partial e}l_{\tilde{x},\dot{\tilde{x}}}\left(e\right) + \frac{1}{2}h_e\left(e\right)'h_e\left(e\right)\\ -k\frac{\partial V_{\tilde{x}}(e)}{\partial e}g\left(\tilde{x}+e\right)g\left(\tilde{x}+e\right)'\frac{\partial V_{\tilde{x}}(e)}{\partial e}' < 0\end{aligned} \tag{8}$$

for all $x\in X$, for a proper but constraint error region $\|e\|<E$ and for $m\geq\max_{\forall t}\left\|\dot{\tilde{x}}\left(t\right)\right\|_2$, then a robust

tracking controller which solves the RFC problem is defined by

$$k_{fb}(\tilde{x},x) = -\tilde{R}(x)^{-1} r_{\tilde{x}}(e) \tag{9}$$

with

$$r_{\tilde{x}}(e) = g(\tilde{x}+e)' \frac{\partial V_{\tilde{x}}(e)}{\partial e}'$$
$$\tilde{R}(x) = R(x)/\kappa$$

Theorem 2: If there exists a solution for (8) with $\alpha = \kappa / \left(2k\bar{\lambda}_R\right)$ then the controller (9) ensures an H_∞ attenuation level $\|T_{z_e,w}\|_\infty \leq \gamma$, where

$$\gamma = \frac{\bar{\lambda}_R\sqrt{2k}}{\kappa} \tag{10}$$

For the proof of Theorem 1 and Theorem 2, we want to refer to [1]. Furthermore in [1] the following remark is shown.

Remark 3: If there is no solution satisfying Theorem 2 but there is a solution satisfying Theorem 1 which means that $\alpha \neq \kappa / \left(2k\bar{\lambda}_R\right)$ and $0 < \alpha < \kappa / \left(k\bar{\lambda}_R\right)$, it can be concluded that the guaranteed robustness bound is

$$\gamma = \sqrt{\frac{\bar{\lambda}_R}{2\alpha\kappa - 2\alpha^2\bar{\lambda}_R k}} \tag{11}$$

III. Tracking control in presence of constant disturbances

In the previous sections it is assumed that $w \in L_2$. However, this assumption cannot be applied to frequent applications. Therefore, we enlarge the class of disturbances by assuming they can be separated into a square integrable part and a constant part.

$$w(t) = w_{si}(t) + w_o(t)$$

where

$$\int_0^\infty w_{si}(\tau)' w_{si}(\tau)\, d\tau \quad < \quad \infty \quad \text{and} \quad w_o(t) = const.$$

To compensate the constant part of the input disturbance it is necessary to add an integrator to the controller. There are several possibilities to consider this integrator. One of them, the most common method, is to consider the integrator of the controller as a part of the system to be controlled (see [12]). This would increase the system order and would increase the effort for the design of the controller. In this paper we present another method. To be precise, the controller is designed as in the case of square integrable input disturbance and is then extended by an integrator. The advantage of this method is that the problem is very simple to be solved. The disadvantage is that it is not possible to guarantee the same robustness level than without an integrator in the controller. The following theorem explains how to add the integrator to the controller and how robustness of the closed loop is affected. Suppose that the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)(u+w) && (12a)\\ y &= x && (12b)\\ z &= h(x) + d(x)u, && (12c) \end{aligned}$$

for which we use similar assumptions accordingly for $f(x)$, $g(x)$, $h(x)$ and $d(x)$ as in system (1), is controlled using the robust tracking controller mentioned above, described in detail in [1]

$$u = u_\star = k_{ff}\left(\dot{\tilde{x}},\tilde{x}\right) + k_{fb}(\tilde{x},x) \tag{13}$$

where k_{ff} is the inverse control law and k_{fb} the robustifying feedback controller (9) that restricts the L_2- gain (from w to z) as stated in Theorem 1 and Theorem 2 if $w_0 = 0$. Finally, consider the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)(u_\star + \bar{u} + w) && (14a)\\ y &= x && (14b)\\ \bar{z} &= h_e(e) + d(x)u_\star && (14c) \end{aligned}$$

where $e = x - \tilde{x}$ is the tracking error and $h_e(e)$ is a performance function whereas it is assumed that $h_e(e)$ is orthogonal to $d(x)$. We are ready to prove the following result:

Theorem 3: The integral control action

$$\bar{u} = -\frac{1}{\alpha}\int_{t=0}^{\infty} g(\tilde{x}+e)' \frac{\partial V_{\tilde{x}}(e)}{\partial e}' dt \tag{15}$$

with $0 < \alpha < \frac{\kappa}{\bar{\lambda}_R k}$, is such that the L_2 - attenuation level of the closed loop system (14) with (13) and (15) (from w to $\bar{z}$) satisfies

$$\tilde{\gamma} \leq \sqrt{\frac{\bar{\lambda}_R}{2\alpha\kappa - 2\alpha^2\bar{\lambda}_R k}} \tag{16}$$

Proof: Using the robust tracking controller as presented in the theorem taking $\varrho = \frac{1}{\alpha}$ the closed loop error system ($\dot{e} = \dot{x} - \dot{\tilde{x}}$) forms

$$\begin{aligned} \dot{e} &= f(\tilde{x}+e) + g(\tilde{x}+e)\, k_{ff}\left(\tilde{x},\dot{\tilde{x}}\right) + g(\tilde{x}+e)\, w\\ &\quad -\dot{\tilde{x}} - g(\tilde{x}+e)\,\tilde{R}^{-1}(\tilde{x}+e)\, r_{\tilde{x}}(e) + g(\tilde{x}+e)\,\bar{u}\\ \dot{\xi} &= \varrho r_{\tilde{x}}(e)\\ \bar{z} &= h_e(e) - d(x)\tilde{R}^{-1}(\tilde{x}+e)\, r_{\tilde{x}}(e) \end{aligned}$$

where ξ defines the state of the feedback law, or the state of the integrator. The proof follows the rationale of the proof of Theorem 2 in [2] for the augmented closed-loop error system, characterized by the state error vector $[e' \ \eta']'$, where $\eta = w_0 - \xi$. With the notation of the feedforward tracking error vector field this results in

$$\begin{aligned} \dot{e} &= l_{\tilde{x},\dot{\tilde{x}}}(e) + g(\tilde{x}+e)(w_{si} + \eta)\\ &\quad -g(\tilde{x}+e)\,\tilde{R}^{-1}(\tilde{x}+e)\, r_{\tilde{x}}(e) + g(\tilde{x}+e)\,\bar{u}\\ \dot{\eta} &= -\varrho r_{\tilde{x}}(e)\\ \bar{z} &= h_e(e) - d(x)\tilde{R}^{-1}(\tilde{x}+e)\, r_{\tilde{x}}(e) \end{aligned}$$

A consequence of the asymptotic stability of the closed-loop system is that both the tracking error and the offset of the disturbance w_0 is compensated through the controller (as obvious), i.e. η tends to zero. The only thing left is to show the robustness of the tracking controller. Precisely, we want to find a value of $\tilde{\gamma}$ such that

$$J = \frac{1}{2}\int_0^\infty \left(\bar{z}'\bar{z} - \tilde{\gamma}^2 w_{si}' w_{si}\right) dt < 0$$

for all $w_{si} \in L_2$. For this purpose, the following Hamiltonian function is considered:

$$\begin{aligned} H\left(x,\eta,w_{si},\tilde{\mu}\right) &= \tfrac{1}{2} r_{\tilde{x}}\left(e\right)' \tilde{R}^{-1}\left(\tilde{x}+e\right) r_{\tilde{x}}\left(e\right) \\ &+ \tfrac{1}{2} h_e'\left(e\right) h_e\left(e\right) - \tfrac{\tilde{\gamma}^2}{2} w_{si}' w_{si} + \tilde{\mu}_e' l_{\tilde{x},\dot{\tilde{x}}} \\ &+ \tilde{\mu}_e' g\left(x\right) w_{si} - \tilde{\mu}_e' g\left(x\right) \tilde{R}^{-1}\left(\tilde{x}+e\right) r_{\tilde{x}}\left(e\right) \\ &+ \tilde{\mu}_e' g\left(x\right)\eta - \varrho \tilde{\mu}_\eta' r_{\tilde{x}}\left(e\right) + \tilde{\mu}_{\tilde{x}} \dot{\tilde{x}} \le 0 \end{aligned} \tag{17}$$

For being robust it is sufficient that the upper bound of the Hamiltonian function is less than zero for

$$\tilde{\mu}_e = \left(\frac{\partial \tilde{V}_{\tilde{x}}(e,\eta)}{\partial e}\right)' \; \tilde{\mu}_\eta = \left(\frac{\partial \tilde{V}_{\tilde{x}}(e,\eta)}{\partial \eta}\right)' \; \tilde{\mu}_{\tilde{x}} = \left(\frac{\partial \tilde{V}_{\tilde{x}}(e,\eta)}{\partial \tilde{x}}\right)'$$

with $\tilde{V}_{\tilde{x}}\left(e,\eta\right)$ being a local positive definite function (positive definite in every $\tilde{x}$). Now we define

$$\tilde{V}_{\tilde{x}}\left(e,\eta\right) = \frac{1}{\alpha} V_{\tilde{x}}\left(e,\eta\right) + \frac{1}{2}\eta'\eta \quad \text{with} \quad \alpha > 0. \tag{18}$$

Furthermore, from the proof of Theorem 1 and Theorem 2 in [1] it is known that

$$\begin{aligned} &\tfrac{1}{2} r_{\tilde{x}}\left(e\right)' \tilde{R}^{-1}\left(\tilde{x}+e\right) r_{\tilde{x}}\left(e\right) + \tfrac{1}{2} h_e'\left(e\right) h_e\left(e\right) \\ &\tfrac{\tilde{\gamma}^2}{2} w_{si}' w_{si} + \tilde{\mu}_e' l_{\tilde{x},\dot{\tilde{x}}} + \tilde{\mu}_e' g\left(x\right) w_{si} \\ &- \tilde{\mu}_e' g\left(x\right) \tilde{R}^{-1}\left(\tilde{x}+e\right) r_{\tilde{x}}\left(e\right) + \tilde{\mu}_{\tilde{x}} \dot{\tilde{x}} \le 0 \end{aligned} \tag{19}$$

and (11) is true if condition (8) in Theorem 1 is fulfilled. Hence one can see that if we choose a $\varrho = \frac{1}{\alpha}$ then

$$\varrho \tilde{\mu}_\eta' r_{\tilde{x}}\left(e\right) = \tilde{\mu}_e' g\left(x\right)\eta$$

and (17) results in (19) what finishes the proof. ■

Remark 4: In Theorem 3 α is a design parameter to be fixed as a trade off between the speed of the integral action (that increases for small values of α, see (15)) and the robustness requirements (16).

IV. Guaranteed State Penalty for Constant Operating Points

The proposed state feedback controller robustly stabilizes the system in a wide operating range. The main advantage of this controller is that the calculation of one control Lyapunov function, whose Hessian matrix is positive definite in the considered operating range, is sufficient. However one drawback of this method is that the performance is not equal for all operating points of a given set.

The performance of a control loop is usually defined by an objective function which in the optimal L_2 control problem in general is

$$J_2 = \frac{1}{2}\int_0^\infty z'z dt$$

where z is the defined performance output of the considered and often augmented plant (12).

If we concentrate on the performance output (12c) we see that it includes both a state penalty function $h(x)$ and an input penalty function $d(x)$. Since we assume that $d(x)'h_e(x) = 0$ the state penalty function enters the objective function in a square form $m\left(x\right) = h(x)'h(x)$ where $m\left(x\right)$ is called the state penalty. In the following it will be shown that if the input penalty function is kept constant over the considered operating range, it is possible to calculate a lower bound of the state penalty which is valid for any operating point. For tuning the tracking controller this result is quite interesting, since it can be guaranteed that the robust tracking controller in the undisturbed case performs in every operating point of the considered operating range at least as a L_2 optimal controller with the guaranteed bound of the state penalty. Since this performance bound can be calculated offline, it may help finding a feasible control Lyapunov function.

For further progress and for system (1) we consider the following assumptions:

(A1) There is no disturbance, hence $w = 0$.

(A2) There exist positive constants $k > 0$, $\kappa \ge 1$ and an $\alpha = \frac{\kappa}{2k\bar{\lambda}_R}$ such that the condition (8) is true for all $x \in X$ and all $\|e\| < E$ and the robustifying feedback controller is given by (6). The guaranteed robustness bound then is given by (10) (see Theorem 1 and 2).

(A3) For each stationary operating point $x_0 \in X$ it is possible to calculate the feedforward control input $k_{ff}\left(x_0, 0\right) = u_0$ such that $k_{fb}\left(\tilde{x}_0, x_0\right)\big|_{x_0=\tilde{x}_0} = 0$. Hence,

$$u_0 = -\left(g\left(x_0\right)' g\left(x_0\right)\right)^{-1} g\left(x_0\right)' f\left(x_0\right) \tag{20}$$

With these assumptions we are now ready for the following theorem.

Theorem 4: If the assumptions (A1), (A2) and (A3) hold, then for stationary operating points ($\tilde{x} \in X$, $\dot{\tilde{x}} = 0$) the feedback controlled system performs at least as a L_2 optimal controlled system designed with the state penalty $m_{x_0}\left(e\right)$ where

$$m_{x_0}\left(e\right) = \frac{\kappa}{4k\bar{\lambda}_R} h_e\left(e\right)' h_e\left(e\right) \tag{21}$$

Proof: For stationary operating points $\tilde{x} = x_0$, $\dot{\tilde{x}} = 0$ and with assumption (A2) the condition (8) of Theorem 1 yields

$$\begin{aligned} &\tfrac{2k\bar{\lambda}_R}{\kappa} \tfrac{\partial V_{x_0}(e)}{\partial e} l_{x_0,0}\left(e\right) + \tfrac{1}{2} h_e\left(e\right)' h_e\left(e\right) \\ &-k \frac{\partial V_{x_0}(e)}{\partial e} g\left(x_0+e\right) g\left(x_0+e\right)' \frac{\partial V_{x_0}(e)}{\partial e}' < 0 \end{aligned} \tag{22}$$

The feedback control law for stationary operating points is

$$k_{fb}(x_0,e) = -\tilde{R}(x_0+e)^{-1} r_{x_0}(e) \qquad (23)$$

where

$$r_{x_0}(e) = g(x_0+e)' \frac{\partial V_{x_0}(e)}{\partial e}' \\ \tilde{R} = \tfrac{1}{\kappa}R$$

We now apply the idea of inverse optimality to estimate the loacal state penalty achieved by the controller (see also [14] for similar ideas). Therefore, the considered objective function for a given x_0 is

$$J_{x_0} = \int_{t=0}^{\infty} \left(m_{x_0}(e) + \tfrac{1}{2}u'\tilde{R}(x)^{-1}u \right) d\tau \qquad (24)$$

The function $m_{x_0}(e)$ describes the state penalty of the optimization problem. Under the assumptions (A1), (A2) and (A3) the tracking error dynamics equation of the closed loop system $(u = k_{fb}(x_0,e))$ is

$$\dot{e} \;=\; l_{x_0,0}(e) - g(x)\tilde{R}^{-1}(x) r_{x_0}(e)$$

Assuming the control law (23) is optimal in the sense that it minimizes the costs of (24) the according HJB equation is

$$\frac{\partial V_{x_0}(e)}{\partial e} l_{x_0,0}(e) - \frac{\partial \tilde{V}_{x_0}(e)}{\partial e} g(x)\tilde{R}(x)^{-1} g(x)' \frac{\partial V_{x_0}(e)}{\partial e}' + \\ + \tfrac{1}{2}\frac{\partial V_{\tilde{x}}(e)}{\partial e} g(x)\tilde{R}(x)^{-1} g(x)' \frac{\partial V_{x_0}(e)}{\partial e}' + m_{x_0}(e) = 0$$

Define

$$\tilde{V}_{x_0}(e) = V_{x_0}(e) = \int_0^e v_{x_0}(e)\,de$$

and the state penalty function becomes

$$m(e) = -\frac{\partial V_{x_0}(e)}{\partial e} l_{x_0,0}(e) \\ + \tfrac{1}{2}\frac{\partial V_{x_0}(e)}{\partial e} g(x)\tilde{R}(x)^{-1} g(x)' \frac{\partial V_{x_0}(e)}{\partial e}' \qquad (25)$$

Applying assumption (A2) it is possible to see that

$$\frac{\partial V_{x_0}(e)}{\partial e} l_{\tilde{x},x_0}(e) < \alpha k \frac{\partial V_{\tilde{x}}(e)}{\partial e} g(x) g(x)' \frac{\partial V_{x_0}(e)}{\partial e}' \\ - \tfrac{\alpha}{2} h_e(e)' h_e(e) \qquad (26)$$

This can be used to estimate the lower bound of the state penalty. From (25) and (26) it results

$$m(e) > \tfrac{\alpha}{2} h_e(e)' h_e(e) + \\ + \frac{\partial V_{x_0}(e)}{\partial e} \Big(\tfrac{1}{2} g(x)\tilde{R}(x)^{-1} g(x)' \\ - \alpha k g(x) g(x)' \Big) \frac{\partial V_{x_0}(e)}{\partial e}'$$

If the matrix in brackets in the second line of the last equation is positive definite, a lower bound of the state penalty is given by

$$\tfrac{\alpha}{2} h_e(e)' h_e(e).$$

So we need to show that

$$\frac{1}{2} g(x)\tilde{R}^{-1}(x) g(x)' - \alpha k g(x) g(x)' \geq 0 \qquad (27)$$

After some computations it is possible to show that the latter equation is true if the following statement holds:

$$\frac{1}{2}\frac{\kappa}{\bar{\lambda}_R} - \alpha k \geq 0$$

With $\lambda_R > 0$, the result is

$$\kappa - 2\alpha k \bar{\lambda}_R \geq 0 \qquad (28)$$

By using $\alpha = \kappa / \left(2k\bar{\lambda}_R\right)$ it is easy to deduce that if

$$\kappa - 2\alpha k \bar{\lambda}_R = 0 \geq 0 \qquad (29)$$

and then the lower bound of the performance is

$$m_{x_0}(e) \geq \frac{\kappa}{4k\bar{\lambda}_R} h_e(e)' h_e(e) \qquad (30)$$

■

V. Simulation Example

We will now consider a system already treated in [1] and in [10]. For this example we will show that the extension for constant input disturbances works efficient and we will further show how it is possible to estimate the lower bound of the state penalty.

$$\begin{aligned} \dot{x} &= x^2 + w + 2u \\ y &= x \\ z &= (x,u)' \end{aligned}$$

The authors in [10] also mentioned that the positive definite solution of the HJB equation arising in the H_2 optimal control problem is

$$V(x) = \frac{1}{6}\left(x^3 + \left(x^2+4\right)\sqrt{x^2+4} \right) - \frac{4}{3}$$

We now want the output to track a given trajectory which changes continuously from one operating point to another operating point. The feedforward control law of the robust tracking controller is

$$k_{ff}\left(\tilde{x},\dot{\tilde{x}}\right) = \frac{1}{2}\left(\tilde{x}^2 + \dot{\tilde{x}}\right)$$

Now it is possible to show that with $k = 2$, $\kappa = 4$ and with the error performance variable $z_e = e$ the conditions (8) in Theorem 1 and (22) in Theorem 4 are satisfied. Hence, referring to (9) the robustifying state feedback controller becomes

$$k_{fb}(x_i,x) = 2\left(x_i^2 - x^2 - x\sqrt{x^2+4} + x_i\sqrt{x_i^2+4} \right)$$

and this controller guarantees according to (10) in Theorem 2 an attenuation level from w to z_e which is less than 0.5. The input penalty $\tilde{R}$ is $\frac{1}{4}$ which is lower than the input penalty of the central controller. From (21) in

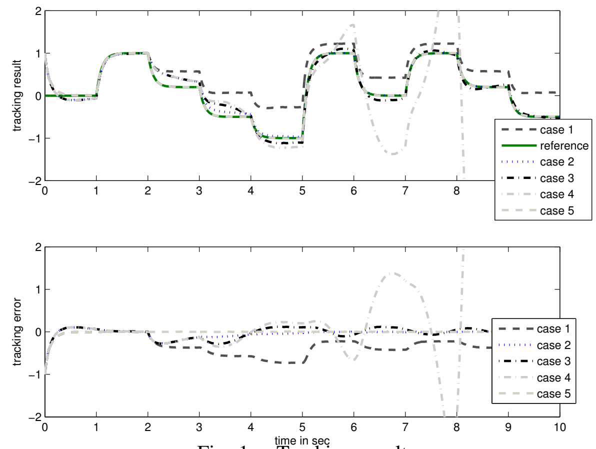

Fig. 1. Tracking results

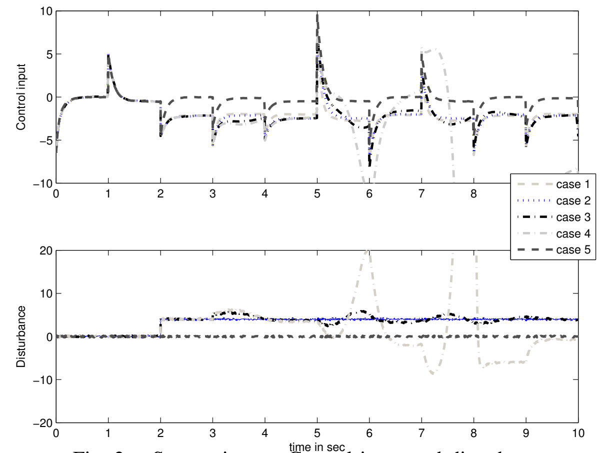

Fig. 2. System inputs: Control input and disturbance

Theorem 4 it results that the lower level of the state penalty is $\frac{1}{4}$. Hence, the controller in the undisturbed case and for constant operating points would at least act as an optimal controller with the performance index

$$J = \int_{0}^{\infty} \left(\tfrac{1}{4} \left(x\left(\tau\right) - x_i \right)^2 + \tfrac{1}{4} u\left(\tau\right)^2 \right) d\tau$$

Following Theorem 3 the integral part to compensate the constant part of the disturbances is

$$\bar{u} = \frac{1}{\alpha} \int_{t=0}^{\infty} k_{fb}\left(x_i, x\right) dt \tag{31}$$

where α can be tuned from 0 to 2 but as long as α is different from 1 the robustness will be lowered.

In Figure 1 the simulation tracking results and in Figure 2 the system inputs (control inputs and disturbances) are shown. The plots show the tracking results of five different test cases. Case 1 only considers the robust tracking controller without integral action. In case 2 the controller also includes the integral action. In case 3 an additional disturbance acting in the counter direction of the controller is added. This disturbance is slightly less than the worst case disturbance. In case 4 the additional disturbance is slightly greater than the worst case disturbance and in case 5 the disturbance is only a white noise and the controller is without integral action. This simulation results show quite well that the presented theory is fine for practical applications.

VI. Conclusions

In this paper we have extended an existing approach for the robust tracking problem of nonlinear uncertain systems. It has been shown that the already known solution can easily be extended by an integral part in order to compensate constant offsets or disturbances while robustness in terms of the L_2 - gain from input disturbance to the performance error is still guaranteed. Furthermore we have shown a possibility for tuning the controller in order to estimate the state penalty of an equivalent optimal L_2 controller. Simulation results underlined the theoretical issues.

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