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Averaging sampling: models and properties

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Abstract— **In the paper we study the properties of a sampling method for stochastic signals corrupted by a wide-band stochastic noise where samples are taken as average values of the signal over the sampling interval to diminish the influence of noise. We also study possible improvement attained by discrete-time Kalman filtering applied to the sampled signal. We compare the results with two competitive methods: classical point-wise sampling followed by discrete-time filtering, and continuous-time Kalman filtering prior to sampling possibly followed by digital filtering. The study is performed for a wide range of sampling periods and noise-to-signal ratios and leads to important practical conclusions.**

I. INTRODUCTION

In the paper we consider sampling of a signal corrupted by a wide-band noise, whose model is depicted in Fig. 1 where $s(t)$ denotes the signal of interest,and $n(t)$ denotes a wide-band noise, $K_s(s)$ and $K_n(s)$ denote forming filters, and $\dot{\xi}_s(t)$ and $\dot{\xi}_n(t)$ are white noises. The literature, e.g. [1], [3], [5] requires that the measured signal $y(t) = s(t) + n(t)$ is passed through a so called anti-aliasing filter before being sampled. In the paper we assume that $y(t)$ is sampled in such way that average values of $y(t)$ are calculated over a constant sampling interval. This method is used by certain analog-to digital converters and is supposed to diminish the influence of noise. It is obvious that for appropriately chosen sampling interval the averaged samples can be closer to the samples of the original signal $s(t)$ than the instantaneous samples of $y(t)$. The results can further be improved using an appropriately designed Kalman filter. Competitive to this method is the use of a continuous-time Kalman filter to the signal $y(t)$ prior to sampling, possibly followed by a discrete-time Kalman filter. The simplest approach consists in applying discrete-time Kalman filter directly to point-wise samples of $y(t)$. These configurations are shown in Fig. 2.

The aim of the paper is to compare these methods and to give practical recommendations to their use.

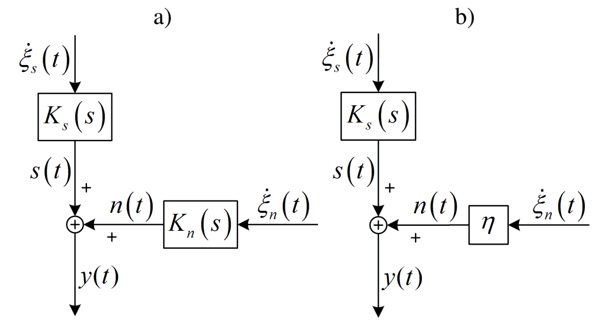

Fig. 1. a) Model of signal $s(t)$ corrupted by noise, and b) Simplified model with white noise

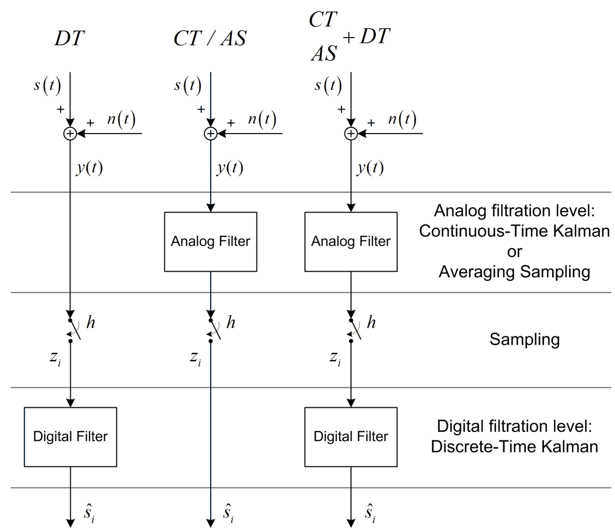

Fig. 2. Configurations of filters and samplers

II. MODELS OF SAMPLING

A. State-space model of signal contaminated by noise

To analyse the properties of sampling we will use state-space models of the system in Fig. 1 consisting of signal

$$\dot{\boldsymbol{x}}_s(t) = \boldsymbol{A}_s\boldsymbol{x}_s(t) + \boldsymbol{c}_s\dot{\xi}_s(t), \tag{1}$$
$$s(t) = \boldsymbol{d}_s'\boldsymbol{x}_s(t), \tag{2}$$

and noise model

$$\dot{\boldsymbol{x}}_n(t) = \boldsymbol{A}_n\boldsymbol{x}_n(t) + \boldsymbol{c}_n\dot{\xi}_n(t), \tag{3}$$
$$n(t) = \boldsymbol{d}_n'\boldsymbol{x}_n(t), \tag{4}$$

where $\dim \boldsymbol{x}_s = n_s$, $\dim \boldsymbol{x}_n = n_n$, $\boldsymbol{x}_s(t)$, $\boldsymbol{x}_n(t)$ are state vectors, $\boldsymbol{A}_s$, $\boldsymbol{A}_n$ are matrices, $\boldsymbol{c}_s$, $\boldsymbol{c}_n$, $\boldsymbol{d}_s$, and $\boldsymbol{d}_n$ are vectors of appropriate dimensions. The initial conditions $\boldsymbol{x}_s(0)$ and $\boldsymbol{x}_s(0)$ are assumed to be a normal random vectors, $\boldsymbol{x}_s(0) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_{s,0})$, $\boldsymbol{x}_n(0) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_{n,0})$. Processes $\dot{\xi}_s(t)$ and $\dot{\xi}_s(t)$ are independent continuous-time white noises with zero means and covariance functions defined as unit Dirac pulse functions, i.e.:

$$\mathrm{E}\,[\dot{\xi}_s(t)] = 0, \qquad \mathrm{E}\,[\dot{\xi}_s(t)\dot{\xi}_s(\tau)] = \delta(t-\tau); \tag{5}$$
$$\mathrm{E}\,[\dot{\xi}_n(t)] = 0, \qquad \mathrm{E}\,[\dot{\xi}_n(t)\dot{\xi}_n(\tau)] = \delta(t-\tau). \tag{6}$$

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The disturbed signal $y(t)$ is the sum of the signal of interest $s(t)$ and noise $n(t)$:

$$y(t) = s(t) + n(t). \tag{7}$$

System (1)-(4) with (7) can be aggregated to:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{C}\dot{\boldsymbol{\xi}}(t), \tag{8}$$
$$s(t) = \boldsymbol{d}_0'\boldsymbol{x}(t), \tag{9}$$
$$y(t) = \boldsymbol{d}'\boldsymbol{x}(t), \tag{10}$$

where:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_s & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_n \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{c}_s & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{c}_n \end{bmatrix},$$
$$\boldsymbol{d}_0 = \begin{bmatrix} \boldsymbol{d}_s \\ \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{d} = \begin{bmatrix} \boldsymbol{d}_s \\ \boldsymbol{d}_n \end{bmatrix},$$
$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{x}_s(t) \\ \boldsymbol{x}_n(t) \end{bmatrix}, \qquad \dot{\boldsymbol{\xi}}(t) = \begin{bmatrix} \dot{\xi}_s(t) \\ \dot{\xi}_n(t) \end{bmatrix}.$$

B. Instantaneous sampling

Simple instantaneous sampling with sampling period h consists in taking the values of the sampled signal at discrete time instants $t_i = ih, i = 0, 1, \ldots$. Available measurements z_i are expressed as

$$z_i = y(t_i). \tag{11}$$

Then the problem defined by measurement equation (11) and state equation (8) is equivalent with the following discrete-time system:

$$\boldsymbol{x}_{i+1} = \boldsymbol{F}\boldsymbol{x}_i + \boldsymbol{w}_i, \tag{12}$$
$$z_i = \boldsymbol{d}'\boldsymbol{x}_i, \tag{13}$$

where:

$$\boldsymbol{F} = \mathrm{e}^{\boldsymbol{A}h}, \tag{14}$$

and $\boldsymbol{w}_i$ is a zero mean vector Gaussian noise with $\mathrm{E}\{\boldsymbol{w}_i\boldsymbol{w}_i'\} = \boldsymbol{W}$, and

$$\boldsymbol{W} = \int_0^h \mathrm{e}^{\boldsymbol{A}s}\boldsymbol{C}\boldsymbol{C}'\mathrm{e}^{\boldsymbol{A}'s}ds. \tag{15}$$

Vectors $\boldsymbol{x}_0$ and $\boldsymbol{w}_i$ are independent for all $i \geq 0$.

The limiting Kalman filter, [2], that provides ($\hat{\boldsymbol{x}}_{i|i} = \mathrm{E}[\boldsymbol{x}_i|\vec{\boldsymbol{z}}_i]$) for the discrete-time system in (12)-(13) as $i \to \infty$ has the form:

$$\hat{\boldsymbol{x}}_{i+1|i+1} = \hat{\boldsymbol{x}}_{i+1|i} + \boldsymbol{k}^f(z_{i+1} - \boldsymbol{d}'\hat{\boldsymbol{x}}_{i+1|i}), \tag{16}$$
$$\hat{\boldsymbol{x}}_{i+1|i} = \boldsymbol{F}\hat{\boldsymbol{x}}_{i|i}, \qquad \boldsymbol{x}_{0|-1} = 0, \tag{17}$$
$$\hat{s}_{i|i} = \boldsymbol{d}_0'\hat{\boldsymbol{x}}_{i|i}, \tag{18}$$

where

$$\boldsymbol{k}^f = \frac{\boldsymbol{\Sigma}\boldsymbol{d}}{\boldsymbol{d}'\boldsymbol{\Sigma}\boldsymbol{d}}, \quad \boldsymbol{\Sigma} = \boldsymbol{W} + \boldsymbol{F}\left(\boldsymbol{\Sigma} - \frac{\boldsymbol{\Sigma}\boldsymbol{d}\boldsymbol{d}'\boldsymbol{\Sigma}'}{\boldsymbol{d}'\boldsymbol{\Sigma}\boldsymbol{d}}\right)\boldsymbol{F}'. \tag{19}$$

The limiting variance $\sigma^2(\Delta s) = \lim_{i\to\infty} \mathrm{var}\{\Delta s(i)\}$ of the estimation error

$$\Delta s(i) = s_i - \hat{s}_{i|i} = \boldsymbol{d}_0'(\boldsymbol{x}_i - \hat{\boldsymbol{x}}_{i|i}), \tag{20}$$

can be calculated from

$$\sigma^2(\Delta s) = \boldsymbol{d}_o'\boldsymbol{V}^o\boldsymbol{d}_o + \boldsymbol{d}_o'\boldsymbol{V}^f\boldsymbol{d}_o - 2\boldsymbol{d}_o'\boldsymbol{V}^{of}\boldsymbol{d}_o, \tag{21}$$

where $\boldsymbol{V}^o$, $\boldsymbol{V}^f$, end $\boldsymbol{V}^{fo}$ are submatrices of matrix $\boldsymbol{V}$

$$\boldsymbol{V} = \mathrm{E}\left\{\begin{bmatrix} \boldsymbol{x}_i \\ \hat{\boldsymbol{x}}_{i|i} \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_i' & \hat{\boldsymbol{x}}_{i|i}' \end{bmatrix}\right\} = \begin{bmatrix} \boldsymbol{V}^o & \boldsymbol{V}^{of} \\ \boldsymbol{V}^{fo} & \boldsymbol{V}^f \end{bmatrix} \tag{22}$$

which is a solution of the following matrix Lyapunov equation:

$$\boldsymbol{V} = \boldsymbol{\Phi}\boldsymbol{V}\boldsymbol{\Phi}' + \boldsymbol{J}\boldsymbol{W}\boldsymbol{J}', \tag{23}$$

with

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{F} & \boldsymbol{0} \\ \boldsymbol{k}^f\boldsymbol{d}'\boldsymbol{F} & (\boldsymbol{I} - \boldsymbol{k}^f\boldsymbol{d}')\boldsymbol{F} \end{bmatrix}, \ \boldsymbol{J} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{k}^f\boldsymbol{d}' \end{bmatrix}.$$

Indeed, inserting (17) into (16) gives

$$\begin{aligned}\hat{\boldsymbol{x}}_{i+1|i+1} &= (\boldsymbol{I} - \boldsymbol{k}^f\boldsymbol{d}')\boldsymbol{F}\hat{\boldsymbol{x}}_{i|i} + \boldsymbol{k}^f z_{i+1} = \\ &= (\boldsymbol{I} - \boldsymbol{k}^f\boldsymbol{d}')\boldsymbol{F}\hat{\boldsymbol{x}}_{i|i} + \boldsymbol{k}^f\boldsymbol{d}'\boldsymbol{F}\boldsymbol{x}_i + \boldsymbol{k}^f\boldsymbol{d}'\boldsymbol{w}_i,\end{aligned} \tag{24}$$

which together with (12) leads to (23). Finally, (19) and (22) give (21).

C. Averaging sampling

Let us denote

$$\frac{dz(t)}{dt} = \frac{1}{h}y(t) = \frac{1}{h}\boldsymbol{d}'\boldsymbol{x}(t). \tag{25}$$

Then the mean value of $y(t)$ over the sampling interval h between the sampling times t_i and t_{i+1} is

$$z_{i+1} = \frac{1}{h}\int_{t_i}^{t_{i+1}} y(t)dt = \int_{t_i}^{t_{i+1}} z(t)dt. \tag{26}$$

The state equation (8) can be extended as follows

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & 0 \\ \frac{\boldsymbol{d}'}{h} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{x}(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{C} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{\xi}} \\ 0 \end{bmatrix}. \tag{27}$$

Integrating it between the i-th and $(i+1)$-th sampling instants yields

$$\boldsymbol{x}_{i+1} = \boldsymbol{F}\boldsymbol{x}_i + \boldsymbol{w}_i, \tag{28}$$
$$z_{i+1} = \boldsymbol{f}'\boldsymbol{x}_i + v_i, \tag{29}$$

with

$$\boldsymbol{F} = \mathrm{e}^{\boldsymbol{A}h}, \qquad \boldsymbol{f}' = \frac{1}{h}\boldsymbol{d}'\int_0^h \mathrm{e}^{\boldsymbol{A}s}ds, \tag{30}$$

and

$$\mathrm{E}\begin{bmatrix} \boldsymbol{w}_i\boldsymbol{w}_j' & \boldsymbol{w}_i v_j \\ v_i\boldsymbol{w}_j' & v_i v_j \end{bmatrix} = \begin{bmatrix} \boldsymbol{W} & \gamma \\ \gamma' & \rho^2 \end{bmatrix}\delta_{ij}, \tag{31}$$

where

$$\begin{bmatrix} \boldsymbol{W} & \gamma \\ \gamma' & \rho^2 \end{bmatrix} = \int_0^h \mathrm{e}^{\bar{\boldsymbol{A}}s}\begin{bmatrix} \boldsymbol{C}\boldsymbol{C}' & \boldsymbol{0} \\ 0 & 0 \end{bmatrix}\mathrm{e}^{\bar{\boldsymbol{A}}'s}ds, \ \bar{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \frac{1}{h}\boldsymbol{d}' & 0 \end{bmatrix}.$$

Let us denote $\Delta s^*(i) = s_i - z_i$ the difference between samples of average values z_i and signal s_i, and $\sigma^2(\Delta s^*) = \lim_{i\to\infty} \mathrm{var}\{\Delta s^*(i)\}$, where

$$\Delta s^*(i+1) = (\boldsymbol{d}_0'\boldsymbol{F} - \boldsymbol{f}')\boldsymbol{x}_i + \boldsymbol{d}_0'\boldsymbol{w}_i - v_i. \tag{32}$$

We then have

$$\sigma^2(\Delta s^*) = (\boldsymbol{d}_0'\boldsymbol{F} - \boldsymbol{f}')\boldsymbol{V}^o(\boldsymbol{F}'\boldsymbol{d}_0 - \boldsymbol{f}) + \boldsymbol{d}_0'\boldsymbol{W}\boldsymbol{d}_0 - 2\boldsymbol{d}_0'\boldsymbol{\gamma}, \tag{33}$$

where

$$\boldsymbol{V}^o = \boldsymbol{F}\boldsymbol{V}^o\boldsymbol{F}' + \boldsymbol{W}. \tag{34}$$

D. Discrete-time Kalman filter for averaging sampling

The results of averaging sampling can further be improved by using a discrete-time Kalman filter. We have the following:

Lemma 1: Denote

$$\bar{\boldsymbol{d}} = \frac{\boldsymbol{\gamma}}{\rho^2}, \quad \bar{\boldsymbol{F}} = \left(\boldsymbol{F} - \bar{\boldsymbol{d}}\boldsymbol{f}'\right), \quad \overline{\boldsymbol{W}} = \boldsymbol{W} - \frac{\boldsymbol{\gamma}\boldsymbol{\gamma}'}{\rho^2}. \tag{35}$$

Then the Kalman filter for (28)-(29) that provides ($\hat{\boldsymbol{x}}_{i|i} = \mathrm{E}[\boldsymbol{x}_i|\vec{\boldsymbol{z}}_i]$) has the following form

$$\hat{\boldsymbol{x}}_{i|i+1} = \hat{\boldsymbol{x}}_{i|i} + \boldsymbol{k}^f(z_{i+1} - \boldsymbol{f}'\hat{\boldsymbol{x}}_{i|i}), \tag{36}$$

$$\hat{\boldsymbol{x}}_{i+1|i+1} = \bar{\boldsymbol{F}}\hat{\boldsymbol{x}}_{i|i+1} + \bar{\boldsymbol{d}}z_{i+1}, \quad \hat{\boldsymbol{x}}_{0|0} = 0 \tag{37}$$

$$\hat{s}_i = \boldsymbol{d}_0'\hat{\boldsymbol{x}}_{i|i}, \tag{38}$$

where

$$\boldsymbol{k}^f = \boldsymbol{\Sigma}\boldsymbol{f}\left(\boldsymbol{f}'\boldsymbol{\Sigma}\boldsymbol{f} + \rho^2\right)^{-1}, \tag{39}$$

and $\boldsymbol{\Sigma}$ is a solution of the matrix Riccati equation

$$\boldsymbol{\Sigma} = \overline{\boldsymbol{W}} + \bar{\boldsymbol{F}}\left(\boldsymbol{\Sigma} + \frac{\boldsymbol{\Sigma}\boldsymbol{f}\boldsymbol{f}'\boldsymbol{\Sigma}}{\boldsymbol{f}'\boldsymbol{\Sigma}\boldsymbol{f} + \rho^2}\right)\bar{\boldsymbol{F}}'. \tag{40}$$

Proof Since $\boldsymbol{w}_i$ and v_i are correlated, we can introduce $\bar{\boldsymbol{w}}_i$ defined as

$$\bar{\boldsymbol{w}}_i = \boldsymbol{w}_i - \frac{\boldsymbol{\gamma}}{\rho^2}v_i, \tag{41}$$

such that $\bar{\boldsymbol{w}}_i$ and v_i are independent, and

$$\mathrm{cov}\{\bar{\boldsymbol{w}}_i, v_j\} = \mathrm{E}\begin{bmatrix}\bar{\boldsymbol{w}}_i\bar{\boldsymbol{w}}_j' & \bar{\boldsymbol{w}}_i v_j \\ v_i\bar{\boldsymbol{w}}_j' & v_i v_j\end{bmatrix} = \begin{bmatrix}\overline{\boldsymbol{W}} & \boldsymbol{0} \\ \boldsymbol{0}' & \rho^2\end{bmatrix}\delta_{ij}. \tag{42}$$

Inserting

$$\boldsymbol{w}_i = \bar{\boldsymbol{w}}_i + \frac{\boldsymbol{\gamma}}{\rho^2}v_i, \tag{43}$$

from (42), and

$$v_i = z_{i+1} - \boldsymbol{f}'\boldsymbol{x}_i, \tag{44}$$

from (29) into (27) results in

$$\boldsymbol{x}_{i+1} = \bar{\boldsymbol{F}}\boldsymbol{x}_i + \bar{\boldsymbol{d}}z_{i+1} + \bar{\boldsymbol{w}}_i. \tag{45}$$

From (45), Kalman filter equations (36)-(38) follow.

Equation (37) together with (36) give:

$$\hat{\boldsymbol{x}}_{i+1|i+1} = \bar{\boldsymbol{F}}(\boldsymbol{I} - \boldsymbol{k}^f\boldsymbol{f}')\hat{\boldsymbol{x}}_{i|i} + (\bar{\boldsymbol{d}} + \bar{\boldsymbol{F}}\boldsymbol{k}^f)z_{i+1}, \; \hat{\boldsymbol{x}}_{0|0} = 0. \tag{46}$$

Limiting Variance of the filtration error at sampling points results from:

$$\sigma^2(\Delta s) = \boldsymbol{d}_0'\boldsymbol{V}^o\boldsymbol{d}_0' + \boldsymbol{d}_0'\boldsymbol{V}^f\boldsymbol{d}_0 - 2\boldsymbol{d}_0'\boldsymbol{V}^{of}\boldsymbol{d}_0, \tag{47}$$

where matrices $\boldsymbol{V}^o$, $\boldsymbol{V}^f$, and $\boldsymbol{V}^{of}$ are blocs constituting matrix $\boldsymbol{V}$ as in (22), being a solution of

$$\boldsymbol{V} = \boldsymbol{\Phi}\boldsymbol{V}\boldsymbol{\Phi}' + \boldsymbol{J}\boldsymbol{W}\boldsymbol{J}' + \boldsymbol{J}\boldsymbol{\gamma}\boldsymbol{D}' + \boldsymbol{D}\boldsymbol{\gamma}'\boldsymbol{J}' + \boldsymbol{D}\boldsymbol{D}'\rho^2, \tag{48}$$

with:

$$\boldsymbol{\Phi} = \begin{bmatrix}\boldsymbol{F} & \boldsymbol{0} \\ \boldsymbol{\delta}\boldsymbol{f}' & \boldsymbol{\Gamma}\end{bmatrix}, \quad \boldsymbol{J} = \begin{bmatrix}\boldsymbol{I} \\ \boldsymbol{0}\end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix}\boldsymbol{0} \\ \boldsymbol{\delta}\end{bmatrix},$$

$$\boldsymbol{\Gamma} = \bar{\boldsymbol{F}}(\boldsymbol{I} - \boldsymbol{k}^f\boldsymbol{f}'), \qquad \boldsymbol{\delta} = (\bar{\boldsymbol{d}} + \bar{\boldsymbol{F}}\boldsymbol{k}^f).$$

III. Simplified models

Very often the power spectrum $S_n(\omega)$ of noise $n(t)$ defined by equations (3)-(4), or by transfer function $K_n(s)$, is much wider than that of the signal of interest $s(t)$. In such case it can be modeled as white noise $n(t)$

$$\mathrm{E}[n(t)] = 0, \qquad \mathrm{E}[n(t)n(\tau)] = \eta^2\delta(t-\tau), \tag{49}$$

with constant spectral density η^2 independent of frequency ω.

The model in (8)-(10) simplifies to

$$\dot{\boldsymbol{x}}_s(t) = \boldsymbol{A}_s\boldsymbol{x}_s(t) + \boldsymbol{c}_s\dot{\xi}_s(t), \tag{50}$$

$$y(t) = \boldsymbol{d}_s'\boldsymbol{x}_s(t) + \eta\dot{\xi}_n(t), \tag{51}$$

$$s(t) = \boldsymbol{d}_s'\boldsymbol{x}_s(t), \tag{52}$$

and the model presented in Fig. 1a) simplifies to that of Fig. 1b), with

$$\eta = |K_n(0)| = |\boldsymbol{d}_n'\boldsymbol{A}_n^{-1}\boldsymbol{c}_n|. \tag{53}$$

While appropriate for continuous-time signal processing modeling, it is completely inadequate for discrete-time models. This is because sampling of physically nonexisting continuous-time white noise with infinite variance can not be defined reasonable. To this end, we propose a discrete-time model of instantaneously sampled noisy signal

$$\boldsymbol{x}_{i+1}^s = \boldsymbol{F}_s\boldsymbol{x}_i^s + \boldsymbol{w}_i^s, \tag{54}$$

$$z_i = \boldsymbol{d}_s'\boldsymbol{x}_i^s + n_i, \tag{55}$$

$$s_i = \boldsymbol{d}_s'\boldsymbol{x}_i^s, \tag{56}$$

with

$$\boldsymbol{F}_s = \mathrm{e}^{\boldsymbol{A}_s h}, \qquad \boldsymbol{W}_s = \int_0^h \mathrm{e}^{\boldsymbol{A}_s v}\boldsymbol{c}_s\boldsymbol{c}_s'\mathrm{e}^{\boldsymbol{A}_s' v}dv, \tag{57}$$

in which noise is presented as discrete-time white noise n_i whose variance ρ^2 equals to the variance of $n(t)$ of the original system, i.e. $\rho^2 = \mathrm{var}\{n_i\} = \mathrm{var}\{n(t)\}$, and can be calculated as

$$\rho^2 = \boldsymbol{d}_n'\boldsymbol{Q}_n\boldsymbol{d}_n, \tag{58}$$

where $\boldsymbol{Q}_n$ fulfills the following Lyapunov equation:

$$\boldsymbol{A}_n\boldsymbol{Q}_n + \boldsymbol{Q}_n\boldsymbol{A}_n' = -\boldsymbol{d}_n\boldsymbol{d}_n'. \tag{59}$$

A. Discrete-time Kalman filter

Kalman filter equations for system in (54)–(55) have formally the same for as in (16)-(18), except for $\dim \boldsymbol{x}^s_{i|i} = n_s$, and

$$\boldsymbol{k}^f = \boldsymbol{\Sigma}\boldsymbol{d}_s\left(\boldsymbol{d}_s'\boldsymbol{\Sigma}\boldsymbol{d}_s + \rho^2\right)^{-1}, \tag{60}$$

where $\boldsymbol{\Sigma}$ is a solution of

$$\boldsymbol{\Sigma} = \boldsymbol{W}_s + \boldsymbol{F}_s\left(\boldsymbol{\Sigma} - \frac{\boldsymbol{\Sigma}\boldsymbol{d}_s\boldsymbol{d}_s'\boldsymbol{\Sigma}'}{\boldsymbol{d}_s'\boldsymbol{\Sigma}\boldsymbol{d}_s + \rho^2}\right)\boldsymbol{F}_s'. \tag{61}$$

When applying this filter to the system in (12)-(13) then the variance $\sigma^2(\Delta s^*) = \lim_{i\to\infty} \operatorname{var}\{s_i - \hat{s}_{i|i}\}$ of the estimation error can be calculated from

$$\sigma^2(\Delta s^*) = \boldsymbol{d}_0'\boldsymbol{V}^o\boldsymbol{d}_0 + \boldsymbol{d}_s'\boldsymbol{V}^f\boldsymbol{d}_s - 2\boldsymbol{d}_0'\boldsymbol{V}^{of}\boldsymbol{d}_s + \rho^2. \tag{62}$$

where $\boldsymbol{V}^o$, $\boldsymbol{V}^f$, end $\boldsymbol{V}^{fo}$ are, as in (22) submatrices of matrix $\boldsymbol{V}$ being a solution of

$$\boldsymbol{V} = \boldsymbol{\Phi}\boldsymbol{V}\boldsymbol{\Phi}' + \boldsymbol{J}\boldsymbol{W}\boldsymbol{J}', \tag{63}$$

with

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{F} & \boldsymbol{0} \\ \boldsymbol{k}^f\boldsymbol{d}'\boldsymbol{F} & (\boldsymbol{I} - \boldsymbol{k}^f\boldsymbol{d}_s')\boldsymbol{F}_s \end{bmatrix}, \ \boldsymbol{J} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{k}^f\boldsymbol{d}' \end{bmatrix}.$$

Consider an exemplary system defined by

$$K_s(s) = \frac{1}{(1+3s)^2}, \quad K_n(s) = \frac{k_n}{T_n^2 s^2 + 2\zeta_n T_n s + 1} \tag{64}$$

$$\zeta_n = 0.2,\ T_n = 0.1, \tag{65}$$

Figure 4 shows that there is almost no difference between the exact and the approximate discrete Kalman filters, perhaps except for very small sampling periods.

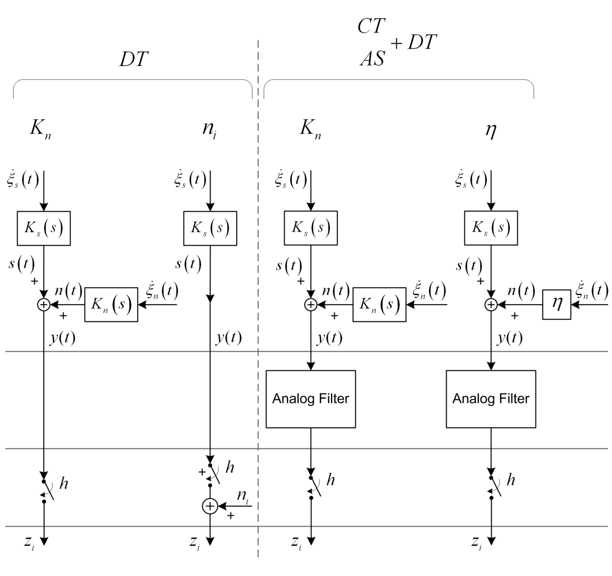

Fig. 3. Models of signals and CT filters assumed for DT Kalman filter design

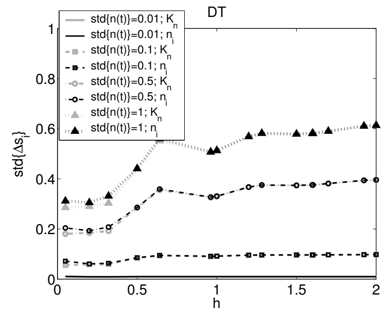

Fig. 4. Comparison of exact and approximate DT Kalman filter

B. Averaging Sampling

Let us denote

$$\frac{dz(t)}{dt} = \frac{1}{h}y(t) = \frac{1}{h}\boldsymbol{d}_s'\boldsymbol{x}_s(t) + \frac{\eta}{h}\dot{\xi}_n, \tag{66}$$

and

$$z_{i+1} = \frac{1}{h}\int_{t_i}^{t_{i+1}} y(t)dt = \int_{t_i}^{t_{i+1}} z(t)dt. \tag{67}$$

Then equation (66) together with (50) form the system:

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{x}_s(t) \\ z(t)\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}_s & 0 \\ \frac{\boldsymbol{d}_s'}{h} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_s(t) \\ z(t)\end{bmatrix} + \begin{bmatrix}\boldsymbol{c}_s & 0 \\ 0 & \frac{\eta}{h}\end{bmatrix}\begin{bmatrix}\dot{\xi}_s(t) \\ \dot{\xi}_n(t)\end{bmatrix}. \tag{68}$$

Solving equation (68) on the interval $ih \le t < (i+1)h$ gives:

$$\boldsymbol{x}^s_{i+1} = \boldsymbol{F}_s\boldsymbol{x}^s_i + \boldsymbol{w}^s_i, \tag{69}$$

$$z_{i+1} = \boldsymbol{f}_s'\boldsymbol{x}^s_i + v^s_i, \tag{70}$$

where:

$$\boldsymbol{F}_s = \mathrm{e}^{\boldsymbol{A}_s h}, \qquad \boldsymbol{f}_s' = \frac{1}{h}\boldsymbol{d}_s'\int_0^h \mathrm{e}^{\boldsymbol{A}_s u}du, \tag{71}$$

and $\boldsymbol{w}^s_i$ i v^s_i are discrete-time white noise signals such that

$$\mathrm{E}\begin{bmatrix}\boldsymbol{w}^s_i\boldsymbol{w}^{s\prime}_j & \boldsymbol{w}^s_i v^s_j \\ v^s_i\boldsymbol{w}^{s\prime}_j & v^s_i v^s_j\end{bmatrix} = \begin{bmatrix}\boldsymbol{W}_s & \boldsymbol{\gamma}_s \\ \boldsymbol{\gamma}_s' & \rho_s^2\end{bmatrix}\delta_{ij}, \tag{72}$$

$$\begin{bmatrix}\boldsymbol{W}_s & \boldsymbol{\gamma}_s \\ \boldsymbol{\gamma}_s' & \rho_s^2\end{bmatrix} = \int_0^h \mathrm{e}^{\bar{\boldsymbol{A}}_s u}\begin{bmatrix}\boldsymbol{c}_s\boldsymbol{c}_s' & \boldsymbol{0} \\ \boldsymbol{0} & \frac{\eta^2}{h^2}\end{bmatrix}\mathrm{e}^{\bar{\boldsymbol{A}}_s' u}du, \tag{73}$$

with:

$$\bar{\boldsymbol{A}}_s = \begin{bmatrix}\boldsymbol{A}_s & \boldsymbol{0} \\ \frac{\boldsymbol{d}_s'}{h} & \boldsymbol{0}\end{bmatrix}. \tag{74}$$

Similar results, expressed in δ-operator, are derived in [3]. Unfortunately there is an error in calculating the matrix in (72). Equations (69)–(70) have formally the same form as those in equations (28)–(29). In [4] a model similar to (69)-(70) is presented, however with z_i instead of z_{i+1} in output equation (70), and simplified covariance matrix (73) with $\boldsymbol{W}_s = \boldsymbol{c}_s'\boldsymbol{c}_s \cdot h$, $\rho_s^2 = \eta^2/h$ and $\boldsymbol{\gamma}_s = \boldsymbol{0}$.

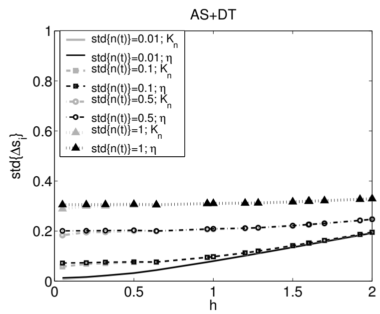

Fig. 5. Avaraging sampling + DT Kalman: exact vs approximate

C. Kalman filter for averaging sampling

Since equations (69)-(70) modeling averaging sampling for simplified noise model have the same form as those for the exact noise model, (28)-(29), the Kalman filter equations are formally the same as those in (36) - (40), i.e.

$$\hat{\boldsymbol{x}}^s_{i|i+1} = \hat{\boldsymbol{x}}^s_{i|i} + \boldsymbol{k}^f(z_{i+1} - \boldsymbol{f}'_s\hat{\boldsymbol{x}}^s_{i|i}), \tag{75}$$

$$\hat{\boldsymbol{x}}^s_{i+1|i+1} = \bar{\boldsymbol{F}}_s\hat{\boldsymbol{x}}^s_{i|i+1} + \bar{\boldsymbol{d}}_s z_{i+1},\ \hat{\boldsymbol{x}}^s_{0|0} = 0 \tag{76}$$

$$\hat{s}_i = \boldsymbol{d}'_s\hat{\boldsymbol{x}}^s_{i|i}, \tag{77}$$

with

$$\boldsymbol{k}^f = \boldsymbol{\Sigma}\boldsymbol{f}_s\left(\boldsymbol{f}'_s\boldsymbol{\Sigma}\boldsymbol{f}_s + \rho^2_s\right)^{-1}, \tag{78}$$

and $\boldsymbol{\Sigma}$ being a solution of the matrix Riccati equation

$$\boldsymbol{\Sigma} = \overline{\boldsymbol{W}}_s + \bar{\boldsymbol{F}}_s\left(\boldsymbol{\Sigma} + \frac{\boldsymbol{\Sigma}\boldsymbol{f}_s\boldsymbol{f}'_s\boldsymbol{\Sigma}}{\boldsymbol{f}'_s\boldsymbol{\Sigma}\boldsymbol{f}_s + \rho^2_s}\right)\bar{\boldsymbol{F}}'_s, \tag{79}$$

where

$$\bar{\boldsymbol{d}}_s = \frac{\boldsymbol{\gamma}_s}{\rho^2_s},\ \bar{\boldsymbol{F}}_s = \left(\boldsymbol{F}_s - \bar{\boldsymbol{d}}_s\boldsymbol{f}'_s\right),\ \overline{\boldsymbol{W}}_s = \boldsymbol{W}_s - \frac{\boldsymbol{\gamma}_s\boldsymbol{\gamma}'_s}{\rho^2_s}, \tag{80}$$

with values of $\boldsymbol{W}_s$, $\boldsymbol{\gamma}_s$ and ρ^2_s taken from (73).

Variance of the filtration error at sampling points is determined by:

$$\sigma^2(\Delta s) = \boldsymbol{d}'_o\boldsymbol{V}^o\boldsymbol{d}'_o + \boldsymbol{d}'_s\boldsymbol{V}^f\boldsymbol{d}_s - 2\boldsymbol{d}'_o\boldsymbol{V}^{of}\boldsymbol{d}_s, \tag{81}$$

where the covariance matrix $\boldsymbol{V}_i$ of the form Eq.(22) is a solution of

$$\boldsymbol{V} = \boldsymbol{\Phi}\boldsymbol{V}\boldsymbol{\Phi}' + \boldsymbol{J}\boldsymbol{W}\boldsymbol{J}' + \boldsymbol{J}\boldsymbol{\gamma}\boldsymbol{D}' + \boldsymbol{D}\boldsymbol{\gamma}'\boldsymbol{J}' + \boldsymbol{D}\boldsymbol{D}'\rho^2, \tag{82}$$

with:

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{F} & \boldsymbol{0} \\ \boldsymbol{\delta}_s\boldsymbol{f}' & \boldsymbol{\Gamma}_s \end{bmatrix}, \boldsymbol{J} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\delta}_s \end{bmatrix},$$

$$\boldsymbol{\Gamma}_s = \bar{\boldsymbol{F}}_s(\boldsymbol{I} - \boldsymbol{k}^f\bar{\boldsymbol{f}}'_s),\ \boldsymbol{\delta}_s = (\bar{\boldsymbol{d}}_s + \bar{\boldsymbol{F}}_s\boldsymbol{k}^f).$$

Fig. 5 displaying results for exemplary system in (64) shows that the simplified Kalman filter gives almost the same results as the exact one.

D. Continuous-time Kalman filter

The Kalman filter for system in (51) - (52), displayed in Fig. 1b), is defined as follows:

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A}_s\hat{\boldsymbol{x}}(t) + \boldsymbol{k}^f_c\left[y(t) - \boldsymbol{d}'_s\hat{\boldsymbol{x}}(t)\right], \tag{83}$$

with:

$$\boldsymbol{k}^f_c = \frac{\boldsymbol{P}\boldsymbol{d}_s}{\eta^2} \text{ and } \boldsymbol{A}_s\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}'_s - \frac{\boldsymbol{P}\boldsymbol{d}_s\boldsymbol{d}'_s\boldsymbol{P}}{\eta^2} + \boldsymbol{c}_s\boldsymbol{c}'_s = 0 \tag{84}$$

The filtered value $\hat{s}(t)$ of $s(t)$ is determined by

$$\hat{s}(t) = \boldsymbol{d}'_s\hat{\boldsymbol{x}}(t). \tag{85}$$

Samples z_i of the signal in (85) can be further processed by a discrete-time Kalman filter.

IV. Properties of averaging sampling

In this section we will study the properties of averaging sampling based on our example.

Comparison of purely discrete Kalman filter with averaging sampling displayed in Fig. 6 shows that while DT Kalman behaves more regularly when changing sampling period h and noise level std $\{n(t)\}$ than the averaging sampling itself. In particular, for very small sampling periods there is no use of AS, while DT exhibits very good performance. Increasing sampling period worsens the quality of DT and improves that of AS but only for noise level great enough. For small noise levels AS behaves rather badly.

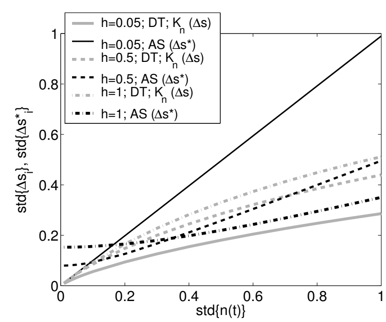

Fig. 6. Discrete-time Kalman vs averaging sampling

Fig. 7 shows that performance of sampled output from analog Kalman filter does not depend on h, and that this limiting performance is gradually attained by discrete Kalman filter when increasing sampling frequency.

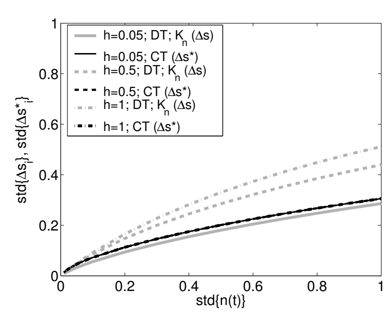

Fig. 7. Discrete-time vs continuous-time Kalman

Decreasing quality of AS for h smaller than certain optimal value depending on noise level is seen in Fig. 8. It is interesting to note that for optimal h the quality of AS is similar to that of CT Kalman. For h higher than optimal the quality of AS degrades due to "smoothing" signal $s(t)$. For small levels of noise this can overweight, making the filtration error greater than the noise itself. As seen from Fig. 9–10, things get improved for h smaller than optimal when AS is followed by DT Kalman filter but, unfortunately, this filter does not help for larger values of h.

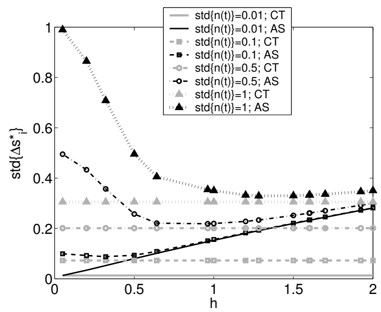

Fig. 8. Averaging sampling vs continuous-time Kalman

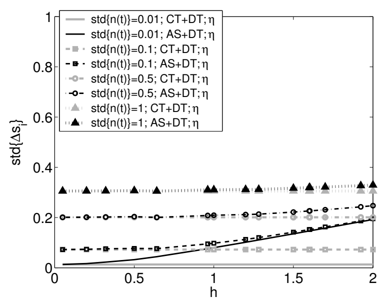

Fig. 9. AS + DT Kalman vs CT Kalman + DT Kalman

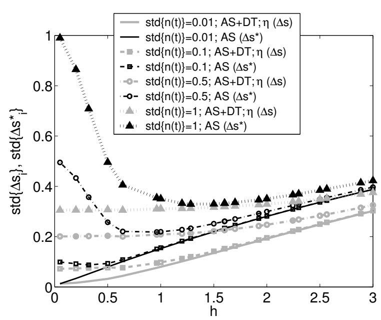

Fig. 10. AS vs AS + DT Kalman

Superiority of DT Kalman filter over AS for small h is seen in Fig. 11, and Fig. 12 shows that even the combination of AS and DT Kalman is not better than purely discrete-time Kalman filter for h small enough. Exemplary realizations of noisy signals and their samples after averaging and additionally discrete-time filtering are shown in Fig. 13.

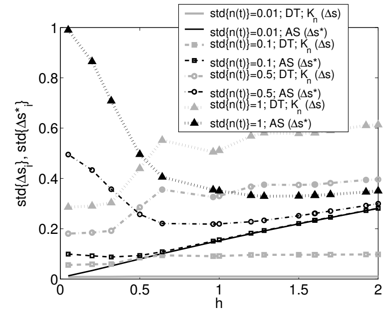

Fig. 11. AS vs DT Kalman

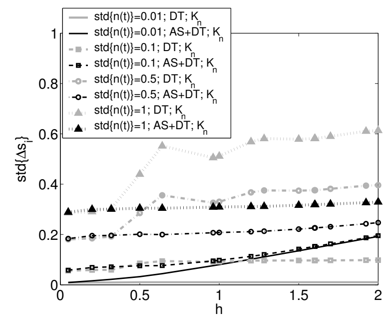

Fig. 12. AS + DT Kalman vs DT Kalman

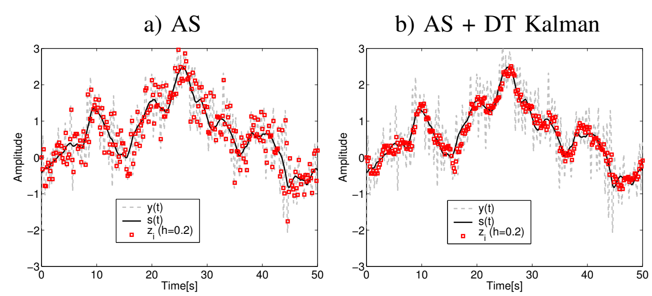

Fig. 13. Signal, noise, AS, and AS + DT Kalman

V. Conclusion

The range of reasonable sampling periods for averaging sampling is rather small, and stretches around optimal value that depends on the noise level. Augmenting the sampler with a discrete Kalman filter improves the results for smaller sampling periods bringing the filtration error close to the lower limit provided by an analog Kalman filter. It does however almost not help for sampling periods greater than optimal. Small noise level results in a small value of optimal sampling period which makes averaging sampling useless.

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