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Uniting a high performance, local controller with a global controller: the output feedback case for linear systems with input saturation

Christophe Prieur, and Andrew R. Teel

***Abstract*— We consider linear control systems with saturation in the input for which we know two stabilizing output feedback controllers. One is a nonlinear globally asymptotically stabilizing controller, while the other one is a linear only locally asymptotically stabilizing controller. We look for a composite output feedback control law that is equal to the linear controller for initial conditions in a neighborhood of the origin and that is globally asymptotically stabilizing. We suggest a constructive approach to solve this uniting problem and to modify the linear local controller into a nonlinear global controller. Moreover since we want some robustness with respect to measurement noise, actuator errors and external disturbances, we consider hybrid output feedback controllers, following recent developments in the literature on robustness in hybrid systems. We illustrate our main result by means of numerical examples.**

I. Introduction

Over the years, research in control of nonlinear dynamical systems has fallen into two major, and usually distinct, categories: 1) control for local (e.g., linear) performance, and 2) control for global attractivity. A practical example of such framework is given by [14]. In this paper we address the problem of local performance with global asymptotic stability. To do that we suggest an algorithm that unites a predesigned local high performance with a global controller. One class of systems for which this problem is crucial is the class of systems with input saturation. Saturation is one of the most important nonlinearities that limits control systems performance in many applications. It is known that the use of linear controllers for systems that are subject to amplitude-limitation in the input may reduce the performance of the closed-loop system or even lead to instability (this is usually called the windup phenomena). One way to ensure local performance with a global attractivity is to unite a (optimal) linear local output feedback controller with a globally stabilizing nonlinear output feedback controller. Our focus in this paper is to solve this problem by considering hybrid output feedback controllers. This solution is based on the explicit construction of an hybrid controller, assuming some numerically tractable conditions. Our result can be seen as an anti-windup result since, from a locally stabilizing controller, we build a global stabilizer. But our approach shows also how we can piece together arbitrary local and global controllers.

Christophe Prieur is with LAAS–CNRS, Université de Toulouse, 7, Avenue du Colonel Roche 31077 Toulouse, Cedex 4, France, Christophe.Prieur@laas.fr. His research is partly done in the framework of the HYCON Network of Excellence, contract number FP6-IST-511368. Andrew R. Teel is with the Center for Control, Dynamical Systems & Computation (CCDC), Department of Electrical and Computer Engineering, University of California Santa Barbara, CA 93106-9560; teel@ece.ucsb.edu. His research was supported in part by NSF under Grant no. CCR-0311084, ECS-0622253 and by AFOSR under Grant no. AFOSR F9550-06-1-0134.

Many different approaches exist in the literature for the design of static and dynamic linear anti-windup compensators (see e.g. [9], [5], [2]). See also [16] where a nonlinear scheduling technique is proposed, using a switching among a family of linear gains. The contribution of this paper is to state conditions, such that, for the anti-windup problem, we may unite a prescribed global, nonlinear controller with a predesigned local, linear controller.

When considering a locally asymptotically stabilizing state feedback and a globally asymptotically stabilizing state feedback, the uniting problem has been already studied and solved in [10] (cf. also [15]). In [10] it is proved that this uniting problem cannot be solved by considering only continuous feedbacks. When using discontinuous feedbacks, we may introduce sensitivity to arbitrary small measurement noise. However the uniting problem can be solved while achieving robustness to small measurement noise by means of a hybrid controller (see [10], and [11]). When only output feedback controllers are considered, we need an extra property to unite a local output feedback controller with a global one. In [12], under an extra assumption (more precisely an input-output-to-state stability assumption), hysteresis is introduced in the region where both (local and global) output feedbacks are appropriate, and the existence of a hybrid output feedback controller solving the uniting problem is established. This extra assumption is not explicitely needed in the present paper as soon as the linear system is detectable. Moreover we give sufficient and numerically tractable conditions for the construction of a solution of the uniting problem.

In the present work, we focus on the following class of control systems

$$\dot{x} = Ax + B\mathrm{sat}(u)\ ,\ y = Cx \qquad (1)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$, A, B, and C are matrices of appropriate dimensions, and "sat" denotes the usual (decentralized and symmetric) saturation map sat : $\mathbb{R}^m \to \mathbb{R}^m$ defined by, for all $u \in \mathbb{R}^m$, and for all $1 \le i \le m$,

$$\mathrm{sat}(u)_{(i)} = \begin{cases} -\overline{u}_{(i)} & \text{if } u_{(i)} < -\overline{u}_{(i)}\ , \\ u_{(i)} & \text{if } -\overline{u}_{(i)} \le u_{(i)} \le \overline{u}_{(i)}\ , \\ \overline{u}_{(i)} & \text{if } \overline{u}_{(i)} < u_{(i)}\ . \end{cases}$$

In the previous, $\overline{u} \in \mathbb{R}^m$ is a given vector, with positive components $\overline{u}_{(i)}$, for $i = 1, \ldots, m$. We assume that two different continuous dynamic output feedback stabilizers are given: a linear one $u = C_0\zeta_0 + D_0 y$, $\dot{\zeta}_0 = A_0\zeta_0 + B_0 y$, and a nonlinear one $u = \alpha_1(Cx, \zeta_1)$, $\dot{\zeta}_1 = \varphi_1(Cx, \zeta_1)$.

In the present paper, we make explicit numerical conditions to design a hybrid output feedback controller solving the uniting problem, i.e. which is equal to the linear local controller for initial conditions in a neighborhood of the origin, and with a global basin of attraction.

The said class of hybrid controllers has been introduced in [3] and depends only on the output. It has a dynamic state ζ, continuous dynamics $\dot{\zeta} = v(Cx, \zeta)$, if $\zeta \in \mathcal{C}$, and discrete dynamics $\zeta^+ = w(Cx, \zeta)$, if $\zeta \in \mathcal{D}$, for given sets $\mathcal{C}$ and $\mathcal{D}$. The system (1) will be in closed loop with the output of the hybrid controller, i.e. $u = u(Cx, \zeta)$ if $\zeta \in \mathcal{C}$.

In [11] it is shown that the class of asymptotically stable hybrid systems has, under appropriate regularity properties, a robustness with respect to small measurement noise, actuator errors and external disturbances (see also [4]). The system in closed loop with the hybrid controller, that is designed in the present paper, has the regularity properties required in [11], and thus we get also a robustness with respect to small measurement noise, actuator errors and external disturbances.

This paper is organized as follows. In Section II, we state the existence of a solution of our uniting problem. In Section III, we make our controller construction explicit and we state numerically tractable conditions. We illustrate our result by some simulations in Section IV. Section V contains some concluding remarks.

II. Problem statement / existence of a solution

Let us consider two continuous dynamic output feedback controllers for (1). One is assumed to be linear:

$$\begin{aligned} \dot{\zeta}_0 &= A_0\zeta_0 + B_0 y \ , \\ u &= C_0\zeta_0 + D_0 y \ , \end{aligned} \tag{2}$$

where $\zeta_0 \in \mathbb{R}^{l_0}$ is the state of the controller, and A_0, B_0, C_0 and D_0 are matrices of appropriate dimensions. The second controller is a nonlinear output feedback controller:

$$\begin{aligned} \dot{\zeta}_1 &= \varphi_1(Cx, \zeta_1) \ , \\ u &= \alpha_1(Cx, \zeta_1) \ , \end{aligned} \tag{3}$$

where $\varphi_1 : \mathbb{R}^p \times \mathbb{R}^{l_1} \to \mathbb{R}^{l_1}$, and $\alpha_1 : \mathbb{R}^p \times \mathbb{R}^{l_1} \to \mathbb{R}^m$ are continuous functions vanishing at the origin.

Assumption 2.1: 1. (local linear controller) *The origin of $\mathbb{R}^n \times \mathbb{R}^{l_0}$ is locally asymptotically stable for the system*

$$\begin{aligned} \dot{x} &= Ax + Bsat(C_0\zeta_0 + D_0 y) \ , \\ \dot{\zeta}_0 &= A_0\zeta_0 + B_0 y \ ; \end{aligned} \tag{4}$$

2. (global nonlinear controller) *the origin of $\mathbb{R}^n \times \mathbb{R}^{l_1}$ is globally attractive for the system*

$$\begin{aligned} \dot{x} &= Ax + Bsat(\alpha_1(Cx, \zeta_1)) \ , \\ \dot{\zeta}_1 &= \varphi_1(Cx, \zeta_1) \ . \end{aligned} \tag{5}$$

In this paper we consider a dynamic hybrid output feedback controller $(\mathcal{C}, \mathcal{D}, u, v, w)$ where, for a given integer l, $\mathcal{C} \subset \mathbb{R}^l$ and $\mathcal{D} \subset \mathbb{R}^l$ are closed sets, while $u : \mathbb{R}^p \times \mathcal{C} \to \mathbb{R}^m$, $v : \mathbb{R}^p \times \mathcal{C} \to \mathbb{R}^l$ and $w : \mathbb{R}^p \times \mathcal{D} \to \mathbb{R}^l$ are continuous functions. The closed-loop system lies in the class of hybrid systems as considered in e.g., [1], [8]. Here we consider the notion of trajectories as studied in [3], [4], [11]. First we recall that a set $S \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if $S = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ ... \leq t_J$. The set S is a *hybrid time domain* if for all $(T, J) \in S$, $S \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid domain.

The system (1) in closed loop with $(\mathcal{C}, \mathcal{D}, u, v, w)$ is defined as the hybrid system

$$\left. \begin{aligned} \dot{x} &= Ax + B\text{sat}(u(Cx, \zeta)) \\ \dot{\zeta} &= v(Cx, \zeta) \end{aligned} \right\} \text{if } \zeta \in \mathcal{C} \ , \qquad \left. \begin{aligned} x^+ &= x \\ \zeta^+ &= w(Cx, \zeta) \end{aligned} \right\} \text{if } \zeta \in \mathcal{D} \ . \tag{6}$$

Given $(x^0, \zeta^0) \in \mathbb{R}^n \times (\mathcal{C} \cup \mathcal{D})$, we say that (x, ζ) is a *trajectory* to (6) starting at (x^0, ζ^0) if (x, ζ) is defined on a hybrid time domain $\text{dom}(x, \zeta)$, takes values in $\mathbb{R}^n \times (\mathcal{C} \cup \mathcal{D})$ and satisfies:

(S1) for all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \text{dom}(x, \zeta)$, we have $\zeta(t, j) \in \mathcal{C}$, $\dot{x}(t, j) = Ax(t, j) + B\text{sat}(u(Cx(t, j), \zeta(t, j)))$, and $\dot{\zeta}(t, j) = v(Cx(t, j), \zeta(t, j))$.

(S2) for all $(t, j) \in \text{dom}(x, \zeta)$ such that $(t, j + 1) \in \text{dom}(x, \zeta)$, we have $\zeta(t, j) \in \mathcal{D}$, $x(t, j + 1) = x(t, j)$, and $\zeta(t, j + 1) = w(Cx(t, j), \zeta(t, j))$.

(S3) $(x, \zeta)(0, 0) = (x^0, \zeta^0)$.

Denoting the Euclidean norms by $|.|$, we recall that the origin is *globally asymptotically stable* for the system (6), if

- (local stability) for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all (x^0, ζ^0) satisfying $|x^0| + |\zeta^0| \leq \delta$ and $(x^0, \zeta^0) \in \mathbb{R}^n \times (\mathcal{C} \cup \mathcal{D})$, every trajectory of (6) starting at (x^0, ζ^0) satisfies $|x(t, j)| + |\zeta(t, j)| \leq \varepsilon$, for all (t, j) in $\text{dom}(x, \zeta)$;
- (global convergence) for all $(x^0, \zeta^0) \in \mathbb{R}^n \times (\mathcal{C} \cup \mathcal{D})$, every trajectory of (6) starting at (x^0, ζ^0) satisfies $\lim_{t+j \to \infty} |x(t, j)| + |\zeta(t, j)| = 0$.

Let us now define our *uniting problem*. We look for:

- an integer $l \geq l_0$, and a dynamic hybrid output feedback controller $(\mathcal{C}, \mathcal{D}, u, v, w)$ such that, $\mathcal{C}$ and $\mathcal{D}$ are closed sets, u, v and w are continuous functions, and such that the origin of (6) is globally asymptotically stable;
- a matrix $M \in \mathbb{R}^{l_0 \times l}$ and $r > 0$ such that for all initial conditions $(x^0, \zeta^0) \in \mathbb{R}^n \times (\mathcal{C} \cup \mathcal{D})$, satisfying $|x^0| + |\zeta^0| \leq r$, every trajectory of (6) starting at (x^0, ζ^0) has the hybrid time domain $[0, \infty) \times \{0\}$ and $(x(t, 0), M\zeta(t, 0)) = (\bar{x}(t), \bar{\zeta}_0(t))$ for some trajectory $(\bar{x}, \bar{\zeta}_0)$ of (4).

Let us remark that, combining the local asymptotic stability and the fact that for small initial conditions the trajectories match those of the local controller, all trajectories of (6) are trajectories of (4) for sufficiently large time. Moreover the closed-loop system is globally attractive.

Theorem 2.2: (existence of a solution of the uniting problem) *Under Assumption 2.1, there exists a dynamic hybrid output feedback controller $(\mathcal{C}, \mathcal{D}, u, v, w)$ solving the uniting problem.*

This theorem is an existence result. To construct a hybrid feedback solving our uniting problem, we need to make our assumption more "quantitative". This is done in the next section, where we introduce a new set of assumptions which is valid as soon as Assumption 2.1 holds. We may then deduce Theorem 2.2.

III. EXPLICIT SOLUTION

In this section, we consider two continuous dynamic output feedback controllers: the linear controller (2), and the nonlinear controller (3).

We denote the usual matrix norms by $\|\cdot\|$ (without specifying the dimensions), and by I_n, I_p, ... the identity matrix in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{p\times p}$, ... respectively. We need the following[1]:

Assumption 3.1: *There exist symmetric positive definite matrices $P \in \mathbb{R}^{m\times m}$, $P_1 \in \mathbb{R}^{n\times n}$, $P_0 \in \mathbb{R}^{(n+l_0)\times(n+l_0)}$ and $Q_0 \in \mathbb{R}^{l_0\times l_0}$, a symmetric positive semidefinite matrix $N \in \mathbb{R}^{(m+p+l_0)\times(m+p+l_0)}$, matrices $H \in \mathbb{R}^{m\times(n+l_0)}$, $L \in \mathbb{R}^{n\times p}$, and positive values $\varepsilon_{0a} < \varepsilon_{0b}$, $\varepsilon_{1a} < \varepsilon_{1b}$, ε_2, and ε_3 such that:*

1. (local linear controller) *the origin of $\mathbb{R}^n \times \mathbb{R}^{l_0}$ is asymptotically stable for with a basin of attraction containing the set $\{(x,\zeta_0),\ (x',\zeta_0')P_0(x',\zeta_0')' \le \varepsilon_3\}$, the value $(x',\zeta_0')P_0(x',\zeta_0')'$ is non-increasing along the trajectories of (4) starting in this set, and we have*
$$P_1(A+LC)+(A+LC)'P_1 \le -2P_1\ ; \qquad (7)$$
2. (global nonlinear controller) *by defining $V_1 : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ with $V_1(x) = x'P_1x$, for all trajectories of (5), we have $\lim_{t\to\infty} V_1(x(t)) + |\zeta_1(t)| \le \varepsilon_2$;*
3. *for each trajectory of (5) starting from $\{(x,\zeta_1),\ V_1(x)+|\zeta_1| \le \varepsilon_2\}$, we have $\rho_1(\alpha_1(Cx(t),\zeta_1(t)),Cx(t)) < \varepsilon_{1a}$, for all $t \ge 0$, where $\rho_1 : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}_{\ge 0}$ is defined by $\rho_1(u,y) = 2\begin{pmatrix} u \\ y\end{pmatrix}'\begin{pmatrix} B'P_1B & 0 \\ \star & L'P_1L\end{pmatrix}\begin{pmatrix} u \\ y\end{pmatrix}$, for all $(u,y) \in \mathbb{R}^m \times \mathbb{R}^p$;*
4. *the inequalities*
$$\varepsilon_3\varepsilon_{0b} \ge \varepsilon_{1b}\ , \qquad (8)$$
$$\begin{pmatrix} \frac{1}{\varepsilon_{0b}}P_1 & 0 \\ \star & \frac{1}{\varepsilon_{0b}}Q_0 \end{pmatrix} \ge P_0\ , \qquad (9)$$
$$\begin{pmatrix} \frac{\varepsilon_{0b}}{\varepsilon_{1b}}P_0 & \overline{u}_iH'_{(i)} \\ \star & 1 \end{pmatrix} \ge 0\ ,\ i=1,\dots,m\ , \qquad (10)$$
$$N \ge \begin{pmatrix} 2B'P_1B & 2B'P_1BD_0 & 2B'P_1BC_0 \\ \star & 2(L'P_1L+D_0'B'P_1BD_0) & D_0'B'P_1BC_0+B_0'Q_0 \\ \star & \star & 2C_0'B'P_1BC_0+A_0'Q_0+Q_0A_0+Q_0 \end{pmatrix}\ , \qquad (11)$$
$$\begin{pmatrix} 2P & P(K-H) \\ \star & \frac{\varepsilon_{0a}\varepsilon_{0b}}{\varepsilon_{1b}}P_0 \end{pmatrix} > diag(I_m,C',I_{l_0})N diag(I_m,C,I_{l_0})\ , \qquad (12)$$
hold, where $H_{(i)}$ denotes the ith row of H and $K = (D_0C\ C_0)$.

Theorem 3.2: (explicit solution of the uniting problem) *Under Assumption 3.1, a dynamic hybrid output feedback controller $(\mathcal{C},\mathcal{D},u,v,w)$ solving the uniting problem is designed as follows.*

Letting $l = l_0+l_1+3$, and decomposing all $\zeta \in \mathbb{R}^l$ as $\zeta = (\zeta_0,\zeta_1,z_0,z_1,q)$ where $(\zeta_0,\zeta_1,z_0,z_1,q) \in \mathbb{R}^{l_0}\times\mathbb{R}^{l_1}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, the following hybrid output feedback controller
$$\begin{array}{rrcl} & \multicolumn{3}{c}{\mathcal{C} = \mathcal{C}_0\cup\mathcal{C}_1\ ,\ \mathcal{D} = \mathcal{D}_0\cup\mathcal{D}_1} \\ u: & \mathbb{R}^p\times\mathcal{C} & \to & \mathbb{R}^m \\ & (y,\zeta) & \mapsto & \alpha_q(y,\zeta_q) \\ v: & \mathbb{R}^p\times\mathcal{C} & \to & \mathbb{R}^l \\ & (y,\zeta) & \mapsto & ((1-q)\varphi_0(y,\zeta_0), q\varphi_1(y,\zeta_1), \\ & & & (1-q)(-z_0+\rho_0(y,\zeta_0)), \\ & & & -z_1+\rho_1(\alpha_q(y,\zeta_q),y),0) \\ w: & \mathbb{R}^p\times\mathcal{D} & \to & \mathbb{R}^l \\ & (y,\zeta) & \mapsto & (q\zeta_0,(1-q)\zeta_1,0,z_1,1-q) \end{array} \qquad (13)$$
where
$$\begin{array}{rcl} \alpha_0(y,\zeta_0) & = & C_0\zeta_0 + D_0y\ , \\ \varphi_0(y,\zeta_0) & = & A_0\zeta_0+B_0y\ , \\ \rho_0(y,\zeta_0) & = & (sat(\alpha_0(y,\zeta_0)))'-\alpha_0(y,\zeta_0)',\ y',\ \zeta_0') \\ & & \times N(sat(\alpha_0(y,\zeta_0)))'-\alpha_0(y,\zeta_0)',\ y',\ \zeta_0')'\ , \\ \mathcal{C}_0 & = & \{\zeta : 0\le z_0\le\varepsilon_{0a},\ 0\le z_1,\ \zeta_1=0,\ q=0\}\ , \\ \mathcal{C}_1 & = & \{\zeta : z_0=0,\ \varepsilon_{1a}\le z_1, \zeta_0=0,\ q=1\}\ , \\ \mathcal{D}_0 & = & \{\zeta : \zeta_1=0,\ \varepsilon_{0a}\le z_0,\ 0\le z_1,\ q=0\}\ , \\ \mathcal{D}_1 & = & \{\zeta : \zeta_0=0,\ z_0=0,\ 0\le z_1\le\varepsilon_{1a},\ q=1\} \end{array}$$
solves the uniting problem.

Let us sketch the proof of Theorem 3.2. First, we use (7) in Assumption 3.1, to state an input-output-to-state stable (IOSS) property for (1). To be self-contained, we recall that (1) is IOSS if there exist functions β of class $\mathcal{KL}$ and γ of class $\mathcal{K}$, such that, for all[2] $u : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$, and for all trajectories x of (1), we have, for all $t \in [0,\infty)$, $|x(t)| \le \max\{\beta(|x(0)|,t), \gamma(\sup_{s\in[0,t]}|y(s)|), \gamma(\sup_{s\in[0,t]}|u(s)|)\}$. This notion and its connection to the detectability for linear systems are studied in [7]. We use (7) in Assumption 3.1, to compute, for all $(x,u) \in \mathbb{R}^n\times\mathbb{R}^m$, $\nabla V_1(x)\cdot(Ax+Bu) \le \frac{\varepsilon_3}{2}|x|^2 + \rho_1(u,Cx)$. This function V_1 is an IOSS-Lyapunov function for (1) as introduced in [7]. We introduce a global norm-observer for the system (1) satisfying the continuous dynamics $\dot{z}_1 = -z_1 + \rho_1(u,y)$.

In a similar way, defining $V_0 : \mathbb{R}^n\times\mathbb{R}^{l_0} \to \mathbb{R}_{\ge 0}$ with $V_0(x,\zeta_0) = x'P_1x + \zeta_0'Q_0\zeta_0$, we check that V_0 is an IOSS-Lyapunov of (4), and we introduce a local norm-observer z_0 for the system (4) satisfying the continuous dynamics $\dot{z}_0 = -z_0 + \rho_0(Cx,\zeta_0)$, when the local controller is used. Both norm-observers allow us to estimate an upper bound on the magnitude of the state. The complete proof is partly based on a result of [12] and is omitted due to space limitation.

Let us denote $\mathbf{A} = \begin{pmatrix} A & 0 \\ B_0C & A_0\end{pmatrix}$, and $\mathbf{B} = \begin{pmatrix} B \\ 0\end{pmatrix}$. Let us note that, above, the assumption on the global,

[1]For each matrix $\mathbf{M}$, the notation $\mathbf{M} > 0$ (resp. $\mathbf{M} \ge 0$) means that the matrix $\mathbf{M}$ is symmetric positive definite (resp. positive semi-definite). For any symmetric matrix, we will denote the symmetric terms by $\star$.

[2]Here and in what follows all controls are assumed to be measurable and essentially bounded functions.

nonlinear output feedback relaxes those of Assumption 2.1 (compare item 2 and item 3 of Assumption 3.1 with item 2 of Assumption 2.1). More precisely, we may prove that Assumption 3.1 holds if Assumption 2.1 is satisfied (see Proposition 3.3 below). Moreover, the problem of the computation of the variables considered in Assumption 3.1 is not convex due to the products appearing in (11). However, we state below a numerical algorithm to compute a solution of the uniting problem by solving LMIs only:

Proposition 3.3: *Assumption 3.1 holds if Assumption 2.1 is satisfied. Moreover the data allowing to define the hybrid controller (13) are computed solving only LMIs as follows:*

Algorithm 3.4: 1. *Compute a symmetric positive definite matrix* $P_1 \in \mathbb{R}^{n\times n}$, *and a matrix* $\mathcal{L} \in \mathbb{R}^{n\times p}$ *solution of* $P_1A + \mathcal{L}C + A'P_1 + C'\mathcal{L}' < -2P_1$, *and let* $L = P_1^{-1}\mathcal{L}$;

2. *compute a symmetric definite matrix* W_0 *in* $\mathbb{R}^{(n+l_0)\times(n+l_0)}$, *a matrix* $Z \in \mathbb{R}^{m\times(n+l_0)}$, *a diagonal positive matrix* $S \in \mathbb{R}^{(n+l_0)\times(n+l_0)}$ *in* $\mathbb{R}^{l_0\times l_0}$ *satisfying*

$$\begin{pmatrix} W_0\mathcal{A}' + \mathcal{A}W_0 & \mathbf{B}S - Z' \\ \star & -2S \end{pmatrix} < 0 \ , \qquad (14)$$

$$\begin{pmatrix} W_0 & W_0K'_{(i)} - Z'_{(i)} \\ \star & \overline{u}^2_{(i)} \end{pmatrix} \geq 0, \ i = 1, \ldots, m \ (15)$$

where $\mathcal{A} = \mathbf{A} + \mathbf{B}K$, *and* $K_{(i)}$ *(resp.* $Z_{(i)}$*) denotes the* i*th row of* K *(resp.* Z*);*

3. *let* $P_0 = W_0^{-1}$ *and* $\varepsilon_3 = 1$, *and compute a symmetric positive matrix* $R \in \mathbb{R}^{l_0\times l_0}$, *and a positive value* ε *satisfying*

$$\begin{pmatrix} \varepsilon P_1 & 0 \\ \star & R \end{pmatrix} \geq P_0 \ . \qquad (16)$$

Let $0 < \varepsilon_{0a} < \varepsilon_{0b} = 1/\varepsilon$ *and* $Q_0 = \varepsilon_{0b}R$;

4. *compute a matrix* $H \in \mathbb{R}^{m\times(n+l_0)}$, *and a positive value* $\hat{\varepsilon}$ *satisfying*

$$\begin{pmatrix} \hat{\varepsilon}P_0 & \overline{u}_iH'_{(i)} \\ \star & 1 \end{pmatrix} \geq 0 \ , i = 1, \ldots, m \ ; \qquad (17)$$

5. *compute a symmetric positive definite matrix* $P \in \mathbb{R}^{m\times m}$, *a symmetric positive semidefinite matrix* $N \in \mathbb{R}^{(m+p+l_0)\times(m+p+l_0)}$ *and a positive value* $\widetilde{\varepsilon}$ *satisfying (11) and*

$$\begin{pmatrix} 2P & P(K-H) \\ \star & \widetilde{\varepsilon}P_0 \end{pmatrix} > diag(I_m, C', I_{l_0})Ndiag(I_m, C, I_{l_0}) \ . \qquad (18)$$

Let $\varepsilon_{1b} = \min(\varepsilon_{0b}\varepsilon_3, \frac{\varepsilon_{0a}\varepsilon_{0b}}{\widetilde{\varepsilon}}, \frac{\varepsilon_{0b}}{\hat{\varepsilon}})$ *and* $0 < \varepsilon_{1a} < \varepsilon_{1b}$.

In the proof of Proposition 3.3, the basin of attraction of (4) is estimated. To do that, we used the modified condition of [2], but other approaches are possible (consider e.g., [6]).

Corollary 3.5: *Under Assumption 2.1, the dynamic hybrid output feedback controller (13) solves the uniting problem and is defined solving only LMIs by following Algorithm 3.4 and by defining* $\rho_1(u, y) = 2(u', y')diag(B'P_1B, L'P_1L)(u', y')'$, *for all* $(u, y) \in \mathbb{R}^m \times \mathbb{R}^p$ *and* l, α_0, φ_0 *and* ρ_0 *as in Theorem 3.2.*

IV. Numerical simulations

A. Making the local controller's region large

For a performance purpose, it may be important to maximize the size of the region where the local controller is used. For this aim, due to the expression of $\mathcal{C}_0$ in Theorem 3.2, we have to maximize the value ε_{0a}. To maximize this value, it is possible to consider some convex optimization problems derived from Algorithm 3.4. First at step 2 of Algorithm 3.4 maximizing the estimate of the basin of attraction of (4) can be accomplished by solving the following convex optimization problem:

$$\min \mu \text{ subject to (14), (15) and } \begin{pmatrix} \mu I_{n+l_0} & I_{n+l_0} \\ \star & W_0 \end{pmatrix} > 0 \ .$$

Also at step 3 of Algorithm 3.4, it possible to maximize the value ε_{0a} by solving the convex optimization problem: $\min \varepsilon$ *subject to (16)* .

B. An academic example

Let us consider the following two-dimensional system:

$$\dot{x} = Ax + B\text{sat}(u) \ , \ y = Cx \ , \qquad (19)$$

with $A = \begin{pmatrix} 0 & 1 \\ 0 & -0.1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and sat is the saturation function with level equal to 10. The equations model a positioning system where the position x_1 is assumed to be measured, and the force which is applied on the system may saturate. The speed x_2 is subject to friction.

For the local controller we consider the following linear controller:

$$\dot{\zeta}_0 = A_0\zeta_0 + B_0y \ , u = C_0\zeta_0 + D_0y \ , \qquad (20)$$

with $A_0 = \begin{pmatrix} -14 & 0 \\ 1 & 0 \end{pmatrix}$, $B_0 = \begin{pmatrix} 16 \\ 0 \end{pmatrix}$, $C_0 = (7.5, \ -0.625)$ and $D_0 = -10$. We easily check that the origin is asymptotically stable for the system (19) in closed-loop with (20) linearized around the origin. However the origin of the nonlinear closed-loop system is not globally asymptotically stable (consider e.g., the trajectory starting from $(x^{0\prime}, {\zeta_0}^{0\prime}) = (10, \ 10, \ 10, \ 10)$ which diverges as the time goes to the infinity).

For the second controller we consider the following static position feedback: $\alpha_1(y) = Ky$ with $K = -0.1$. Using the positive definite function $V : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ defined by $V(x_1, x_2) = \frac{1}{2}x_2^2 - \int_0^{x_1} \text{sat}(Ks)ds$, for all $(x_1, x_2) \in \mathbb{R}^2$, and the Invariance Principle, we may check that the origin of the system (19) in closed loop with α_1 is globally asymptotically stable. Note however that this closed-loop may produce oscillations (consider e.g., the trajectory starting from $x^{0\prime} = (10, \ 10)$).

Now we solve the uniting problem using Corollary 3.5, and we compute the variables allowing to define the hybrid controller (13) considering only LMIs by following Algorithm 3.4 and the optimization issues of Section IV-A. We compute

$$P_1 = \begin{pmatrix} 1.9655 & -0.7578 \\ \star & 0.6014 \end{pmatrix},\ L = \begin{pmatrix} -3.0951 \\ -4.7740 \end{pmatrix},$$
$$\begin{pmatrix} \varepsilon_{0a} \\ \varepsilon_{1a} \end{pmatrix} = \begin{pmatrix} 0.5645 \\ 1.5033 \times 10^{-4} \end{pmatrix}$$
$$N = \begin{pmatrix} 55.0216 & -275.94 & 206.94 & -17.250 \\ \star & 12069 & -8673.7 & 934.16 \\ \star & \star & 7275.6 & -164.14 \\ \star & \star & \star & 347.19 \end{pmatrix},$$

and we study system (19) in closed loop with the dynamic hybrid output feedback controller (13). Let us first consider the following initial condition: $x^{0\prime} = (0;\ 0.05;\ 0;\ 0)$, $q^0 = 1$, ${z_1}^0 = 0.05$, and ${z_0}^0 = 0.05$. We note on Figure 1 that we start using the global controller until the time $t = 60$. After this time instant we use the local controller and see that the trajectory tends to the origin (see Figure 1). We see on Figure 1 that this switch is due to the fact that the value of z_1 becomes lower than ε_{1a} at the switching time.

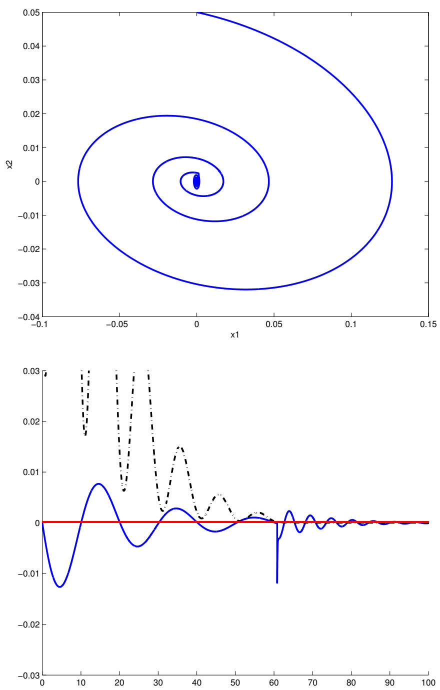

Fig. 1. Up: time evolution of (x_1, x_2) of the system in closed loop with the hybrid controller. Down: time evolution of u (in plain line) and of z_1 (in dashdotted line), the value ε_{1a} is given by the horizontal line.

Now we consider the initial condition $x^{0\prime} = (0,\ 0.05,\ 0,\ 0)$, $q^0 = 0$, ${z_1}^0 = 0.05$, and ${z_0}^0 = 0.05$. We note on Figure 2 that the local controller is used until the time $t = 0.84$, where $z_0(0.84) = 0.5668 > \varepsilon_{0a}$. Thus the global controller is used. We eventually switch to the local controller (after the time $t = 39$) and the trajectory converges to the origin (see Figure 2).

Combining [11, Theorem 4.3] and the regularity of the hybrid controller of Theorem 3.2, we also get a robustness with respect to small measurement noise, actuator errors and external disturbances. To illustrate this on numerical simulations, let us consider the same initial condition than in the previous simulation, and an additive small noise in the output. This noise is a uniform distribution between -0.01 and 0.01. On Figure 3 we note that the state is practically stabilized to the origin, and that, even for large time, we may use the global controller.

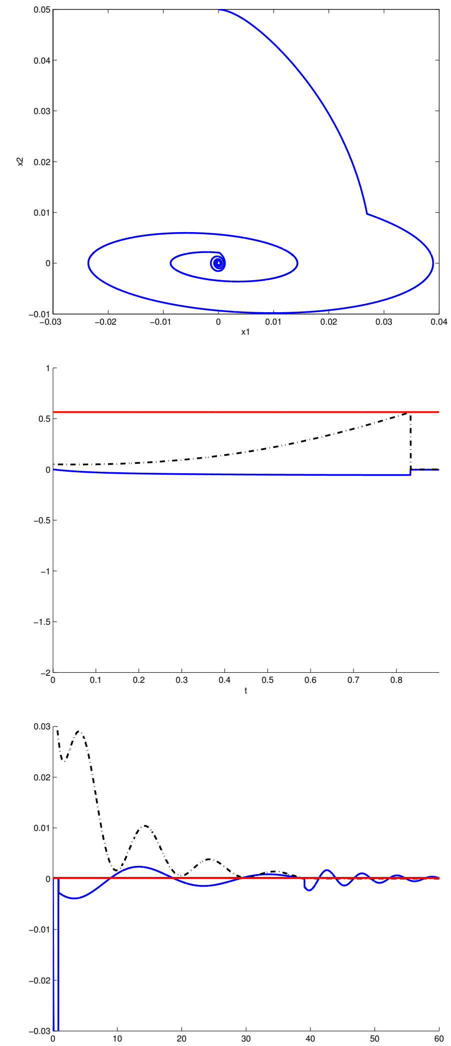

Fig. 2. Up: time evolution of (x_1, x_2) of the system in closed loop with the hybrid controller. Middle: time evolution of u (in plain line) and of z_0 (in dashdotted line), the value ε_{0a} is given by the horizontal line before the first switching time. Down: time evolution of u (in plain line) and of z_1 (in dashdotted line), the value ε_{1a} is given by the horizontal line.

C. An example borrowed from the literature

We consider the following nonlinear two-dimensional system (see [13]):

$$\dot{x} = Ax + B\mathrm{sat}_1(u)\ ,\ y = Cx\ , \tag{21}$$

with $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $C = I_2$, and sat_1 is the saturation function with level equal to 1. Note that the output of the plant is equal to the state x, but the jump condition of the hybrid controller will depend only on the controller state (namely ζ) and not on x.

It is proven in [13] that (21) in closed loop with $\alpha_1(x) = A_1 x + B_1 \mathrm{sat}_2(A_2 x)$, where

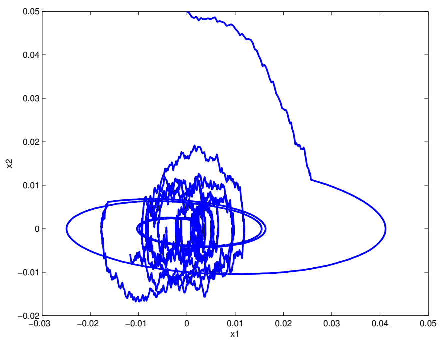

Fig. 3. Up: time evolution of x_1 (in plain line) and of x_2 (in dotted line) of the system in closed loop with the hybrid controller corrupted with noise.

$A_1 = \begin{pmatrix} -0.0264 & -0.3423 \end{pmatrix}$, $B_1 = 1$, $A_2 = \begin{pmatrix} -0.1422 & -0.5441 \end{pmatrix}$, and sat_2 is the saturation function with level u_2 equal to 1, is globally asymptotically stable. The eigenvalues of the nonlinear system (21) in closed loop with α_1, linearized around zero, are $-0.5568 \pm 0.7221i$.

Now we compute a state-feedback matrix A_0 such that the eigenvalues of $A + BA_0$ are -6 and -7. This is done by choosing the following *faster* controller $\alpha_0(x) = A_0 x$, where $A_0 = (41\ 11)$.

By following Algorithm 3.4, we compute

$$P_1 = \begin{pmatrix} 0.3602 & 0.0001 \\ \star & 0.7733 \end{pmatrix}, \; L = \begin{pmatrix} -1.5733 & -0.0001 \\ 0.5343 & 0.7328 \end{pmatrix},$$

$$\begin{pmatrix} \varepsilon_{0a} \\ \varepsilon_{1a} \end{pmatrix} = \begin{pmatrix} 0.5809 \\ 7.410 \times 10^{-5} \end{pmatrix}$$

$$N = \begin{pmatrix} 2.5774 & 63.4076 & 17.0139 \\ \star & 2717 & 583.7 \\ \star & \star & 301.3 \end{pmatrix},$$

and we study system (21) in closed loop with the hybrid controller (13). Let us consider the following initial condition: $x^{0\prime} = (0.01;\ 0.01)$, $q^0 = 1$, ${z_1}^0 = 0.01$, and ${z_0}^0 = 0.6$. We note on Figure 4 that we start using the global controller until the time $t = 5.2$. After this time instant we use the local controller and see that the trajectory tends to the origin (see Figure 4).

V. Conclusion

In this paper we have considered linear systems with saturation in the input. Given two stabilizing output feedbacks controllers (one being a linear but only locally asymptotically stabilizing, the other being nonlinear and globally attractive), we constructed a hybrid output feedback controller that is equal to the local controller for initial condition in a neighborhood of the origin, and that is globally asymptotically stabilizing. Combining [11, Theorem 4.3] and the regularity of the data of our hybrid feedback, we may also state a robustness issue, as illustrated on simulations. The approach suggested in this paper is constructive and is written in terms of numerically computable conditions.

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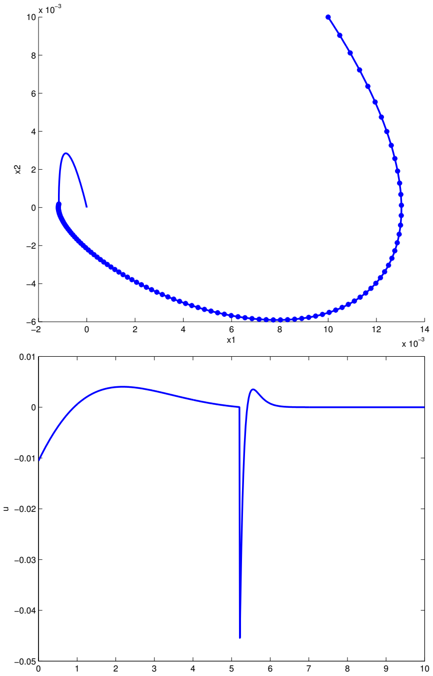

Fig. 4. Up: time evolution of '(x_1, x_2) of the system in closed loop with the hybrid controller (in plain line, the local controller is used, in starred line, the global controller is used). Down : time evolution of u

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