$\mathscr{H}_\infty$ robust memory controllers for networked control systems: uncertain sampling rates and time delays in polytopic domains

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Abstract— In this paper, the problem of controller design for networked control systems with time-varying sampling rates and time delays is investigated. By using a memory at the feedback loop, a digital robust controller that minimizes an upper bound to the $\mathscr{H}_\infty$ performance of the closed loop system is determined. The design conditions are obtained from the Finsler's Lemma combined with the Lyapunov theory and expressed in terms of bilinear matrix inequalities. Extra variables introduced by the Finsler's Lemma are explored in order to provide a better system behavior. The time-varying uncertainties are modelled using polytopic domains. The controller is obtained by the solution of an optimization problem formulated only in terms of the vertices of the polytope, avoiding grids in the parametric space. Numerical examples illustrate the efficiency of the proposed approach.

I. INTRODUCTION

One of the most challenging problems facing electrical engineering nowadays is concerned with the perfect operation of dynamical systems immersed in a hostile environment. There is no denying the fact that linear models, both continuous and discrete in time, with uncertain parameters is an important topic which is much investigated by the control community in the last years all over the world. Among countless reasons, it is sagacious to seek better characterizations of model uncertainties, not only to guarantee stability, but also robustness against practical disturbances and perturbations when filters or controllers are being synthesized.

Taking into consideration the above framework, the Lyapunov theory (more precisely the Lyapunov Second Method [1]) has been one of the main tools to deal with stability analysis and synthesis of controllers. In a large number of cases, the design conditions can be expressed as an optimization problem in terms of linear matrix inequalities (LMIs), which can be numerically handled by powerful softwares [2, 3]. Bilinear matrix inequalities (BMIs) have also been applied in the study of stability of linear systems. Despite nonconvex, optimization problems expressed in terms of BMIs may represent a good strategy to face those cases in which either no solution or only sufficient conditions are available in the literature, for instance see [4–7] and references therein.

Recently, technological advances have enabled the extensive use of communication channels in the control of dynamic systems [8, 9]. Using a real-time network to exchange information between control system components (sensors, actuators, filters, etc.), these systems, known as networked control systems (NCSs), have represented a good alternative to implement distributed control and interconnected systems, among others. To illustrate the importance of NCSs, one can cite the following benefits: reduced system wiring, plug and play devices, ease of system diagnosis and maintenance [9]. Unfortunately, some drawbacks also arise. NCSs suffer from packet size and sampling rates constraints, time delays, sensor signals saturation, etc. The study of control strategies to surpass these difficulties has received considerable attention recently, as for instance [10–17]. Many efforts have been made in order to bring together the advances of the control theory and the benefits of a communication network.

The Lyapunov theory, $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance control and LMIs have soon begun to be used in the NCS framework. Recent works include [18] where a feedback controller for a discrete-time Markovian jump system with random delays is constructed via a set of LMIs conditions, [19] where the control problem for the multipoint-packet system is solved using $\mathscr{H}_2$ optimization techniques, [20] in which the stabilization of an NCS is obtained by means of a packet-loss dependent Lyapunov function and [21] where a Lyapunov-Krasovskii functional is used to design a state feedback controller for a time-delay sampled system.

Despite all these advances, many results appeared so far concerned with NCS still fail in providing robustness against practical issues. Important aspects of the real network are neglected during the design step. For example, controllers are designed without taking into consideration any index of performance, as in [15], what may lead to poor system behaviors. Some methods do not apply when the system is subject to both time-varying sampling rates and network-induced delays, while others require the *a priori* knowledge of the exact values for the sampling rates and time delays applied at run-time. Conservatism is introduced by the use of quadratic stability. BMI techniques seem to be barely explored in NCSs, as well as different structures to model time-varying uncertainties, besides others.

The aim here is to provide robust memory controllers to stabilize NCS subject to both time-varying sampling rates and network-induced delays. The stability conditions of the closed-loop system are certified by the Lyapunov theory and the robustness of the controller by an $\mathscr{H}_\infty$ guaranteed cost. A parameter dependent Lyapunov function rather than the

quadratic stability is applied in order to reduce the conservatism of the proposed method. Using the extra variables introduced by the Finsler's Lemma, the design conditions are expressed in terms of BMIs, that can be freely explored in the search for better performance of the NCSs. Some results appeared in the literature concerned with LMIs can be obtained as a particular case from the conditions proposed here, for instance the ones in [22]–[24]. Using a memory in the feedback loop it is possible to cope with time-varying delays without making of more complex Lyapunov functionals (what may require a bigger computational burden) as the one in [21]. All the sampled system matrices are supposed to be affected by the time-varying parameters, which are modelled inside polytopic domains. The robust memory controller is then obtained by the solution of an optimization problem that minimizes an upper bound to the $\mathscr{H}_\infty$ index of performance subject to a finite number of BMI constraints formulated only in terms of the vertices of a polytope. No grids in the parametric space are used. Numerical examples illustrate the efficiency of the proposed results not only in the NCS framework but also when compared to other techniques for robust control of time-varying discrete-time systems appeared in the literature.

II. PRELIMINARIES AND PROBLEM STATEMENT
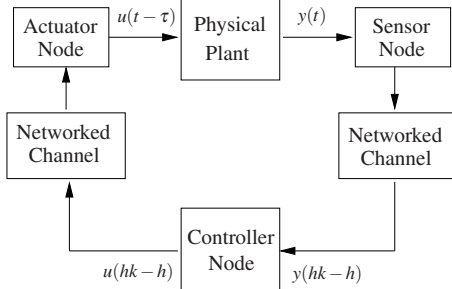
Consider the model described in Figure 1. The physical

Fig. 1.   NCS Model.

plant is given by the following equations, for $t \geq 0$

$$\dot{x}(t) = Ax(t) + Bu(t-\tau)$$
$$y(t) = Cx(t) + Du(t) + D_d u(t-\tau) \qquad (1)$$
$$x(0) = 0, \ u(\theta) = 0, \ \theta \in \{-\tau, 0\}$$

where τ represents the network-induced time delay, $x(t) \in \mathbb{R}^n$ is the state space vector, $u(t) \in \mathbb{R}^m$ is the control signal and $y(t) \in \mathbb{R}^q$ is the output. All matrices are real, with appropriate dimensions.

System (1) is sampled with a period $h \geq \tau$, yielding the discrete-time model, for $k \in \mathbb{Z}_+$, $x(0) = 0$ and $u(\theta) = 0$, $\theta \in \{-h, 0\}$ [25]

$$x(kh+h) = A_s(h)x(kh) + B_{su0}(h,\tau)u(kh)$$
$$+ B_{su1}(h,\tau)u(kh-h) + B_{sw}w(kh) \qquad (2)$$
$$y(kh) = C_s x(kh) + D_{su}u(kh) + D_{sd}u(kh-h) + D_{sw}w(kh)$$

where $w(kh) \in \mathbb{R}^r$ is an extra input, belonging to $l_2[0,\infty)$, used to model possible noise in the process. The system

matrices $A_s(h)$, $B_{su0}(h,\tau)$, $B_{su1}(h,\tau)$, C_s, D_{su} and D_{sd} are given by

$$A_s(h) = e^{Ah}, \ B_{su0}(h,\tau) = \int_0^{h-\tau} e^{As}ds B, \ D_{sd} = D_d$$
$$\qquad (3)$$
$$B_{su1}(h,\tau) = e^{A(h-\tau)}\int_0^{h-\tau} e^{As}ds B, \ C_s = C, \ D_{su} = D$$

As shown in [15], the sampling period h, as far as the time delay τ, may change their values at run-time due to different reasons, as for example bandwidth allocation and scheduling decisions. Nevertheless, bounds in such variations can be determined, guaranteeing that the actual values of h and τ at each instant k (i.e., h_k and τ_k) lie inside finite discrete sets as specified below

$$h_k \in \{h_{min}, \dots, h_{max}\}, \ h_k = \kappa \cdot g, \ \kappa \in \mathbb{N}$$
$$\tau_k \in \{\tau_{min}, \dots, \tau_{max}\}, \ \tau_k = \kappa \cdot g, \ \kappa \in \mathbb{N} \qquad (4)$$

It is assumed that the real values of h_k and τ_k are not known at time instant k, but only that they belong to (4) and vary in such a way that $h_k \geq \tau_k$, $\forall k \in \mathbb{Z}_+$, which implies that $h_{max} \geq \tau_{max}$. One possible way to guarantee that $h_k \geq \tau_k$ is by setting the nominal sampling period h_0 so that the intervals on (4) do not overlap, in other words, $h_{min} \geq \tau_{max}$. Nevertheless, this choice will depend on each system to be controlled and may not be feasible. The number of possible values of these sets depends on the processor/network clock granularity g [15]. The sampled system is then considered as a system that depends on uncertain parameters that are time-varying.

In order to guarantee the stability of the networked system shown in Figure 1, a memory state feedback controller is designed. Using an extra state variable $z(kh)$ to store the last value of the control signal, $u(kh-h)$, system (2) can be rewritten as follows [25]

$$\tilde{x}(kh+h) = \tilde{A}(\alpha(kh))\tilde{x}(kh) + \tilde{B}_u(\alpha(kh))u(kh)$$
$$+ \tilde{B}_w(\alpha(kh))w(kh)$$
$$y(kh) = \tilde{C}(\alpha(kh))\tilde{x}(kh) + \tilde{D}_u(\alpha(kh))u(kh)$$
$$+ \tilde{D}_w(\alpha(kh))w(kh) \qquad (5)$$

where $\tilde{x}(kh) = [x(kh)'\ z(kh)']'$, $\alpha(kh)$ represents the time-varying uncertainties and[1]

$$\tilde{A}(\alpha) = \begin{bmatrix} A_s(h) & B_{su1}(h,\tau) \\ 0 & 0 \end{bmatrix}, \ \tilde{B}_u(\alpha) = \begin{bmatrix} B_{su0}(h,\tau) \\ I \end{bmatrix},$$
$$\tilde{B}_w(\alpha) = \begin{bmatrix} B_{sw} \\ 0 \end{bmatrix}, \tilde{C}(\alpha) = \begin{bmatrix} C_s' \\ D_{sd}' \end{bmatrix}', \tilde{D}_u(\alpha) = D_{su}, \tilde{D}_w(\alpha) = D_{sw}$$
$$\qquad (6)$$

In the case where there is no time delay ($\tau = 0$), the state space vector becomes $\tilde{x}(kh) = x(kh)$ and the augmented system matrices simplify in a standard way.

The control signal is given by

$$u(kh) = K_x x(kh) + K_d u(kh-h) = \begin{bmatrix} K_x & K_d \end{bmatrix} \begin{bmatrix} x(kh) \\ z(kh) \end{bmatrix} \qquad (7)$$

In order to represent the set of all possible matrices in system (5) due to the time-varying uncertainties (4), a

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[1]The time dependence of α(kh) will be omitted to lighten the notation.

polytopic model is considered. More specifically, the system matrices, for any time $kh \geq -h$, are described as a convex combination of well-defined vertices, which are given by the arrangements of the extreme values of (4). In general, each uncertain parameter defines two vertices, one given by its minimum value and the other by its maximum value. This will not be the case, as for instance, when the functions of the uncertain parameter appeared in the system matrices are not strictly monotonic.

The closed-loop system is given by

$$
\begin{aligned}
\tilde{x}(kh+h) &= \tilde{A}_{cl}(\alpha)\tilde{x}(kh) + \tilde{B}_{wcl}(\alpha)w(kh) \\
y(kh) &= \tilde{C}_{cl}(\alpha)\tilde{x}(kh) + \tilde{D}_{wcl}(\alpha)w(kh)
\end{aligned}
\tag{8}
$$

with

$$
\begin{aligned}
\tilde{A}_{cl}(\alpha) &= \tilde{A}(\alpha) + \tilde{B}_u(\alpha)K, \ \tilde{B}_{wcl}(\alpha) = \tilde{B}_w(\alpha), \\
\tilde{C}_{cl}(\alpha) &= \tilde{C}(\alpha) + \tilde{D}_u(\alpha)K, \ \tilde{D}_{wcl}(\alpha) = \tilde{D}_w(\alpha)
\end{aligned}
\tag{9}
$$

where $K = [K_x \ K_d]$. The whole of possible outcomes for the set (9) belongs to the polytope

$$
\mathscr{P} \triangleq \left\{ \left[\begin{array}{c|c} \tilde{A}_{cl}(\alpha) & \tilde{B}_{wcl}(\alpha) \\ \hline \tilde{C}_{cl}(\alpha) & \tilde{D}_{wcl}(\alpha) \end{array} \right] = \sum_{i=1}^{N} \alpha_i \left[\begin{array}{c|c} \tilde{A}_{cli} & \tilde{B}_{wcli} \\ \hline \tilde{C}_{cli} & \tilde{D}_{wcli} \end{array} \right] \right\}
\tag{10}
$$

with the time-varying vector α lying inside the unit simplex

$$
\mathscr{U} = \left\{ \alpha \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1, \ \alpha_i \geq 0, \ i = 1,\ldots,N \right\}
$$

for all $kh \geq -h$.

The control problem to be dealt with can be stated as follows.

Problem 1: Find constant matrices $K_x \in \mathbb{R}^{m \times n}$ and $K_d \in \mathbb{R}^{m \times n}$ of the state feedback control (7), such that the closed-loop system (8) is asymptotically stable, and an upper bound γ to the $\mathscr{H}_\infty$ performance is minimized, that is, for all $kh \in [-h, \infty)$

$$
\sup_{w(kh) \neq 0} \frac{\|y(kh)\|_2^2}{\|w(kh)\|_2^2} < \gamma^2
\tag{11}
$$

with $w(kh) \in l_2[0, \infty)$.

Before proceeding to the solution of Problem 1, a previous result is needed.

Lemma 1: (Finsler) Let $\xi \in \mathbb{R}^a$, $\mathscr{Q} = \mathscr{Q}' \in \mathbb{R}^{a \times a}$, $\mathscr{B} \in \mathbb{R}^{b \times a}$ with $rank(\mathscr{B}) < a$, and $\mathscr{B}^\perp$ a basis for the null-space of $\mathscr{B}$ (i.e. $\mathscr{B}\mathscr{B}^\perp = 0$). The following statements are equivalent.

i) $\xi' \mathscr{Q} \xi < 0$, $\forall \mathscr{B}\xi = 0$, $\xi \neq 0$;
ii) $\mathscr{B}^{\perp'} \mathscr{Q} \mathscr{B}^\perp < 0$;
iii) $\exists \mu \in \mathbb{R} : \mathscr{Q} - \mu \mathscr{B}' \mathscr{B} < 0$;
iv) $\exists \mathscr{X} \in \mathbb{R}^{a \times b} : \mathscr{Q} + \mathscr{X}\mathscr{B} + \mathscr{B}'\mathscr{X}' < 0$.

Proof: See [26]. ∎

By applying the Bounded Real Lemma [2], combined with Lemma 1, condition (11) can be guaranteed as follows.

Lemma 2: For a given $\gamma > 0$, if there exists a parameter-dependent matrix $P(\alpha)' = P(\alpha) > 0$ such that the statements

of Lemma 1 are satisfied for

$$
\mathscr{Q} = \begin{bmatrix} P(\alpha_+) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -P(\alpha) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
$$

$$
+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma^{-1}\tilde{B}_{wcl}(\alpha)\tilde{B}_{wcl}(\alpha)' & \gamma^{-1}\tilde{B}_{wcl}(\alpha)\tilde{D}_{wcl}(\alpha)' \\ \mathbf{0} & \gamma^{-1}\tilde{D}_{wcl}(\alpha)\tilde{B}_{wcl}(\alpha)' & \gamma^{-1}\tilde{D}_{wcl}(\alpha)\tilde{D}_{wcl}(\alpha)' - \gamma\mathbf{I} \end{bmatrix}
$$

$$
\mathscr{B} = \begin{bmatrix} -\mathbf{I} & \tilde{A}_{cl}(\alpha)' & \tilde{C}_{cl}(\alpha)' \end{bmatrix},
$$

$$
\mathscr{B}^\perp = \begin{bmatrix} \tilde{A}_{cl}(\alpha)' & \tilde{C}_{cl}(\alpha)' \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \ \xi = \begin{bmatrix} \tilde{x}(k+1)' & \tilde{x}(k)' & w(k)' \end{bmatrix}'
$$

where $\alpha_+ = \alpha(k+1)$, then the closed-loop system (8) is asymptotically stable with an upper bound $\gamma > 0$ to the $\mathscr{H}_\infty$ performance.

Proof: Let $v(k) = \tilde{x}(k)'P(\alpha)\tilde{x}(k)$ be a parameter-dependent Lyapunov function. Considering the dual system (i.e. $\tilde{A}_{cl} = \tilde{A}'_{cl}$, $\tilde{B}_{wcl} = \tilde{C}'_{wcl}$, $\tilde{C}_{cl} = \tilde{B}'_{wcl}$ and $\tilde{D}_{wcl} = \tilde{D}'_{wcl}$), it is straightforward from statement *i)* of Lemma 1 that Lemma 2 ensures $v(k) > 0$ and

$$
\Delta v(k) < -\gamma^{-1}y(k)'y(k) + \gamma w(k)'w(k)
$$

with the choice $\xi = [\tilde{x}(k+1)' \ \tilde{x}(k)' \ w(k)']'$. The last inequality comes from $\Delta v(k) < 0$ and

$$
y(k)'y(k) - \gamma^2 w(k)'w(k) < 0
$$

by applying the Bounded Real Lemma. Therefore, system (8) has an upper bound γ to the $\mathscr{H}_\infty$ performance and, from the Lyapunov theory [1], is asymptotically stable. ∎

The conditions of Lemma 2 appear as nonlinearities that must be tested at all points of the simplex $\mathscr{U}$, *i.e.*, at an infinite number of points. Hence, the main goal hereafter is to obtain finite-dimensional conditions in terms of the vertices of the polytope $\mathscr{P}$ to solve Problem 1. Using Schur complement, change of variables and exploring the extra variables provided by Lemma 1, parameter-dependent BMIs assuring the existence of such controllers are given in the next section.

III. MAIN RESULTS

Theorem 1: ($\mathscr{H}_\infty$ ROBUST MEMORY CONTROLLER) Given the augmented sampled system (5), if there exist matrices $L \in \mathbb{R}^{m \times (n+m)}$, $H_i \in \mathbb{R}^{q \times (n+m)}$, F, G_i, $P_i = P'_i > 0 \in \mathbb{R}^{(n+m) \times (n+m)}$, $i = 1,\ldots,N$ and a scalar $\gamma > 0$ such that[2]

$$
\Xi_{ij} \triangleq \begin{bmatrix} P_j - F - F' & \mathscr{F}_{12} & F\tilde{C}'_i + L'\tilde{D}_{ui} - F'H'_j & \mathbf{0} \\ (\star) & \mathscr{F}_{22} & \mathscr{F}_{23} & \tilde{B}_{wi} \\ (\star) & (\star) & \mathscr{F}_{33} & \tilde{D}_{wi} \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0
\tag{12}
$$

$$
i = 1,\ldots,N, \ j = 1,\ldots,N
$$
$$
\mathscr{F}_{12} = F\tilde{A}'_i + L'\tilde{B}'_{ui} - F'G'_j,
$$
$$
\mathscr{F}_{22} = G_j F\tilde{A}'_i + \tilde{A}_i F'G'_j + G_j L'\tilde{B}'_{ui} + \tilde{B}_{ui}LG'_j - P_i,
$$
$$
\mathscr{F}_{23} = G_j F\tilde{C}'_i + G_j L'\tilde{D}'_{ui} + \tilde{A}_i F'H'_j + \tilde{B}_{ui}LH'_j,
$$
$$
\mathscr{F}_{33} = H_j F\tilde{C}'_i + \tilde{C}_i F'H'_j + H_j L'\tilde{D}'_{ui} + \tilde{D}_{ui}LH'_j - \gamma\mathbf{I}
$$

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[2]The term $(\star)$ indicates symmetric blocks in the LMIs.

then the closed-loop system (8) is asymptotically stable with a memory controller $K = [K_x \; K_d] = L \, (F')^{-1}$ and an $\mathscr{H}_\infty$ guaranteed cost γ.

Proof: Firstly, applying the following operation

$$\Xi(\alpha) = \sum_{j=1}^{N} \alpha_j \left\{ \sum_{i=1}^{N} \alpha_i \Xi_{ij} \right\} \qquad (13)$$

in the BMIs (12) with the change of variables $L = KF'$ it follows that

$$\Xi(\alpha) = \begin{bmatrix} \hat{\mathscr{F}}_{11} & \hat{\mathscr{F}}_{12} & F\tilde{C}_{cl}(\alpha)' - F'H(\alpha_+)' & \mathbf{0} \\ (\star) & \hat{\mathscr{F}}_{22} & \hat{\mathscr{F}}_{23} & \tilde{B}_{wcl}(\alpha) \\ (\star) & (\star) & \hat{\mathscr{F}}_{33} & \tilde{D}_{wcl}(\alpha) \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0 \qquad (14)$$

$$\hat{\mathscr{F}}_{11} = P(\alpha_+) - F - F', \; \hat{\mathscr{F}}_{12} = F\tilde{A}_{cl}(\alpha)' - F'G(\alpha_+)'$$
$$\hat{\mathscr{F}}_{22} = G(\alpha_+)F\tilde{A}_{cl}(\alpha)' + \tilde{A}_{cl}(\alpha)F'G(\alpha_+)' - P(\alpha)$$
$$\hat{\mathscr{F}}_{23} = G(\alpha_+)F\tilde{C}_{cl}(\alpha)' + \tilde{A}_{cl}(\alpha)F'H(\alpha_+)'$$
$$\hat{\mathscr{F}}_{33} = H(\alpha_+)F\tilde{C}_{cl}(\alpha)' + \tilde{C}_{cl}(\alpha)F'H(\alpha_+)' - \gamma\mathbf{I}$$

Using Schur complement, inequality (14) can be rewritten as follows

$$\begin{bmatrix} \hat{\mathscr{F}}_{11} & \hat{\mathscr{F}}_{12} & F\tilde{C}_{cl}(\alpha)' - F'H(\alpha_+)' \\ (\star) & \hat{\mathscr{F}}_{22} & \hat{\mathscr{F}}_{23} \\ (\star) & (\star) & \hat{\mathscr{F}}_{33} \end{bmatrix}$$
$$+ \gamma^{-1}\hat{\mathscr{F}}_4(\alpha)\hat{\mathscr{F}}_4(\alpha)' < 0 \quad (15)$$

where

$$\hat{\mathscr{F}}_4(\alpha) = \begin{bmatrix} \mathbf{0} & \tilde{B}_{wcl}(\alpha)' & \tilde{D}_{wcl}(\alpha)' \end{bmatrix}'$$

Defining $\mathscr{X} = [F' \; F'G(\alpha_+)' \; F'H(\alpha_+)']'$ inequality (15) yields statement *iv)* of Lemma 1 with $\mathscr{Q}$, $\mathscr{B}$ and ξ given by Lemma 2. Lastly, the controller K is obtained by the change of variables $L = KF'$, what concludes the proof. ∎

Corollary 1: The minimum γ attainable by the conditions of Theorem 1 is given by the optimization problem

$$\min \gamma \quad \text{s.t. (12)} \qquad (16)$$

From this point, some remarks are in order.

Remark 1: Stability of NCSs with time-varying sampling rates and time-delays described by (4) was also addressed in [15] in the context of interval models. The solution is given in two independent steps. Firstly, a state feedback controller is designed considering the fixed nominal values of sampling rate and time-delay. Then, the robustness of the closed-loop system against the uncertainties (4) is verified by means of an interval system without applying any index of performance. In other words, the effect of the time-varying parameters (h_k and τ_k) is not considered during the design step. Theorem 1, however, provides a memory robust controller in a single step taking into consideration all possible outcomes of h_k and τ_k in (4). Further, an extra input $w(kh)$ was added in order to characterize possible $l_2[0,\infty)$ noise in the sampling process. An upper bound to the $\mathscr{H}_\infty$ performance of the closed-loop system with respect to $w(kh)$ is minimized.

Remark 2: By choosing $G(\alpha_+) = \mathbf{0}$ and $H(\alpha_+) = \mathbf{0}$ the conditions of Theorem 1 reduce to the ones proposed in [23] in the context of poly-quadratic stability [22]. In this paper,

differently from [23], statement *iv)* in Lemma 1 was applied to reach more general BMI conditions with multipliers defined as in Lemma 2 and $\mathscr{X} = [F' \; F'G(\alpha_+)' \; F'H(\alpha_+)']'$. The advantages of this approach are due to the extra variables that can be used in the search for better performance of the closed-loop system. For example, a lower $\mathscr{H}_\infty$ guaranteed cost may be obtained exploring the new variables $G(\alpha_+)$ and $H(\alpha_+)$. In this sense, Lemma 2 encompasses the conditions in [22].

Remark 3: Although other methods could be applied in the solution of the BMI problem (16), the following algorithm is proposed. Fix the variables H_i and G_i, minimize γ w.r.t. F, L and P_i. Then, fix the variables F, L and P_i, minimize γ w.r.t. H_i and G_i, get the new values of H_i and G_i. Repeat this procedure until no significant changes in the value of γ occur. This approach is sometimes called an Alternating Semi-Definite Programming (or Gauss-Seidel) method [4]. At each step a convex optimization problem in terms of LMI conditions is solved.

Remark 4: The use of memory controller brings some advantages when dealing with discrete time-delay systems. Using a new variable to store the past values of the control signal, it was possible to cope with Problem 1 without applying more complex Lyapunov functions, (for instance, the Lyapunov-Krasovskii functional). Sophisticated Lyapunov functionals may lead to conditions that require a bigger computational effort to be solved.

Remark 5: By setting the variables $G(\alpha_+)$ and $H(\alpha_+)$ at time $kh + h$ ($\alpha_+ = \alpha(kh + h)$) all products between parameter-dependent matrices appeared at the BMIs (12) occur at different instants of time. As a consequence, the number of BMIs and the computational time required to solve the optimization problem (16) are reduced. If Theorem 1 was written with $G(\cdot)$ and $H(\cdot)$ at time kh, a more sophisticated procedure, as the one proposed in [27], should be applied in order to get the BMI conditions expressed just in terms of the vertices of the polytope, resulting in a larger number of BMIs.

Remark 6: The conditions of Theorem 1 are directly applicable to NCS whose matrices depend affinely on the vector of time-varying parameters, since this class of systems has a polytopic representation whenever the parameters are bounded [28].

Remark 7: Finally, it is important to emphasize that the use of time-varying uncertainties in polytopic domains represents an interesting strategy to face Problem 1. Firstly, it does not require the knowledge of the processor/network clock granularity g, since the only information used to derive the polytopic model are the extreme values of sets (4). Secondly, the time-varying uncertainties, introduced during the sampling stage, can be completely modelled by a polytope of the form (10). Once one has defined the vertices of the closed-loop polytope, there will exist a vector $\alpha(kh_k)$ such that (10) holds for each time instant k. The only condition on vector $\alpha(kh_k)$ is that it belongs to the unit simplex $\mathscr{U}$. Furthermore, and the most interesting property, the number of values in the sets (4) does not influence the computational

burden; in other words, a larger number of h_k and τ_k does not require a bigger computational effort, what allows the clock granularity to be as small as possible.

IV. NUMERICAL EXPERIMENTS

Example I

This example, borrowed from [15], consists of an electric motor. The aim is to design an $\mathscr{H}_\infty$ robust memory control of the speed through a communication network. The continuous-time system is given by

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{\rho} \end{bmatrix} = \begin{bmatrix} -\dfrac{b}{J} & \dfrac{K}{J} \\ -\dfrac{K}{L} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \rho \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} v$$

$$\dot{\theta} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \rho \end{bmatrix} \qquad (17)$$

where ρ is the current, θ the rotor angle, J the moment of inertia of the rotor, b the damping ratio of the mechanical system, K the electromotive force constant, R the electric resistance and L the electric inductance. The control signal v is the voltage applied to the motor.

Considering zero delay and time-varying sampling rates in the sensor (as in [15]), system (17) can be rewritten in the form (5) with the following system matrices

$$\tilde{A}(\alpha) = \begin{bmatrix} e^{-10h_k} - 0.0003e^{-2h_k} & 0.125(e^{-2h_k} - e^{-10h_k}) \\ 0.002(e^{-10h_k} - e^{-2h_k}) & -0.0003e^{-10h_k} + e^{-2h_k} \end{bmatrix},$$

$$\tilde{B}_u(\alpha) = \begin{bmatrix} 0.025e^{-10h_k} - 0.125e^{-2h_k} + 0.099 \\ 0.0000626e^{-10h_k} - 0.99e^{-2h_k} + 0.99 \end{bmatrix}, \qquad (18)$$

$$\tilde{B}_w(\alpha) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tilde{C}(\alpha) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}', \tilde{D}_u(\alpha) = 0, \ \tilde{D}_w(\alpha) = 1.$$

The sampling rate is allowed to vary within the interval $h_k \in [0.001 \ 0.099]$. Closing the loop with (7), system (18) is expressed by the polytope (10) with two vertices ($N = 2$), where the parameters α_i are related to h_k. The first vertex is given by (18) with $h_k = h_{min}$ and the second one with $h_k = h_{max}$. For $J = 0.01 kgm^2/s^2$, $b = 0.1Nms$, $K = 0.01Nm/Amp$, $R = 1\Omega$ and $L = 0.5H$. Theorem 1 provides the result shown in Table I.

TABLE I

$\mathscr{H}_\infty$ ROBUST MEMORY CONTROLLER FOR EXAMPLE I.

Method	$\mathscr{H}_\infty$ Upper Bound γ	Gain Matrix K
Theorem 1	124.6270	[0.0005 −0.9921]

Although not mentioned in [15], system (18) is already stable (*i.e.* the stabilizing gain $K = [0 \ 0]$ could be applied). Nevertheless, Theorem 1 is still useful in providing a gain matrix that guarantees robustness against unmodeled $l_2[0, \infty)$ perturbations.

Example II

Consider a discrete-time system (5) with $\tau = 0$ and vertices

$$\tilde{A}_1 = \begin{bmatrix} 0.28 & -0.315 \\ 0.63 & -0.84 \end{bmatrix}, \ \tilde{A}_2 = \begin{bmatrix} 0.52 & 0.77 \\ -0.7 & -0.07 \end{bmatrix},$$

$$\tilde{B}_{w1} = \tilde{B}_{w2} = \begin{bmatrix} 1 & 0 \end{bmatrix}', \ \tilde{B}_{u1} = \begin{bmatrix} 1 & 2 \end{bmatrix}', \ \tilde{B}_{u2} = \begin{bmatrix} 9 & 21 \end{bmatrix}',$$

$$\tilde{C}_1 = \begin{bmatrix} 1 & 3 \end{bmatrix}, \ \tilde{C}_2 = \begin{bmatrix} 2 & 1 \end{bmatrix}, \ \tilde{D}_{w1} = \tilde{D}_{w2} = \tilde{D}_{u1} = \tilde{D}_{u2} = \begin{bmatrix} 0 \end{bmatrix}$$

This system was also studied in [23], but in a simpler case where matrix $\tilde{C}$ was fixed and time-invariant ($\tilde{C}_1 = \tilde{C}_2$), and the time-varying parameters of matrix $\tilde{B}_u$ had a lower variance. Here, the results from Theorem 1 were compared to [23, Theorem 5]. Using the proposed conditions, the $\mathscr{H}_\infty$ upper bound γ was reduced in approximately 49.13%, providing better rejection of disturbances, as it can be seen in Table II.

TABLE II

$\mathscr{H}_\infty$ ROBUST MEMORY CONTROLLER FOR EXAMPLE II.

Method	$\mathscr{H}_\infty$ Upper Bound γ	Gain Matrix K
Theorem 1	24.2184	[0.0303 0.0005]
Theorem 5 in [23]	47.6082	[0.0286 − 0.0013]

V. CONCLUSION

The $\mathscr{H}_\infty$ robust memory controller for networked control systems with time-varying sampling rate and time delays belonging to a polytope has been addressed in this paper. A sufficient condition, where all system matrices are considered to be affected by time-varying parameters, has been stated in terms of BMIs described only at the vertices of the polytope. The controller design is accomplished by means of an optimization problem, which can be faced by efficient numerical algorithms. A new state-space variable, representing the memory of the controller, was added in order to model time delays in the control signal. Extra variables provided by the Finsler's Lemma were used to derive the BMI conditions. The proposed approach also provides some improvements when compared with other methods from the literature in the context of discrete-time systems with time-varying uncertainties, what increases its reliability when applied in NCSs.

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