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On Transmitter Design in Power Constrained LQG Control

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Abstract— **We consider a linear dynamic system to be controlled using feedback information that has to be transmitted over a power constrained channel with additive noise. We propose a novel approach to the transmitter design in order to minimize the cost function for the linear quadratic Gaussian (LQG) control problem when the standard state estimator and linear controller are used. We show that the well known lower bound on transmit power is tight for our solution and derive a transmission scheme that achieves this lower bound.**

I. Introduction

Over the last decade, there has been considerable interest in the investigation of control problems that take into account constraints on the communication links which are used for information exchange. The recent survey papers [1], [2] and the extensive lists of references therein are examples which document this development. In this period, some fundamental insights have been gained like the minimal data rate [3]–[5] or minimal transmit power [6] necessary for the stabilization of linear systems. Publications related to information theory analyzed the information that can be transmitted using closed control loops [7] or refined information theoretic quantities describing the requirements of closed loop control [8]. Concerning the communication channel, there exist mainly two different viewpoints: restrictions of the transmission rate due to noiseless but discrete channels (with finite quantization levels) or real valued channels with additive noise. There are only few results on discrete channels with errors [9]–[11] which harder to handle. There have also been contributions to distributed control systems with communication constraints which inspired new algorithms based on the classical assumption that quantization behaves like independent additive noise [12], [13].

Some of the approaches in the field of control under communication constraints have been developed in the LQG context [12]–[17]. This paper also focuses on this framework, with the communication link to be an additive white Gaussian noise (AWGN) channel and the constraint of limited transmit power which, in combination with the channel noise, leads to a finite signal to noise ratio. There have already been previous attempts to consider this type of problem. In [18], the control signal is transmitted directly over the AWGN channel and the controller is designed to generate a control signal with constrained variance. The disadvantage hereby is that no receiver is specified which, among other details, results in the fact that available transmit power may not be fully exploited even if available. The authors of [19] consider the introduction of a scaling factor at the transmitter for scalar systems and include this factor in the LQG cost function. This results in a bounded transmit power, but does not allow for the specification of a hard power constraint since the power actually used depends on the weighting factor assigned to the transmit scaling.

We will follow a similar idea as in [19], which is based on the optimization of the LQG cost using a scaling. However, we consider a single-input single-output (SISO) system with vector valued state implying the design of a transmit filter vector instead of a scalar. The optimization is performed under a hard transmit power constraint which is formulated such that the validity of the solution of the standard LQG problem, i. e., the optimal state estimator (Kalman filter) and a linear controller, is guaranteed. Consequently, the transmitter is designed to minimize the resulting cost after the optimal estimator (which depends on the transmitter) and controller (independent on the transmitter) are applied.

The paper is organized as follows. In Section II, we introduce the system and the channel model as well as the cost function to be optimized and the solution to the standard LQG problem. Section III presents the definition of the transmit power constraint and motivates the choice for it. We propose a suboptimal solution to the resulting optimization problem and give an interpretation. It is shown that the well known lower bound for the transmit power is tight for the solution. The last part of the section describes how this bound can be achieved by transmit processing for noiseless systems.

Notation: Vectors and matrices are denoted by lower and upper case bold letters (e. g., $\boldsymbol{a}$ and $\boldsymbol{A}$), whereas scalars are lower case letters (e. g., a). The operators $\mathrm{E}[\bullet]$, $\mathrm{E}[\bullet|\,\boldsymbol{a}]$, $(\bullet)^{\mathrm{T}}$, and $\mathrm{tr}[\bullet]$ are expectation, expectation conditioned on the vector $\boldsymbol{a}$, transpose and trace of a matrix, respectively. $\mathbf{e}_i$ is the ith column of the $N \times N$ identity matrix $\mathbf{I}_N$. The all-zeros vector of dimension N is denoted by $\mathbf{0}_N$. $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{C}_{\boldsymbol{a}})$ denotes the Gaussian distribution of the real random vector $\boldsymbol{a}$ with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{C}_{\boldsymbol{a}} = \mathrm{E}\left[(\boldsymbol{a}-\boldsymbol{\mu})(\boldsymbol{a}-\boldsymbol{\mu})^{\mathrm{T}}\right]$.

II. Preliminaries and Problem Formulation

The system under consideration is depicted in Fig. 1 and its components will be presented in Section II-A and II-B.

A. System Model

We consider the following linear time invariant, discrete time system in state space representation:

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{b}u_k + \boldsymbol{w}_k, \quad k \in \{0,1,2,\ldots\}, \tag{1}$$

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where $\boldsymbol{x}_k \in \mathbb{R}^M$ is the system state at time index k and $\boldsymbol{A} \in \mathbb{R}^{M\times M}$ is the state transition matrix. The initial state $\boldsymbol{x}_0 \sim \mathcal{N}(\boldsymbol{0}_M, \boldsymbol{C}_{\boldsymbol{x}_0})$ and the stationary process noise $\boldsymbol{w}_k \sim \mathcal{N}(\boldsymbol{0}_M, \boldsymbol{C}_{\boldsymbol{w}})$, $k \in \{0,1,2,\ldots\}$, are assumed to be mutually independent Gaussian random vectors. The system has a scalar input $u_k \in \mathbb{R}$, $k \in \{0,1,2,\ldots\}$, and $\boldsymbol{b} \in \mathbb{R}^M$ is the system input vector.

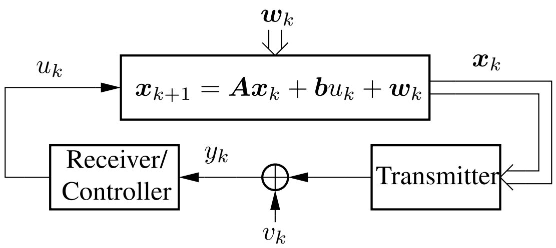

Figure 1. Closed loop system.

B. Channel Model

A typical assumption is that the system state is not directly observable. Instead, only the noisy observation

$$y_k = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_k + v_k \in \mathbb{R}, \quad k \in \{0,1,2,\ldots\}, \tag{2}$$

is available. Here, $\boldsymbol{c} \in \mathbb{R}^M$ is the system output vector and $v_k \sim \mathcal{N}(0, c_v)$, $k \in \{0,1,2,\ldots\}$, is the stationary observation noise which is assumed to be mutually independent and independent to the process noise and initial state.

Eq. (2) allows for a second interpretation. A standard channel model in communication theory is the AWGN channel, which can carry a real number and provides a noisy version of it at the channel output. Assume now that we have access to the system state, but are only able to transmit signals to the controller over such an AWGN channel. Thus, we have the degree of freedom to choose the vector $\boldsymbol{c}$ (under the assumption that $(\boldsymbol{A}, \boldsymbol{c})$ is observable) and treat the joint communication and control problem in the LQG framework which is referred to as transmitter design. The goal is to determine the control sequence u_k, $k \in \{0,1,2,\ldots\}$, and the vector $\boldsymbol{c}$ such that the cost function presented in Section II-C is minimized.

In Section III-D, we will consider a more general transmitter in order to achieve the minimum transmit power possible. In this case, the observation equation has the form

$$y_k = \boldsymbol{c}^{\mathrm{T}}(\boldsymbol{x}_k - \boldsymbol{\delta}_k) + v_k \in \mathbb{R}, \quad k \in \{0,1,2,\ldots\}, \tag{3}$$

where $\boldsymbol{\delta}_k$ has to be determined in addition to $\boldsymbol{c}$.

Note that together with Eq. (2) or (3), respectively, the system given in Eq. (1) describes a SISO system.

C. Cost Function

We consider the LQG control problem with infinite horizon. In this case, the cost function which describes the average cost per stage is given by [20]

$$J_\infty = \lim_{N\to\infty} \frac{1}{N} \mathrm{E}\left[\boldsymbol{x}_N^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}_N + \sum_{n=0}^{N-1} \boldsymbol{x}_n^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}_n + r u_n^2\right]. \tag{4}$$

where $\boldsymbol{Q} \in \mathbb{R}^{M\times M}$ is a positive (semi)definite weight matrix and $r > 0$ is the control weight.

D. Solution of the LQG Control Problem

A well know result is that the cost function (4) is minimized by the control values

$$u_k = \boldsymbol{l}^{\mathrm{T}}\hat{\boldsymbol{x}}_k^{(k|k)}, \tag{5}$$

where

$$\hat{\boldsymbol{x}}_k^{(k|k)} = \mathrm{E}\left[\boldsymbol{x}_k | y_0, \ldots, y_k, u_0, \ldots, u_{k-1}\right] \tag{6}$$

is the estimate of $\boldsymbol{x}_k$ given the information available at time k. The linear controller is

$$\boldsymbol{l}^{\mathrm{T}} = -\left(\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b} + r\right)^{-1}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{A}, \tag{7}$$

where $\boldsymbol{K}$ is the positive semidefinite solution of the discrete algebraic Riccati equation (DARE)

$$\boldsymbol{K} = \boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{K} - \boldsymbol{K}\boldsymbol{b}\left(\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b} + r\right)^{-1}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\right)\boldsymbol{A} + \boldsymbol{Q}. \tag{8}$$

The conditional mean estimate in Eq. (6) is computed using the Kalman filter. Applying the control given in Eq. (5) to the system (1), the optimal cost reads as

$$J_\infty^* = \mathrm{tr}\left[\boldsymbol{P}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}\right] + \mathrm{tr}\left[\boldsymbol{K}\boldsymbol{C}_{\boldsymbol{w}}\right], \tag{9}$$

where $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}$ is the stationary covariance matrix of the estimation error $\tilde{\boldsymbol{x}}_k = \boldsymbol{x}_k - \hat{\boldsymbol{x}}_k^{(k|k)}$ and

$$\boldsymbol{P} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b}\left(\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b} + r\right)^{-1}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{A}. \tag{10}$$

The stationary error covariance is given by [20]

$$\boldsymbol{C}_{\tilde{\boldsymbol{x}}} = \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} - \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c}\left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c} + c_v\right)^{-1}\boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}. \tag{11}$$

The stationary error covariance matrix of the Kalman filter in the "prediction" step $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$ is the solution of the DARE

$$\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} = \boldsymbol{A}\left(\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} - \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c}\left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c} + c_v\right)^{-1}\boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\right)\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{C}_{\boldsymbol{w}}. \tag{12}$$

III. Transmitter Design

Using the interpretation of the observation equation (2) as the transmission of $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_k$ over an AWGN channel, we now aim at the transmit filter vector $\boldsymbol{c}$ for the system state such that the LQG cost is minimized and a constraint on the transmit power is satisfied. The necessity of such a constraint will be explained in Section III-A.

The cost function in Eq. (4) can be minimized w.r.t. the control sequence $\boldsymbol{u} = [u_0, u_1, \ldots]$ first and then to $\boldsymbol{c}$ because

$$\min_{\boldsymbol{u},\boldsymbol{c}} J_\infty = \min_{\boldsymbol{c}}\left(\min_{\boldsymbol{u}} J_\infty\right), \quad \text{s.t.} \quad \{\boldsymbol{u}, \boldsymbol{c}\} \in \mathcal{G}, \tag{13}$$

with the restriction that the control signal u_k at time k must depend on $y_0, y_1, \ldots, y_k$, and $u_0, u_1, \ldots, u_{k-1}$ only. Here, $\mathcal{G}$ describes the set of values the vector $\boldsymbol{c}$ and the control sequence $\boldsymbol{u}$ are allowed to be chosen from. It will be used in the following to limit the transmit power. If $\mathcal{G}$ solely restricts the choice of $\boldsymbol{c}$, e.g.,

$$\mathcal{G} = \{\boldsymbol{u}, \boldsymbol{c} | g(\boldsymbol{c}) \le 0\}, \tag{14}$$

where $g(\boldsymbol{c}) \in \mathbb{R}$ is a function of $\boldsymbol{c}$ only, the solution of $\boldsymbol{u}$ will be identical to Eq. (5) and (7), i.e., a linear controller with an optimal state estimator. Thus, for this type of constraints, the inner minimization of (13) results in the optimal value given by Eq. (9). Since the second term of (9) containing the process noise covariance matrix is independent on $\boldsymbol{c}$, only the first term can be further minimized w. r. t. $\boldsymbol{c}$. Thus, the objective is to minimize the trace of the weighted error covariance matrix.

A. Transmit Power Constraint

If the cost function J_∞ is minimized w. r. t. $\boldsymbol{c}$ without a constraint, the result will have an infinite norm. This is easy to verify using Eq. (11). Increasing the norm of $\boldsymbol{c}$ is equivalent to decreasing the variance c_v of the observation noise. In the limit for infinite norm, we have a noiseless state estimation problem which clearly results in a smaller estimation error than with observation noise, but the transmit power used will be infinitely large. This shows that at least the norm of $\boldsymbol{c}$ must be restricted in order to ensure a finite transmit power.

The actual transmit power used is given by the stationary variance of $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_k$, which reads as $\mathrm{E}[(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_k)^2] = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}_{\boldsymbol{x}}\boldsymbol{c}$, with the stationary covariance matrix $\boldsymbol{C}_{\boldsymbol{x}}$ of the system state. The restriction of this variance in order to keep the transmit power finite has the major disadvantage that $\boldsymbol{C}_{\boldsymbol{x}}$ is a function of the control sequence $\boldsymbol{u}$. In this case, the solution presented in Eq. (5) and (7) is not optimal anymore since it is the result of an unconstrained optimization of J_∞ w. r. t. $\boldsymbol{u}$. Thus, instead of using $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}_{\boldsymbol{x}}\boldsymbol{c}$, we consider the following constraint:

$$\boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c} \leq P_{\mathrm{Tx}}, \tag{15}$$

where P_{Tx} is the available transmit power. This choice is motivated by the following reasons:

- The error covariance matrix $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$ is independent on any control signal [20]. Referring to Eq. (14), the constraint set $\mathcal{G}$ is given by

$$\mathcal{G} = \left\{\boldsymbol{u}, \boldsymbol{c} \,\middle|\, \boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c} - P_{\mathrm{Tx}} \leq 0 \right\}, \tag{16}$$

  which restricts the choice of $\boldsymbol{c}$ only. Thus, the control given in (5) remains optimal.
- In [15] it has been shown that the optimal linear transmitter at time index $k+1$ that additionally has perfect access to the observations y_ℓ, $\ell \in \{0, 1, \ldots, k\}$, performs an innovation coding which results in a covariance matrix of the signal to be transmitted that is identical to the error covariance matrix $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$. In Section III-D, we show that for the case of a noiseless dynamic system (i. e., $\boldsymbol{w}_k \equiv \boldsymbol{0}_M, \forall k$), this can be achieved without additional feedback from the receiver to the transmitter and without changing the solution obtained in Section III-B which does not take into account the innovation coding.

B. Optimization Problem

Following the preceding discussion, the optimization problem for the determination of the transmit filter vector $\boldsymbol{c}$ is

$$\min_{\boldsymbol{c}} \mathrm{tr}\left[\boldsymbol{P}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}\right] \quad \text{s. t.} \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c} \leq P_{\mathrm{Tx}}, \tag{17}$$

where the expressions for $\boldsymbol{P}$, $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}$ and $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$ are given by Eq. (10), (11) and (12), respectively. The problem is that the error covariance matrix $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}$ does not only depend on $\boldsymbol{c}$, but also on $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$ which is, as the solution of a DARE, an implicit function of $\boldsymbol{c}$.

In order to derive a suboptimum solution to the problem, we assume in a first step that $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$ is *not* a function of $\boldsymbol{c}$ and solve the optimization in (17) which is straight forward using this assumption. Then, $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$ is updated with this solution according to (12) and, if necessary, (17) is solved again using the updated error covariance matrix. This implies an iterative procedure. Nevertheless, we will see that it is not necessary to perform it this way.

Note that this approach can be interpreted in the context of the suboptimal “forward” solution to the minimization of the LQG cost function with a time variant transmit vector. Considering the LQG problem with finite horizon N, the cost function after the application of the optimal LQG controller is a sum of traces of the weighted process noise covariance matrix $\boldsymbol{C}_{\boldsymbol{w}}$ and the weighted covariance matrices $\boldsymbol{C}_{\tilde{\boldsymbol{x}}_k}, k \in \{0, 1, \ldots, N-1\}$, of the state estimation error at time index k (see, e. g., [21]). The optimum strategy would be to determine all transmit vectors $\boldsymbol{c}_k, k \in \{0, 1, \ldots, N-1\}$, jointly in order to minimize the contribution of the estimation errors to the final cost. This could be accomplished backwards in time using a dynamic programming approach. In order to simplify the solution, the suboptimal approach minimizes the contribution of each estimation error separately starting with $k = 0$. This determines the transmit vector $\boldsymbol{c}_0$ and the covariance matrix $\boldsymbol{C}_{\tilde{\boldsymbol{x}}_1}^{\mathrm{P}}$, which is necessary for the determination of $\boldsymbol{c}_1$ etc. Performing the transition to the average cost infinite horizon problem, this gives an idea about the suboptimality of the solution presented in the following with the assumption that $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$ does not depend on $\boldsymbol{c}$.

In order to keep things simple we rewrite the cost function in (17) using the eigenvalue decomposition of $\boldsymbol{P}$,

$$\boldsymbol{P} = \boldsymbol{W}\boldsymbol{\Lambda}\boldsymbol{W}^{\mathrm{T}} = \lambda\boldsymbol{w}\boldsymbol{w}^{\mathrm{T}}, \tag{18}$$

where $\boldsymbol{W}$ is an orthonormal matrix and $\boldsymbol{\Lambda}$ is a diagonal matrix. Since $\boldsymbol{P}$ is positive semidefinite and has rank one (cf. Eq. 10), only one eigenvalue λ is positive, all others are zero. Thus, problem (17) reads as

$$\min_{\boldsymbol{c}} \lambda\boldsymbol{w}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}\boldsymbol{w} \quad \text{s. t.} \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c} \leq P_{\mathrm{Tx}}. \tag{19}$$

The corresponding Lagrange function is

$$L(\boldsymbol{c}, \mu) = \lambda\boldsymbol{w}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}\boldsymbol{w} + \mu\left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c} - P_{\mathrm{Tx}}\right), \tag{20}$$

with the Lagrange multiplier $\mu \geq 0$. Note that the error covariance matrix $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}$ is given by Eq. (11). Taking the derivative of $L(\boldsymbol{c}, \mu)$ w. r. t. $\boldsymbol{c}$ and setting it to zero, we get the condition

$$\begin{aligned}\lambda&\left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c} + c_v\right)\left(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c}\right)\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{w} \\ &= \left(\lambda\left(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c}\right)^2 + \mu\left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c} + c_v\right)^2\right)\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c},\end{aligned} \tag{21}$$

where we have to keep in mind that $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$ is a positive definite matrix assumed to be independent on $\boldsymbol{c}$. Thus, we find that

$$\boldsymbol{c} = \alpha \boldsymbol{w}, \tag{22}$$

where α is a non-zero real scaling factor. Using this result, the Lagrange multiplier can be expressed as

$$\mu = \lambda \left(\alpha^2 \boldsymbol{w}^{\mathrm{T}} \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} \boldsymbol{w} + c_v\right)^{-2} \boldsymbol{w}^{\mathrm{T}} \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} \boldsymbol{w} c_v, \tag{23}$$

which is positive under the assumptions made. Thus, the transmit power constraint is active and we find α by inserting $\boldsymbol{c}$ in (15) using equality:

$$\alpha = \pm \sqrt{\frac{P_{\mathrm{Tx}}}{\boldsymbol{w}^{\mathrm{T}} \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} \boldsymbol{w}}}. \tag{24}$$

Finally, this results in

$$\boldsymbol{c} = \sqrt{\frac{P_{\mathrm{Tx}}}{\boldsymbol{w}^{\mathrm{T}} \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} \boldsymbol{w}}} \boldsymbol{w}, \tag{25}$$

where the positive solution has been chosen since the sign of α has no influence on the cost function. Note that only the scaling of $\boldsymbol{c}$ depends on the error covariance matrix $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$. Thus, the unscaled version of the transmit vector can be computed independently of the error covariance matrix and is given by the eigenvector of $\boldsymbol{P}$ corresponding to the non-zero eigenvalue. It remains to determine the error covariance matrix $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$ for the correct scaling. Inserting Eq. (25) in (12), we get

$$\begin{aligned}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} &= \boldsymbol{A}\left(\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} - \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} \boldsymbol{w}\left(\boldsymbol{w}^{\mathrm{T}} \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} \boldsymbol{w} + \frac{c_v}{\alpha^2}\right)^{-1} \boldsymbol{w}^{\mathrm{T}} \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\right)\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{C}_{\boldsymbol{w}} \\ &= \boldsymbol{A}\left(\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} - \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} \boldsymbol{w}\left(\boldsymbol{w}^{\mathrm{T}} \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} \boldsymbol{w} \frac{P_{\mathrm{Tx}} + c_v}{P_{\mathrm{Tx}}}\right)^{-1} \boldsymbol{w}^{\mathrm{T}} \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\right)\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{C}_{\boldsymbol{w}}.\end{aligned} \tag{26}$$

The solution in Eq. (25) has an interesting interpretation. Since $\boldsymbol{w}$ is the eigenvector belonging to the only non-zero eigenvector of $\boldsymbol{P}$ (cf. Eq. 10), it can be written as

$$\boldsymbol{w} = \boldsymbol{A}^{\mathrm{T}} \boldsymbol{K} \boldsymbol{b} \left\|\boldsymbol{A}^{\mathrm{T}} \boldsymbol{K} \boldsymbol{b}\right\|_2^{-1}, \tag{27}$$

with the eigenvalue $\lambda = \left(\boldsymbol{b}^{\mathrm{T}} \boldsymbol{K} \boldsymbol{b} + r\right)^{-1} \left\|\boldsymbol{A}^{\mathrm{T}} \boldsymbol{K} \boldsymbol{b}\right\|_2^2$. Comparing this with Eq. (7), we see that the transmit vector $\boldsymbol{c}$ is just a scaled version of the optimal control vector $\boldsymbol{l}$. Thus, the transmitter computes the optimal control and scales it in order to meet the power constraint. The receiver reconstructs the state vector from the received scalar signal and applies the unscaled control vector $\boldsymbol{l}$ (cf. Eq. 7).

C. Minimal Transmit Power

A well known result in the literature on control under communication constraints is the lower bound on the transmit power necessary for the stabilization of an unstable linear plant. This bound is given by [6]

$$P_{\mathrm{Tx,min}} = \left(\prod_i \left|\lambda_i^{(\mathrm{u})}\right|^2 - 1\right) c_v, \tag{28}$$

where $\lambda_i^{(\mathrm{u})}$ are eigenvalues of $\boldsymbol{A}$ that lie outside the unit disc. In the following, we will show that this bound is also tight for the proposed transmission scheme. For the proof we utilize the solution given in Eq. (25), which results in no loss of transmit power and allows to achieve the lower bound.

Proposition 1. *The lower bound on the transmit power for the transmitter shown in Eq.* (25) *is given by $P_{Tx,min}$ (cf. Eq. 28) and can be approached arbitrarily close.*

Proof. Considering Eq. (26), we find a DARE with the parameter

$$t = \frac{c_v}{\alpha^2} = \frac{c_v}{P_{\mathrm{Tx}}} \boldsymbol{w}^{\mathrm{T}} \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} \boldsymbol{w}, \tag{29}$$

which depends on the given transmit power and the error covariance matrix $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$. This matrix is a function of P_{Tx} and the unscaled transmit filter vector $\boldsymbol{w}$ (cf. Eq. 26). For $P_{\mathrm{Tx}} \to \infty$, the error covariance matrix will approach its asymptotic value which is identical to a scenario without observation noise. In this case, t will approach zero. On the other hand, for $P_{\mathrm{Tx}} \to P_{\mathrm{Tx,min}}$, the estimation error will grow and, in the limit, approach infinity which results in $t \to \infty$. Rewriting Eq. (29), the transmit power can be expressed as

$$P_{\mathrm{Tx}} = \boldsymbol{w}^{\mathrm{T}} \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}} \boldsymbol{w} t^{-1} c_v = \boldsymbol{w}^{\mathrm{T}} \boldsymbol{X} \boldsymbol{w} c_v, \tag{30}$$

where we used the abbreviation $\boldsymbol{X} = t^{-1} \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$. Using this notation and considering the case $t \to \infty$, Eq. (26) can be rewritten as

$$\boldsymbol{X} = \boldsymbol{A}\left(\boldsymbol{X} - \boldsymbol{X}\boldsymbol{w}\left(\boldsymbol{w}^{\mathrm{T}} \boldsymbol{X} \boldsymbol{w} + 1\right)^{-1} \boldsymbol{w}^{\mathrm{T}} \boldsymbol{X}\right)\boldsymbol{A}^{\mathrm{T}}. \tag{31}$$

This DARE corresponds to the deterministic expensive cost linear quadratic regulator (LQR) problem with state feedback which has the solution

$$\min_{\substack{u_k \\ k=0,1,2,\ldots}} \sum_{k=0}^{\infty} u_k^2 = \boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{X} \boldsymbol{x}_0, \tag{32}$$

subject to the state equation $\boldsymbol{x}_{k+1} = \boldsymbol{A}^{\mathrm{T}} \boldsymbol{x}_k + \boldsymbol{w} u_k$ and $u_k = \boldsymbol{l}^{\mathrm{T}} \boldsymbol{x}_k$. Here, $\boldsymbol{x}_0$ is the initial system state. In [6] and [22] it is shown that

$$\boldsymbol{w}^{\mathrm{T}} \boldsymbol{X} \boldsymbol{w} = \prod_i \left|\lambda_i^{(\mathrm{u})}\right|^2 - 1 \tag{33}$$

Note that this result holds for $t \to \infty$.[1] Inserting Eq. (33) in (30), we finally find the bound given in Eq. (28) which can be approached arbitrarily close by increasing the value of t to infinity with $P_{\mathrm{Tx}} \to P_{\mathrm{Tx,min}}$. □

Note that this result on the transmit power holds for every vector $\boldsymbol{c}$ which satisfies the transmit power constraint and under the observability assumption. It does not depend on the properties of the solution obtained in Section III-B, Eq. (25). Thus, any appropriately scaled transmit vector could be used. Nevertheless, the resulting cost J_∞^* (cf. Eq. 9) will be different due to the influence of $\boldsymbol{c}$ on the estimation error. The solution presented in Section III-B ensures that

[1]The result from [6] and [22] holds here since the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{A}^{\mathrm{T}}$ are identical.

the minimal transmit power is achievable while keeping the LQG cost small. Note that in [22] the result from Eq. (33) is used to derive the coarsest quantizer for a noise free SISO system. This indicates again the interconnection of minimal transmit power in noisy systems and minimal information rate in noise free, quantized systems.

D. Innovation Coding

In the previous subsections, we assumed that the transmit signal is a linear function of the state $\boldsymbol{x}_k$ with stationary covariance matrix $\boldsymbol{C}_{\boldsymbol{x}}$, whereas the transmit power constraint implied that the covariance matrix of the signal to be transmitted is $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$. Due to the stability of the closed loop system, the transmit power is bounded, but the mismatch in the transmit covariance matrices results in an increase of this power if $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_k$ is transmitted (cf. Eq. 2). The goal is to introduce a coding scheme at the transmitter which generates the desired covariance matrix, but ensures that the processing at the receiver that uses the Kalman filter remains optimal. To this end, we recall the Kalman filter equations for the computation of the state estimate,

$$\hat{\boldsymbol{x}}_k^{(k|k)} = \hat{\boldsymbol{x}}_k^{(k|k-1)} + \boldsymbol{g}\left(y_k - \boldsymbol{c}^{\mathrm{T}}\hat{\boldsymbol{x}}_k^{(k|k-1)}\right), \qquad (34)$$

with $\hat{\boldsymbol{x}}_k^{(k|k)} = \mathrm{E}\left[\boldsymbol{x}_k | \, y_0, \ldots, y_k, u_0, \ldots, u_{k-1}\right]$,

$$\hat{\boldsymbol{x}}_k^{(k|k-1)} = \boldsymbol{A}\hat{\boldsymbol{x}}_{k-1}^{(k-1|k-1)} + \boldsymbol{b}u_{k-1}, \qquad (35)$$

and the stationary Kalman gain vector (using the transmit power constraint)

$$\boldsymbol{g} = \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c}\left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c} + c_v\right)^{-1} = \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c}\left(P_{\mathrm{Tx}} + c_v\right)^{-1}. \qquad (36)$$

In Eq. (34) we see that the first step in the Kalman filter algorithm is the computation of the innovation

$$\begin{aligned} z_k &= y_k - \boldsymbol{c}^{\mathrm{T}}\hat{\boldsymbol{x}}_k^{(k|k-1)} \\ &= \boldsymbol{c}^{\mathrm{T}}\left(\boldsymbol{x} - \hat{\boldsymbol{x}}_k^{(k|k-1)}\right) + v_k. \end{aligned} \qquad (37)$$

If the transmitter knows the state estimate $\hat{\boldsymbol{x}}_k^{(k|k-1)}$, it can compute the estimation error $\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k^{(k|k-1)}$ in advance and transmit it using the same filter vector $\boldsymbol{c}$ that has been computed in Section III-B. Referring to Eq. (3), this corresponds to $\boldsymbol{\delta}_k = \hat{\boldsymbol{x}}_k^{(k|k-1)}$. The only modification necessary at the receiver is to omit the computation of the innovation (cf. Eq. 37) but to use directly the channel output which is now identical to z_k.

It remains to discuss how the transmitter gets knowledge about $\hat{\boldsymbol{x}}_k^{(k|k-1)}$. In [15], the existence of a perfect link between receiver and transmitter is assumed that feeds back the channel output. Thus, the transmitter can also run the Kalman filter and compute the state estimate. The drawback of this assumption is that it can hardly be realized. But for a noiseless dynamic system (i.e., $\boldsymbol{w}_k \equiv \boldsymbol{0}_M, \forall k$), such a link is not needed since the state which can be observed at the transmitter contains all necessary information. First, at time index $k+1$, the control u_k can be computed using the observed state sequence by (cf. Eq. 1)

$$u_k = \boldsymbol{b}^{+}\left(\boldsymbol{x}_{k+1} - \boldsymbol{A}\boldsymbol{x}_k\right), \qquad (38)$$

with $\boldsymbol{b}^{+} = \boldsymbol{b}^{\mathrm{T}}/\|\boldsymbol{b}\|_2^2$. Recall that $u_k = \boldsymbol{l}^{\mathrm{T}}\hat{\boldsymbol{x}}_k^{(k|k)}$ (cf. Eq. 5). Since u_k is known, we can solve with Eq. (34) for y_k. With the knowledge of the channel output, it is now possible to determine the state estimate as

$$\hat{\boldsymbol{x}}_k^{(k|k)} = \hat{\boldsymbol{x}}_k^{(k|k-1)} + \boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c}\left(\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}\boldsymbol{c}\right)^{-1}\left(u_k - \boldsymbol{l}^{\mathrm{T}}\hat{\boldsymbol{x}}_k^{(k|k-1)}\right). \qquad (39)$$

With Eq. (35), the estimation error $\boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1}^{(k+1|k)}$ can be computed and transmitted using $\boldsymbol{c}$. Note that this estimation error has the desired covariance matrix $\boldsymbol{C}_{\tilde{\boldsymbol{x}}}^{\mathrm{P}}$.

Putting all parts together, the resulting control loop that transmits feedback information over a power constrained AWGN channel can be depicted as in Fig. 2.

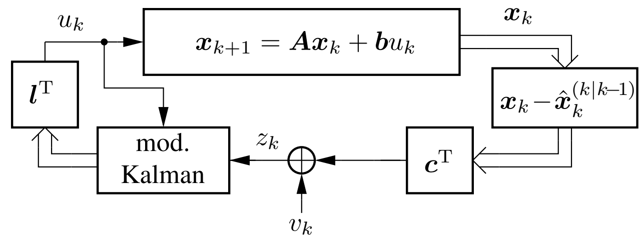

Figure 2. Proposed scheme with innovation coding and modified Kalman filter for noiseless system ($\boldsymbol{w}_k \equiv \boldsymbol{0}_M$).

The drawback of this approach is that the state estimate can be determined at the transmitter only theoretically. Since the estimate is computed recursively using the control variable u_k instead of the observation y_k, any error like round-off etc. will cause problems due to error propagation. Thus, the state estimator at the transmitter and the receiver should be synchronized from time to time.

IV. Numerical Example

In order to evaluate the suboptimality of the solution of the transmitter design found in Section III-B, we applied it to the stabilization problem of an inverted pendulum [13], [23]. The physical parameters of the system are given in Table I.

Mass of cart	0.5 kg
Mass of pendulum	0.2 kg
Friction of cart	$0.1 \, \frac{\mathrm{N}}{\mathrm{m/s}}$
Length to pendulum center of mass	0.3 m
Inertia of pendulum	$6 \cdot 10^{-3} \mathrm{kg\,m}^2$

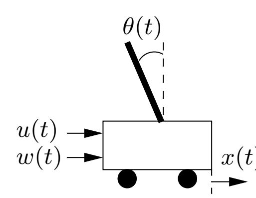

Table I

MODEL PARAMETERS OF THE PENDULUM STABILIZATION PROBLEM.

The state of the continuous dynamics is $[x(t), \dot{x}(t), \theta(t), \dot{\theta}(t)]^{\mathrm{T}}$. The system has been discretized with a sampling period of $T_{\mathrm{s}} = 5\mathrm{ms}$ using zero order hold. The weight matrix for the state of the discrete time LQG problem is chosen to be $\boldsymbol{Q} = \mathbf{e}_1\mathbf{e}_1^{\mathrm{T}} + 10^6\mathbf{e}_3\mathbf{e}_3^{\mathrm{T}}$ in order to keep the angle $\theta(t)$ small, and $r = 1$. The covariance matrix of the process noise in the continuous time domain is determined by the disturbance force $w(t)$ and is given by $\boldsymbol{C}_{\boldsymbol{w},\mathrm{cont.}} = 0.1\mathbf{e}_2\mathbf{e}_2^{\mathrm{T}}$.[2] The discrete observation noise

[2]Due the the presence of process noise, we have to assume that the control values u_k are available to the transmitter (cf. Section III-D).

variance is $c_v = 0.1$. Using the parameters given in Table I, this results in an minimal transmit power $P_{\text{Tx,min}} \approx 0.0057$.

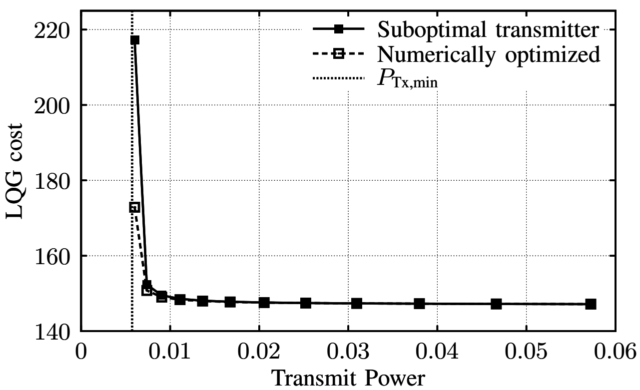

Figure 3. LQG cost for the inverted pendulum.

The solid line in Fig. 3 shows the resulting cost if the solution given in Eq. (25) is applied. Using this as an initial point, we applied a general purpose numerical optimizer to the transmitter optimization problem. The dashed line shows the resulting cost. As expected, we see that the numerically optimized transmitter performs better, but the gap is small for a transmit power larger than approx. $2P_{\text{Tx,min}}$.

V. Conclusion

In this paper, we considered the problem of joint transmitter and controller design for a linear SISO system where the control loop is closed over an AWGN channel with transmit power constraint. Due to the quadratic cost function and the restriction to linear transmitters, the problem could be investigated in the LQG framework. In order to find a solution based on known results, we modified the power constraint not to include the covariance matrix of the system state, but the covariance matrix of the estimation error. Neglecting the impact of the error covariance matrix on the transmit filter vector, we found the solution vector that minimizes the LQG cost which turns out to be a scaled version of the optimal control vector. We showed that the known bound for the minimal transmit power is tight for the proposed scheme and the minimal transmit power can be achieved by innovation coding at the transmitter for noiseless systems. Future work includes the extension of the presented scheme to MIMO systems and to systems with process noise. A further point is the evaluation of the proposed approach taking into account the dependence of the estimation error covariance matrix on the transmit filter.

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