2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Computational Eigenstructure Assignment in Linear Multivariable Systems

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Abstract— **Computational eigenstructure assignment is presented for linear multivariable systems. A complete computational solution - successive mapping and correction - is developed to solve the matrix equations, that arise in eigenstructure assignment. It is shown that the computation algorithm finds a solution for any admissible closed-loop Jordan form. The algorithm can also be used for Jordan pair assignment as well as the reduced- and full-order design.**

Index Terms— **Eigenstructure assignment, Control systems, State space methods, Linear systems.**

I. INTRODUCTION

Eigenstructure assignment is one of the most important problems in multivariable control theory, assigning the closed-loop system a prescribed set of eigenvalues (and eigenvectors). The matrix equations $AX - BKX = XJ$ and $YA - YLC = JY$ with (A, B, C) being known and controllable and observable, and J being in the Jordan form with arbitrary given eigenvalues, has close relations with the eigenstructure assignment problem. The matrix equation $AX - BKX = XJ$ is the state feedback design problem for eigenvalue assignment, while the dual version $YA - YLC = JY$ is the observer design problem. Therefore, the matrix equations are fundamental to all feedback design problems in linear state-space control system theory, such as the eigenvalue assignment problem, the state observer design problem, and the eigenstructure assignment problem.

Direct computation of the state feedback matrix which assigns a prescribed admissible closed-loop eigenstructure was considered in [1]–[5]. In [6], the class of assignable eigenvectors and generalized eigenvectors associated with the assigned eigenvalues was explicitly described by a set of free parameter vectors. The approach of [6], [7] assumed that no open-loop eigenvalue appeared in the set of closed-loop eigenvalues. This assumption was removed by [8]. For all classes of state feedback controllers [9] identified the minimum number of degrees of freedom in the parametric form of the feedback gain matrix that assigns a desired set of closed-loop eigenvalues. Some papers tried to develop computational algorithms to solve $AX - BKX = XJ$, for various forms of J [10], [11]. Unfortunately, the solution X of [10], [11] is not explicit and complete especially in the freedom of J. Because of this reason, the solution of [10], [11] cannot be applied to the function observer design and the state feedback eigenstructure design, since solving these problems requires information about the freedom of X. The only complete analytical solution of $AX - BKX = XJ$ is the solution based on J being in companion form. This solution is not applicable to some basic and important design problems [12]. In [12], an analytical solution of of $AX - BKX = XJ$ was presented carrying out an orthonormal similarity transformation and an inverse matrix, and solving series of linear equation groups. In [13], solutions linearly expressed by a group of parameter vectors are proposed. To obtain solutions, one needs to carry out a series of matrix elementary transformations. In [14], a simple algorithm for eigenstructure assignment by state feedback was presented applying the insights provided by the parametric approach to the problem considered by [1], that provides naturally for the case of common open- and closed-loop right characteristic vectors. Assignment of a common open- and closed-loop characteristic vector requires a corresponding parameter vector to be a null vector. In [15], a parametric solution was presented in a recursive form for descriptor systems. In [16], a parametric solution was presented to the Sylvester equation, adopting coprime matrices satisfying a certain factorization condition. In [17], an algorithm is presented to compute solutions to a Sylvester equation associated with linear descriptor systems, either by eigenstructure assignment or by linear matrix inequalities. In [18], a solution of the constrained Sylvester equation under a certain rank condition associated with linear descriptor systems was presented. In [19], a large Sylvester equation $AX + XB = C$ (the matrix A is large and B is of moderate size) was considered.

This paper is to present a computational solution to eigenstructure assignment in linear multivariable systems. A novel computational algorithm - successive mapping and correction (SMC) - is to be developed to solve the matrix equations that arise in eigenstructure assignment: Jordan pair assignment as well as the reduced- and full-order design. We show that the computation algorithm finds a solution for any admissible closed-loop Jordan form. The algorithm is expected to facilitate a computer based approach, resolving the constraints concerning linear independent eigenvectors and real control gains in matrix transformations.

Throughout the paper, the notation M^+ denotes the pseudo inverse matrix of M such that $MM^+M = M$. For the matrices M and N with the same number of rows, the notation $M\Big|_{\mathcal{I}_c(N)} \Leftarrow N$ denotes replacing some columns of M indicated by $\mathcal{I}_c(N)$ with each column of N in order. For square block diagonal matrices M and N, $M \ominus N$ denotes a reduced-order matrix with common block diagonal elements

This work was supported by Samsung Electronics Co., LTD.

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978-1-4244-2079-7/08/$25.00 ©2008 AACC.

being removed. Also, although not mentioned individually, all the matrix dimensions are assumed to be appropriate for compatible matrix formulations.

II. Problem Formulation

Consider a linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^{n_u}$ is the control, and $y \in \mathbb{R}^{n_y}$ is the output measurement, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$, and $C \in \mathbb{R}^{n_y\times n}$. Without loss of generality, it is assumed that the input matrix B is of full rank n_u and the observation matrix C is of full rank n_y.

The open-loop Jordan form J_{ol} satisfies

$$X_{ol}^{-1} A X_{ol} = J_{ol} \tag{2}$$

where X_{ol} is the right open-loop eigenvector matrix. The Jordan form J_{ol} has Jordan blocks associated with algebraic and geometric multiplicities of each open-loop eigenvalue.

Let us consider a state feedback control

$$u = -Kx$$

resulting in the closed-loop

$$\dot{x} = A_{cl} x = (A - BK)\, x.$$

Then the closed-loop Jordan form J_{cl} satisfies

$$AX_{cl} - BW_{cl} = X_{cl} J_{cl} \tag{3}$$

where $W_{cl} = KX_{cl}$, the parameter matrix.

Definition 1: A closed-loop Jordan form J_{cl} is admissible if there exist K and a nonsingular closed-loop eigenvector matrix X_{cl} such that

$$J_{cl} \in \mathcal{J} = \left\{ J_{cl} \mid X_{cl}^{-1} (A - BK)\, X_{cl} = J_{cl} \right\}.$$

Likewise, a Jordan pair (X_{cl}, J_{cl}) is admissible if there exists K such that

$$(X_{cl}, J_{cl}) \in \mathcal{P} = \left\{ (X_{cl}, J_{cl}) \mid X_{cl}^{-1} (A - BK)\, X_{cl} = J_{cl} \right\}.$$

The problem we are concerned with is, given an admissible closed-loop Jordan form $J_{cl} \in \mathcal{J}$, to find a computational solution (X_{cl}, W_{cl}) such that the matrix equation (3) is satisfied. If X_{cl} is invertible, one can compute K from $K = W_{cl} X_{cl}^{-1}$. Accordingly, the aim is to develop an efficient algorithm to solve the matrix equation (3) for eigenstructure assignment. We also need to show that the computation algorithm finds a solution for any admissible closed-loop Jordan form.

The state observer design yields the dual version matrix equation

$$Y_{cl} A - Y_{cl} LC = J_{cl} Y_{cl}$$

where Y_{cl} is the closed-loop eigenvector matrix. Utilizing the duality between the state feedback control and the state observer designs, we can simply focus on either case. Similar formulations and results can be straightforwardly obtained for its dual problem.

III. Computational Solution

Without loss of generality we focus on the state feedback and reduced-order design case the open-loop system already has some of the desired closed-loop eigenvalues. However, the results must be directly applicable to the full-order design case with common open- and closed-loop eigenvalues.

To begin with, let Λ_{ol} and Λ_{cl} be the open- and closed-loop eigenvalue matrix respectively. It is possible that Λ_{ol} already has some of the desired closed-loop eigenvalues Λ_{cl} such that the reduced-order eigenvalue matrix

$$\Lambda_r = \Lambda_{ol} \ominus \Lambda_{cl} \tag{4}$$

with corresponding reduced-order Jordan form

$$J_r = J_{ol} \ominus J_{cl}. \tag{5}$$

The index vector $\mathcal{I}_c(J_r) \in \mathbb{R}^r$ is chosen such that the reduced-order Jordan form J_r includes Jordan blocks in the closed-loop Jordan form J_{cl}, i.e.

$$J_{cl} = J_{ol}\Big|_{\mathcal{I}_c(J_r)} \Leftarrow J_r.$$

Then the reduced-order Jordan form J_r must satisfy

$$AX_r - BW_r = AX_r - BKX_r = X_r J_r \tag{6}$$

with corresponding reduced-order eigenvector matrix X_r whose columns are the right characteristic vectors of the closed-loop system corresponding to Λ_r arranged in order of chaining, subject to complex conjugate pairing. Furthermore, the Jordan form matrix J_r is nonsingular in stable closed-loop design.

A. Successive Mapping and Correction

Let us consider a successive mapping

$$X_r := \begin{bmatrix} A & -B \end{bmatrix} \begin{bmatrix} X_r \\ W_r \end{bmatrix} J_r^{-1}. \tag{7}$$

The matrix update X_r obtained via the mapping does not satisfy the matrix equation (6). For the matrix equation (7) to be satisfied all the time, we introduce an optimal parameter matrix W_r computed as

$$W_r = B^+ \left(AX_r - X_r J_r \right). \tag{8}$$

Then, the matrix equation (6) is satisfied with an error defined as

$$E_r = AX_r - BW_r - X_r J_r. \tag{9}$$

The key role of the corrections via W_r is to guarantee the matrix equation (6) being satisfied in a least square sense (minimizing the equation error) in recursion such that the solution (X_r^*, W_r^*) the sequential mapping and correction converges must be in the ball with a radius δ centered at the solution. The ball then can include a unique solution to the matrix equation (6), that corresponds to an initial value $\left({}^0X_r, {}^0W_r\right)$. This characterizes the SMC algorithm solution set to the matrix equation (6), the key step for solving the eigenstructure assignment problem. In this way, successive mapping and correction is to be performed starting from an initial value $\left({}^0X_r, {}^0W_r\right)$ to obtain a unique solution set that corresponds to an initial value $\left({}^0X_r, {}^0W_r\right)$. Of course, different initial points $\left({}^0X_r, {}^0W_r\right)$ may yield different solutions (X_r^*, W_r^*).

Furthermore, introducing

$$W_r := \mathrm{Re}\left[W_r X_r^+\right] X_r \tag{10}$$

leads K to be real valued, where Re $[\cdot]$ denotes the complex real part. After complete convergence, we have $E_r = 0$ and

$$W_r X_r^+ = B^+(AX_r - X_r J_r)X_r^+ = B^+BKX_rX_r^+ = KX_rX_r^+$$

that must be real such that

$$W_r X_r^+ X_r = W_r = \mathrm{Re}[W_r X_r^+] X_r.$$

The SMC algorithm thereby generously allows complex eigenvector and parameter matrices X_r and W_r to produce a real gain.

B. Convergent SMC scheme

An essential question that arises at this point is whether the SMC scheme described by (7) and (8) is guaranteed to be convergent or not. However, the scheme does not appear to be convergent at all. This subsection is thereby dedicated to developing a convergent SMC scheme.

Introducing an under relaxation factor α, $0 < \alpha \leq 1$, yields

$$X_r := X_r + \alpha\left(\begin{bmatrix}A & -B\end{bmatrix}\begin{bmatrix}X_r\\ W_r\end{bmatrix} J_r^{-1} - X_r\right). \tag{11}$$

Then, let us analyze how the factor α can contribute to derive a convergent SMC scheme that converges to find a unique solution (X_r^*, W_r^*) corresponding to an initial condition $\left({}^0X_r, {}^0W_r\right)$.

Here, we let

$$\phi = \begin{bmatrix}X_r\\ W_r\end{bmatrix} = (X_r, W_r) \in \Phi,$$

a set together with a metric $\rho : \Phi \times \Phi \to \mathbb{R}$ given by $\rho(\phi, \psi) = \|\phi - \psi\|$ for every $\phi, \psi \in \Phi$. For $\phi \in \Phi$ and $\delta \in \mathbb{R}$, the set $\mathcal{B}_\delta(\phi^*) = \{\phi \in \Phi \mid \rho(\phi, \phi^*) \leq \delta\}$ thereby denotes a closed ball of radius δ centered at the solution ϕ^* corresponding to an initial point $\phi_0 = \left({}^0X_r, {}^0W_r\right) \in \mathcal{B}_\delta(\phi^*)$. Then for the metric space (Φ, ρ), the SMC scheme can be represented by a mapping $T : \Phi \to \Phi$,

$$T(\phi) = \phi + \begin{bmatrix}\alpha E_r(\phi) J_r^{-1}\\ B^+\left((1-\alpha)E_r(\phi) + \alpha A E_r(\phi) J_r^{-1}\right)\end{bmatrix} \tag{12}$$

where the error in recursion

$$E_r(\phi) = \begin{bmatrix}A & -B\end{bmatrix}\phi - \begin{bmatrix}I & 0\end{bmatrix}\phi J_r. \tag{13}$$

Definition 2: The mapping $T : \Phi \to \Phi$ is said to be a *contraction* if $0 < \beta < 1$ such that for all ϕ, $\psi \in \Phi$, we have

$$\rho(T(\phi), T(\psi)) \leq \beta\rho(\phi, \psi).$$

Definition 3: A point ϕ^* is said to be a fixed point of T if it solves the (fixed-point) equation

$$T(\phi^*) = \phi^*.$$

The SMC scheme is to deliver a unique solution for a given initial condition ϕ_0. Observing that the matrix equation (6) is homogeneous and its solution ϕ^* is not unique, one can find that a solution only becomes unique when either V_r or W_r is fixed. The parameter matrix W_r can be used obtain a unique specialized solution V_r^* corresponding to W_r^*. Likewise, choosing different initial points ϕ_0 may result in different solutions ϕ^* such that $\phi_0 \in \mathcal{B}_\delta(\phi^*)$.

Proposition 1: For any nonsingular J_r, there exist some positive real $\alpha \leq 1/\left\|A - B^+BA\right\|$ such that

$$\alpha^k \left\|J_r^{-k}\right\| < \frac{1}{\left\|\begin{bmatrix}\begin{bmatrix}A & -B\end{bmatrix}\\ B^+A\begin{bmatrix}A & -B\end{bmatrix}\end{bmatrix}^k\right\|} \tag{14}$$

for $k \to \infty$. For such α, the metric $\rho\left(T^k(\phi), 0\right)$ decreases/disappears according to k.

Proof: To appear in the full version paper. ■

Theorem 1: Given a system doublet (A, B) and an arbitrary element $\phi_0 \in \mathcal{B}_\delta(\phi^*) \subset \Phi$, the SMC scheme expressed by the mapping $T : \Phi \to \Phi$ in (12) finds a unique solution ϕ^* such that $\phi_0 \in \mathcal{B}_\delta(\phi^*)$ for some positive real $\alpha \leq 1$, as the limit of every sequence generated by the iteration: $\phi_{k+1} = T(\phi_k)$.

Proof: To appear in the full version paper. ■

Corollary 1: For two arbitrary closed balls $\mathcal{B}_\delta(\phi^*)$ and $\mathcal{B}_\epsilon(\phi^{**})$ for $\delta, \epsilon \in \mathbb{R}$ with unique solutions ϕ^* and ϕ^{**} for each, the solution $\phi^{**} = \phi^* \in \mathcal{B}_\epsilon(\phi^{**})$, if $\mathcal{B}_\epsilon(\phi^{**}) \subseteq \mathcal{B}_\delta(\phi^*)$.

Proof: Clear by uniqueness of the solution in the set $\mathcal{B}_\delta\left(\phi^*\right)$. ■

Remark 1: Different initial values $\phi_0' \in \mathcal{B}_\delta(\phi^*)$ and $\phi_0'' \in \mathcal{B}_\delta(\phi^{**})$ result in different solutions $\phi^{**} \neq \phi^*$, if $\mathcal{B}_\delta(\phi^{**}) \nsubseteq \mathcal{B}_\delta(\phi^*)$.

Remark 2: The SMC scheme finds a solution with a smaller positive real α at the expense of slower convergence.

Corollary 2: The mapping $T : \Phi \to \Phi$ is contraction on all of the space Φ, not just on a ball around ϕ^*.

Proof: Clear from the fact that the SMC scheme finds a fixed point for any arbitrary initial point $\phi_0 \in \mathcal{B}_\delta(\phi^*)$. ■

Given an open-loop system doublet (A, B) and an admissible desired closed-loop Jordan form ${}^*J_{cl}$, the SMC scheme to find a solution ϕ^* can be summarized as follows:

Step 1. Choose ${}^*J_r$ and find $\mathcal{I}_c\left(J_r\right)$;
Step 2. Choose $0 < \alpha \leq 1$ to satisfy (14);
Step 3. Choose an initial value ϕ_0;
Step 4. Compute X_r from (11) for successive mapping;
Step 5. Compute W_r from (10) for correction;
Repeat Steps 4 - 5 until a fixed solution is obtained;

If the SMC scheme does not converge with the eigenvector matrix X_r becoming singular, then the desired closed-loop Jordan form ${}^*J_{cl}$ is inadmissible. This is clear from the result of Theorem 1.

Finally, the SMC scheme solves for both X_r and the product KX_r iteratively, which could be a great improvement over [20] where X_r is calculated iteratively, then K is calculated from that. Thus the iterative SMC scheme is computationally cheaper.

C. Control Gain for Eigenstructure Assignment

The SMC algorithm is shown to find a unique solution ϕ^*, corresponding to an initial condition $\phi_0 \in \mathcal{B}_\delta(\phi^*)$, to the matrix equation (6). As mentioned, the corrections via W_r in recursion functionally contribute to identify the closed set $\mathcal{B}_\delta$.

It is not surprising that the SMC scheme always delivers a solution set, the eigenvector matrix X_r and the parameter matrix W_r, provided the closed-loop Jordan form is admissible (Theorem 1). When the desired closed-loop Jordan form J_{cl} violates the admissibility condition (Definition 1), X_r becomes singular with its condition number being infinite. In this case, the SMC scheme hardly converges. Thus, if the SMC scheme does not converge it means that the eigenvector matrix X_r becomes singular and the desired closed-loop Jordan form ${}^*J_{cl}$ is inadmissible.

Once we obtain a solution $(X_r, W_r) \in \mathcal{B}_\delta(X_r^*, W_r^*)$, we can compute

$$X_{cl} = X_{ol}\Big|_{\mathcal{I}_c(J_r)} \Leftarrow X_r \tag{15}$$

and

$$W_{cl} = 0_{n_u \times n}\Big|_{\mathcal{I}_c(J_r)} \Leftarrow W_r. \tag{16}$$

For nonsingular X_{cl} the control gain K is computed from

$$K = W_{cl} X_{cl}^{-1}. \tag{17}$$

For the resulting closed-loop system matrix $A_{cl} = A - BK$ the error in design can be defined as

$$E_d = J_{cl} - X_{cl}^{-1} A_{cl} X_{cl} \tag{18}$$

with the trace function

$$\text{Tr}^{1/2}\left(E_d E_d^T\right) \tag{19}$$

as a design evaluation criterion.

The metric condition $\rho(\phi, \phi^*) = \|\phi - \phi^*\| = 0$ mostly appears to be an impractically excessive convergence criterion, since it may require very long iteration. A solution set, to working precision $\epsilon \ll 1$, $\phi \in \mathcal{B}_\epsilon(\phi^*) \subset \mathcal{B}_\delta(\phi^*) \subset \Phi$ must be good enough in engineering practice. Alternatively, considering $E_r(\phi - \phi^*) = E_r(\phi)$, the trace function $\text{Tr}^{1/2}\left(E_r(\phi)E_r^T(\phi)\right)$ is a good convergence criterion, that appears to be a sort of weighted convergence for spectrums.

The functional contribution of the corrections via W_r is worth while to observe. It contributes to build a closed set $\mathcal{B}_\delta$ in which a unique solution exists, corresponding to an initial condition $\phi_0 \in \mathcal{B}_\delta$. This must be the key step for solving the matrix equation (6) for eigenstructure assignment. As one can expect, there can be many solutions satisfying the matrix equation (6). For a given initial condition, however, the SMC algorithm is shown to find a corresponding unique solution. Introducing a different initial condition, say, in $\mathcal{B}_\delta(\phi^{**})$ may deliver a different solution ϕ^{**}. We must remark that the solution is nonunique in general and, by further inspection, that the nonuniqueness is attributable to the freedom in assigning the associated eigenvectors.

One can thus use the SMC algorithm to find a parameter matrix concerning a desired Jordan pair $({}^*X_{cl}, {}^*J_{cl}) \in \mathcal{P}$ assignment. Applying an (optimal) initial parameter matrix ${}^0W_r$ from (8) and (10) for a desired Jordan pair $({}^*X_r, {}^*J_r)$ provides an optimal initial point $\phi_0 = ({}^*X_r, {}^0W_r) \in \mathcal{B}_\delta({}^*\phi)$ for the SMC algorithm to find a parameter matrix W_r^* to deliver a solution $\phi^* = (X_r^*, W_r^*) \in \mathcal{B}_\delta({}^*\phi)$ such that $X_r^* = {}^*X_r$ (Corollary 1). Moreover if the desired Jordan pair $({}^*X_{cl}, {}^*J_{cl}) \notin \mathcal{P}$ while ${}^*J_{cl} \in \mathcal{J}$, the SMC algorithm can anyway find a true eigenvector matrix X_{cl}^* such that $(X_{cl}^*, {}^*J_{cl}) \in \mathcal{P}$ (Theorem 1).

A dual version SMC algorithm can be simply derived in a similar way, which can be applied to the dual version matrix equation $Y_r A - Y_r LC = J_r Y_r$ with (A, C) being known and observable to find a real gain for the observer design problem.

IV. Application Examples

The application examples are to demonstrate utility of the SMC algorithm for eigenstructure assignment applied to full- and reduced-order Jordan form assignment as well as Jordan pair assignment. The SMC algorithm is applied with a relaxation factor $\alpha = 0.5$ and a convergence criterion $\text{Tr}^{1/2}(E_r E_r^T) < 10^{-15}$, to working precision.

A. Example 1

Consider the fourth-order continuous-time open-loop system shown in [14] where

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

with an open-loop Jordan form

$$J_{ol} = \begin{bmatrix} -0.5 + 1.3229i & 0 & 0 & 0 \\ 0 & -0.5 - 1.3229i & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Case 1: Given a desired closed-loop Jordan form

$${}^*J_{cl} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

we try

$${}^*J_r = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

with $\mathcal{I}_c(J_r) = \begin{bmatrix} 1 & 2 \end{bmatrix}$. The algorithm finds a solution set

$$X_r = \begin{bmatrix} 0.8611 & -0.4286 \\ -0.1389 & 0.5714 \\ -0.1944 & -0.1429 \\ 0.3889 & 0.4286 \end{bmatrix},$$

$$W_r = \begin{bmatrix} 0.5278 & -0.4286 \\ 0.25 & 1.5714 \end{bmatrix}.$$

Then from (15) and (16) we have

$$X_{cl} = \begin{bmatrix} 0.8611 & -0.4286 & -2 & -0.5 \\ -0.1389 & 0.5714 & 0 & -2 \\ -0.1944 & -0.1429 & 0 & 0 \\ 0.3889 & 0.4286 & 0 & 0 \end{bmatrix}$$

and

$$W_{cl} = \begin{bmatrix} 0.5278 & -0.4286 & 0 & 0 \\ 0.2500 & 1.5714 & 0 & 0 \end{bmatrix}.$$

Observe the null parameter vectors associated with common open- and closed-loop characteristic vectors in the reduced-order design. The control gain

$$K = W_{cl} X_{cl}^{-1} = \begin{bmatrix} 0 & 0 & -14.1429 & -5.7143 \\ 0 & 0 & 18.1429 & 9.7143 \end{bmatrix}$$

confirms $X_{cl}^{-1}(A - BK)X_{cl} = {}^*J_{cl}$ with $\mathrm{Tr}^{1/2}(E_d E_d^T) = 5.4563 \times 10^{-15}$.

Case 2: Let us consider a full-order design with a desired closed-loop Jordan form

$${}^*J_r = {}^*J_{cl} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3+i & 0 \\ 0 & 0 & 0 & -3-i \end{bmatrix}$$

with $\mathcal{I}_c(J_r) = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$. The algorithm finds a solution

$$X_{cl} = \begin{bmatrix} 0.58 & -0.41 & -0.08-0.094i & -0.057-0.026i \\ -0.17 & 0.43 & -0.06-0.037i & -0.026-0.02i \\ -0.08 & -0.08 & 0.04+0.065i & 0.04+0.01i \\ 0.17 & 0.24 & -0.19-0.154i & -0.10-0.067i \end{bmatrix}$$

and

$$W_{cl} = W_r = \begin{bmatrix} 0.5278 & -0.4286 & 0.3333 & 1 \\ 0.25 & 1.5714 & 0.3333 & -0.6667 \end{bmatrix}.$$

The control gain

$$K = W_{cl} X_{cl}^{-1} = \begin{bmatrix} 0.4605 & -0.296 & -2.26 & -0.8982 \\ -0.3213 & 0.778 & 4.055 & 3.6597 \end{bmatrix}$$

confirms $X_{cl}^{-1}(A - BK)X_{cl} = {}^*J_{cl}$ with $\mathrm{Tr}^{1/2}(E_d E_d^T) = 2.9343 \times 10^{-15}$.

Case 3: Let us try a desired closed-loop Jordan form

$${}^*J_r = {}^*J_{cl} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

with $\mathcal{I}_c(J_r) = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$.

The SMC scheme does not converge and the condition number for X_{cl} becomes infinite. One thus can not proceed to compute K. The desired Jordan form ${}^*J_{cl}$ is turned out to be inadmissible.

B. Example 2

Consider the third-order continuous-time open-loop system shown in [7] where

$$A = \begin{bmatrix} ccc0 & 1 & 2 \\ -2 & 3 & 0 \\ -2 & -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

with an open-loop Jordan form

$$J_{ol} = \begin{bmatrix} 2.8589 & 0 & 0 \\ 0 & 0.0706+2.3647i & 0 \\ 0 & 0 & 0.0706-2.3647i \end{bmatrix}.$$

Case 1: Consider a complex-valued desired closed-loop Jordan form

$${}^*J_r = {}^*J_{cl} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2+i & 0 \\ 0 & 0 & -2-i \end{bmatrix}$$

with $\mathcal{I}_c(J_r) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. The algorithm finds a solution

$$X_{cl} = X_r = \begin{bmatrix} 1 & -0.0188+0.059i & 0.1817+0.101i \\ 1.5 & 0.0995-0.008i & -0.2622+0.208i \\ 3.5 & 0.0028+0.056i & 0.1226+0.144i \end{bmatrix},$$

$$W_{cl} = W_r = \begin{bmatrix} 4 & 0.52646-0.26i & -1.883+0.577i \\ 2.75 & -0.19996+0.25i & 1.064+0.151i \end{bmatrix}.$$

Then the control gain

$$K = W_{cl} X_{cl}^{-1} = \begin{bmatrix} -4.1291 & 4.5028 & 0.39281 \\ 3.8131 & -1.2976 & 0.25238 \end{bmatrix}$$

confirms $X_{cl}^{-1}(A - BK)X_{cl} = {}^*J_{cl}$ with $\mathrm{Tr}^{1/2}(E_d E_d^T) = 7.0083 \times 10^{-14}$. The gain is real valued even with complex valued eigenvectors and parameter vectors.

Case 2: Let us consider the Jordan pair (X_r, J_r) assignment problem where the desired closed-loop Jordan form

$${}^*J_r = {}^*J_{cl} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

with $\mathcal{I}_c(J_r) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and the corresponding desired eigenvector matrix

$${}^*X_r = \begin{bmatrix} 1 & 0.5 & -0.5 \\ 1.5 & -1 & 0 \\ 3.5 & 0 & -0.5 \end{bmatrix}.$$

The algorithm surely finds a solution

$$X_{cl} = X_r = \begin{bmatrix} 1 & 0.5 & -0.5 \\ 1.5 & -1 & 0 \\ 3.5 & 0 & -0.5 \end{bmatrix} = {}^*X_r$$

and

$$W_{cl} = W_r = \begin{bmatrix} 4 & -5 & 1 \\ 2.75 & 2.25 & -1.5 \end{bmatrix}.$$

We observe that the desired $({}^*X_r, {}^*J_r)$ is achieved. The control gain

$$K = W_{cl} X_{cl}^{-1} = \begin{bmatrix} -2 & 4 & 0 \\ 2.5 & -1 & 0.5 \end{bmatrix}$$

confirms $X_{cl}^{-1}(A - BK)X_{cl} = {}^*J_{cl}$ with $\mathrm{Tr}^{1/2}(E_d E_d^T) = 1.1322 \times 10^{-14}$.

Case 3: Let us consider the Jordan pair (X_r, J_r) assignment problem where the desired closed-loop Jordan form

$$^*J_r = {}^*J_{cl} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

with $\mathcal{I}_c(J_r) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and corresponding desired eigenvector matrix

$$^*X_r = \begin{bmatrix} 0.5 & -0.5 & -0.5 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}.$$

The algorithm however finds a solution

$$X_{cl} = X_r = \begin{bmatrix} 0.5 & -0.5 & -0.5 \\ 1 & 0 & 0 \\ 2 & 1 & -0.5 \end{bmatrix}$$

and

$$W_{cl} = W_r = \begin{bmatrix} 3 & 0 & 1 \\ 1.25 & 0.5 & -1.5 \end{bmatrix}.$$

We observe $X_{cl} \neq {}^*X_{cl}$, i.e., $({}^*X_{cl}, {}^*J_{cl}) \notin \mathcal{P}$ can not be achieved. Instead, the true right eigenvector matrix X_{cl} delivering the control gain

$$K = W_{cl} X_{cl}^{-1} = \begin{bmatrix} -1.3333 & 5 & -0.66667 \\ 1.6667 & -2.25 & 1.3333 \end{bmatrix}$$

confirms $X_{cl}^{-1}(A - BK)X_{cl} = {}^*J_{cl}$ with $\mathrm{Tr}^{1/2}(E_d E_d^T) = 7.5546 \times 10^{-15}$ such that $(X_{cl}, {}^*J_{cl}) \in \mathcal{P}$.

V. Conclusions

This paper has presented a complete computational solution to the eigenstructure assignment problems in linear multivariable systems. The proposed novel computational algorithm - successive mapping and correction - can find solutions for any admissible closed-loop Jordan form. The computation algorithm is complete in a sense that it is mathematically shown to find a solution for any admissible closed-loop Jordan form and it can also be used for Jordan pair assignment as well as the reduced- and full-order design. The examples of various eigenstructure assignment problems verified utilities of the algorithm. The algorithm is expected to facilitate a computer based approach, resolving the constraints concerning linear independent eigenvectors and real control gains in matrix transformation.

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