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Trajectory tracking in linear hybrid systems: an internal model principle approach

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Abstract— **This paper deals with the problem of asymptotically tracking periodic trajectories for a class of hybrid systems (*switched systems*) having linear dynamics in each operating mode, where the reference trajectory and the control action are such that only isolated swithcing events occur. The possible presence of uncertainties in the system description is considered and no assumptions on the uniformity of dimension of the state vectors among modes are made. In order to deal with the hybrid nature of the considered system and the possible presence of discontinuities of its solutions at switching times, a properly amended tracking control problem is defined and a control strategy based on a discontinuous version of the classical internal model principle is proposed.**

I. INTRODUCTION

Dynamical systems that are described by an interaction between continuous and discrete dynamics are usually called *hybrid systems*. In this paper, continuous-time systems with (isolated) discrete switching events are considered. Such systems are referred to as *switched systems* (an excellent reference on analysis and control of such systems is [1]). More precisely, a switched system is an hybrid dynamical system consisting of a family of continuous-time subsystems and a rule that orchestrates the switching between them [2]. Switched systems have numerous applications in control of mechanical systems, automotive industry, aircraft and air traffic control, switching power converters, and many other fields (see, e.g., [1]–[5] and the references therein). Some other examples of switched system are: thermostat, tank system, bouncing ball, Clegg integrator, biological networks [6], chemical process control, engine control (a four-stroke gasoline engine is naturally modeled by using four discrete modes corresponding to the position of the pistons, while combustion and power train dynamics are continuous) [7].

In most cases, the uniformity of continuous state space (i.e., all operating modes have the same continuous state space, the n-dimensional real-valued space) is assumed. Branicky proposed the model of general hybrid dynamical systems [8] as a unified framework which captures various aspects within hybrid dynamics. It was mentioned in his work that failure situations can be modeled as hybrid dynamics by relaxing the common assumption about the dimension of continuous state space. Provided that the re-initialization of continuous states is properly defined, the relaxation is quite natural because each continuous dynamics can be defined separately from others [9]. In [9], well-posedness for a class of bimodal modular hybrid dynamical systems is studied. The notion of modularity (i.e., the system dynamics change according to each alteration of structure (state space) in the form of component/module attachment or detachment) enables to model several interesting phenomena, e.g., component breakdown and hot-swap of modules, which are forbidden in the conventional framework of system theory. In [8], it is remarked that the state space may change in modeling component failures or changes in dynamical description based on autonomous or controlled events which change it. Examples include the collision of two inelastic particles, an aircraft mode transition that changes variables to be controlled [10], the problem to take into account overlapping local coordinate systems on a manifold [11]. In [12], switched dynamical systems with state-space dilation and contraction formed by concatenating the states of a set of local dynamical systems or semi-flows on state spaces with different dimensions at specified time-instants are considered. Such systems arise naturally in many aerospace applications such as multi-body dynamic systems involving changes in the degrees of freedom, and systems composed of multiple spacecraft with docking and undocking capabilities flying in formation. Other examples of hybrid systems with possibly non-uniform state space representations are considered in [13]–[18], where Bond-Graphs are used in order to model physical hybrid systems.

In this paper, the problem of asymptotically tracking periodic trajectories for hybrid systems having linear dynamics in each operating mode is considered. The possible presence of uncertainties in the system description is considered and no assumptions on the uniformity of dimension of the state vectors among modes are made. By following the approach proposed by the same authors in [19], [20] and [21], a properly amended tracking control problem, dealing with the hybrid and discontinuous nature of the system, has been defined by using notions similar to the *quasi stability* concept in [22]. Moreover, a control strategy based on a discontinuous version of the classical internal model principle (see, e.g., [23] and [24]) is proposed.

In the following, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ the set of non-negative real numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Z}^+$ the set of non-negative integers, $\mathbb{N}$ the set of positive integers, $\mathbb{R}^\nu$ the set of vectors of dimension ν, $\mathbb{R}^{\nu\times\mu}$ the set of real matrices of dimensions $\nu\times\mu$, with $\nu,\mu\in\mathbb{N}$. For the sake of brevity, the shorthand notations $\mathbf{g}(\tau^-)$ and $\mathbf{g}(\tau^+)$ are used in place of $\lim_{t\to\tau^-}\mathbf{g}(t)$ and $\lim_{t\to\tau^+}\mathbf{g}(t)$, respectively, for each vector function $\mathbf{g}(t)$. Finally, $\|\cdot\|$ denotes the Euclidean norm of the vector at argument.

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II. THE CONSIDERED HYBRID SYSTEMS AND REFERENCE TRAJECTORIES

A switched system is a particular hybrid system characterized by

(i) the family of *switching surfaces* and the resulting *operating regions*;
(ii) the family of *continuous-time subsystems*, or *modes*, one for each operating region;
(iii) the *reset maps*.

In general terms, in each operating region a continuous-time dynamical system is given. Whenever the system trajectory hits a switching surface, the continuous state jumps instantaneously to a new value, specified by a reset map. The instantaneous jumps of the continuous state are sometimes referred to as *impulse effects* [1]. In the following, (i),(ii) and (iii) are defined for the class of switched systems considered in this paper.

Define the (finite) *index set* as $\mathcal{P} := \{1, \dots, M\}$, where M is the number of modes of the switched system. The switching surface $\mathcal{C}_{ij}$ characterizes the part of the *boundary* of the operating region of mode j from which a transition to mode i occurs, i.e. a *switching* from mode j to mode i. It is defined as follows

$$\mathcal{C}_{ij} := \{\mathbf{x} \in \mathbb{R}^{n_j} : \mathbf{J}_{ij}\mathbf{x} = b_{ij}, \}, \quad i, j \in \mathcal{P}, \tag{1}$$

where $\mathbf{J}_{ij}^T \in \mathbb{R}^{n_j}$, $b_{ij} \in \mathbb{R}$ and $n_j \in \mathbb{N}$, so that the j-th operating region can be expressed as

$$\mathcal{X}_j := \left\{ \mathbf{x} \in \mathbb{R}^{n_j} : \bigcap_{i \in \mathcal{P}} \left(\mathbf{J}_{ij}\mathbf{x} - b_{ij} \leq 0 \right) \right\} \subseteq \mathbb{R}^{n_j}. \tag{2}$$

Remark 1. In (1) and (2), it is assumed that $\mathbf{J}_{ij} = \mathbf{0}$ and $b_{ij} = 1$ for all $i \in \mathcal{P}$ such that the transition from mode j to mode i is not defined. For the sake of simplicity, it is assumed that there is only one transition between mode i and mode j; at the expense of a more cumbersome notation the case of multiple different transitions (corresponding to different switching surfaces) can be dealt with too. Notice also that, since in the sequel we'll only be concerned with the local behavior around a reference trajectory not including any state belonging to the sets $\mathcal{C}_{ij} \cap \mathcal{C}_{hj}$ (for $h \neq i$ and $j = 1, \dots, M$), the fact that in the above definition the transition from mode j to its successor mode is not explicitly and uniquely defined when the state belongs to $\mathcal{C}_{ij} \cap \mathcal{C}_{hj}$ has no relevance for the following discussion.

For each operating region (or *mode*), continuous-time, linear, time-invariant dynamical subsystems are considered. In particular, in the generic i-th mode, the system evolves according to the following dynamics

$$\mathrm{P}_i : \begin{cases} \dot{\mathbf{x}}_i(t) = \mathbf{A}_i(\boldsymbol{\beta})\mathbf{x}_i(t) + \mathbf{B}_i(\boldsymbol{\beta})\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}_i(\boldsymbol{\beta})\mathbf{x}_i(t) + \mathbf{D}_i(\boldsymbol{\beta})\mathbf{u}(t) \end{cases}, \quad i \in \mathcal{P}, \tag{3}$$

where $\mathbf{A}_i(\boldsymbol{\beta}) \in \mathbb{R}^{n_i \times n_i}, \mathbf{B}_i(\boldsymbol{\beta}) \in \mathbb{R}^{n_i \times q}, \mathbf{C}_i(\boldsymbol{\beta}) \in \mathbb{R}^{q \times n_i}$ and $\mathbf{D}_i(\boldsymbol{\beta}) \in \mathbb{R}^{q \times q}$ are matrices with real entries depending on a vector $\boldsymbol{\beta} \in \boldsymbol{\Theta}$ of parameters which are subject to variations and/or uncertainties and play the role of the *physical* parameters of the plant. The nominal value $\bar{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$ is assumed to be an interior point of the bounded set $\boldsymbol{\Theta}$.

Remark 2. notice that subsystems P_i are assumed to be square, i.e., having the same number of inputs and outputs. Such an assumption could be removed at the expense of a more cumbersome set of weaker assumptions and rendering some steps in the derivations and proofs more involved.

Assumption 1. There exists a closed neighborhood $\boldsymbol{\Phi}_a \subseteq \boldsymbol{\Theta}$ such that all the entries of $\mathbf{A}_i(\boldsymbol{\beta}), \mathbf{B}_i(\boldsymbol{\beta}), \mathbf{C}_i(\boldsymbol{\beta})$ and $\mathbf{D}_i(\boldsymbol{\beta})$ are continuous functions of $\boldsymbol{\beta}$ in $\boldsymbol{\Phi}_a$, for all $i \in \mathcal{P}$.

Assumption 2. For all $i \in \mathcal{P}$ and $\boldsymbol{\beta} \in \boldsymbol{\Phi}_b$ with $\boldsymbol{\Phi}_b \subseteq \boldsymbol{\Theta}$ being a neighborhood of $\bar{\boldsymbol{\beta}}$, it is assumed that the triples $(\mathbf{A}_i(\boldsymbol{\beta}), \mathbf{B}_i(\boldsymbol{\beta}), \mathbf{C}_i(\boldsymbol{\beta}))$ are reachable and observable.

Concerning the transition between modes, a time t_k is a *switching time*, if

$$\mathbf{J}_{\sigma(t_k^+)\sigma(t_k^-)}\mathbf{x}_{\sigma(t_k^-)}(t_k^-) - b_{\sigma(t_k^+)\sigma(t_k^-)} = 0, \tag{4a}$$

$$\mathbf{J}_{\sigma(t_k^+)\sigma(t_k^-)}\dot{\mathbf{x}}_{\sigma(t_k^-)}(t_k^-) > 0, \tag{4b}$$

where the piecewise constant function $\sigma : \mathbb{R} \to \mathcal{P}$ is the *switching signal*. The discontinuities of such a function coincide with the switching times and, on every interval between two consecutive switching times, this function takes a constant value. The role of σ is to specify, at each time t, the index $\sigma(t) \in \mathcal{P}$ of the *active subsystem* [1]. It is assumed that $\sigma(t)$ is right-continuous, i.e., $\lim_{t \to t_k^+} \sigma(t) = \sigma(t_k)$.

Remark 3. The manifold of $\mathcal{X}_j$ identified by (4a) represents the condition of *contact* among the dynamical system with the constraint surface $\mathcal{C}_{ij}$. On the other hand, (4b) represents the *transversality* condition which guarantees that, at the intersection, the flow of (3) is not tangent to the constraint (1) [25].

At switching time t_k, the transition from the *departure mode* d_k to the *arrival mode* a_k is given by $\sigma(t_k^-) = d_k$ and $\sigma(t_k^+) = \sigma(t_k) = a_k$, where d_k, $a_k \in \mathcal{P}$. As for the jump in the plant state vector, the following (linear) reset map is considered

$$\mathbf{x}_{\sigma(t_k^+)}(t_k^+) = \boldsymbol{\Gamma}_{\sigma(t_k^+)\sigma(t_k^-)}\mathbf{x}_{\sigma(t_k^-)}(t_k^-), \tag{5}$$

where, letting j and i be the indexes relevant to the departure and the arrival modes, respectively, then $\boldsymbol{\Gamma}_{ij} \in \mathbb{R}^{n_i \times n_j}$.

The reference trajectories considered in this paper for the class of hybrid systems defined above are assumed to be periodic with period $T \in \mathbb{R}^+$ and $N \in \mathbb{Z}^+$ switching events per period. In summary, in a whole period the nominal trajectory can be represented as follows (from now on the *bar*-notation denotes the nominal values)

$$\bar{\mathbf{y}}(t) = \begin{cases} \bar{\mathbf{y}}_1(t), & t \in [\bar{t}_1, \bar{t}_2), \\ \quad\vdots & \\ \bar{\mathbf{y}}_N(t), & t \in [\bar{t}_N, \bar{t}_{N+1}), \end{cases} \tag{6}$$

where $\bar{t}_k$, $k \in \mathbb{Z}^+$ denotes the k-th desired switching time and, in view of the periodicity one has that: $\bar{\mathbf{y}}_{(k \bmod N)}(t + T) = \bar{\mathbf{y}}_k(t)$ and $\bar{t}_{k+N} = \bar{t}_k + T$. For any $i \in \{1, \dots, N\} =: \mathcal{I}_N$, the reference signal $\bar{\mathbf{y}}_i(t)$ is assumed to be generated by

$$\begin{cases} \dot{\mathbf{x}}_i^r(t) = \mathbf{A}_i^r \mathbf{x}_i^r(t) \\ \bar{\mathbf{y}}_i(t) = \mathbf{C}_i^r \mathbf{x}_i^r(t) \end{cases}, \quad \forall t \in [\bar{t}_k, \bar{t}_{k+1}),$$

with some initial state $\mathbf{x}_i^r(\bar{t}_k^+)$ and $k \in \mathcal{I}_N$. Let $\bar{\phi}_i(s)$ be the minimal polynomial of $\mathbf{A}_i^r$ (where all the roots of $\bar{\phi}_i(s)$, i.e. the eigenvalues of A_i, have nonnegative real parts), and let

$$\bar{\phi}(s) = s^m + \alpha_1 s^{m-1} + \cdots + \alpha_m, \tag{7}$$

be the least common multiple of all the polynomials $\bar{\phi}_i(s)$ for $i \in \mathcal{I}_N$; the following assumption is made.

Assumption 3. For all $i \in \mathcal{P}$ and $\boldsymbol{\beta} \in \boldsymbol{\Phi}_c$ with $\boldsymbol{\Phi}_c \subseteq \boldsymbol{\Theta}$ being a neighborhood of $\bar{\boldsymbol{\beta}}$ and for every root λ of $\bar{\phi}(s)$, which is defined in (7), it is assumed that

$$\operatorname{rank}\left(\begin{bmatrix} \mathbf{A}_i(\boldsymbol{\beta}) - \lambda \mathbf{I} & \mathbf{B}_i(\boldsymbol{\beta}) \\ \mathbf{C}_i(\boldsymbol{\beta}) & \mathbf{D}_i(\boldsymbol{\beta}) \end{bmatrix}\right) = n_i + q. \tag{8}$$

Remark 4. In view of Assumption 1, there exists a neighborhood $\boldsymbol{\Phi}_{abc} \subseteq (\boldsymbol{\Phi}_a \cap \boldsymbol{\Phi}_b \cap \boldsymbol{\Phi}_c) \subseteq \boldsymbol{\Theta}$ of $\bar{\boldsymbol{\beta}}$ such that if Assumptions 2 and 3 are satisfied for $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$, then they are automatically satisfied for all $\boldsymbol{\beta} \in \boldsymbol{\Phi}_{abc}$.

Finally, by defining the minimum distance between two consecutive desired switching times as (see Fig. 1)

$$\rho := \min_{k \in \mathcal{I}_N} \{|\bar{t}_k - \bar{t}_{k+1}|\}, \tag{9}$$

the following definition of admissible desired trajectory can be given.

Definition 1. A reference trajectory $\bar{\mathbf{y}}(t)$ in the form (6) is said to be *admissible* for a hybrid system characterized by (1)-(5) if the following properties hold

1) compatibility of the reset values: for all $k \in \mathcal{I}_N$
$$\bar{\mathbf{x}}_{\bar{\sigma}(\bar{t}_k^+)}(\bar{t}_k^+) = \boldsymbol{\Gamma}_{\bar{\sigma}(\bar{t}_k^+)\bar{\sigma}(\bar{t}_k^-)} \bar{\mathbf{x}}_{\bar{\sigma}(\bar{t}_k^-)}(\bar{t}_k^-);$$

2) no degenerate switching times: for all $k \in \mathcal{I}_N$
$$\mathbf{J}_{\bar{\sigma}(\bar{t}_k^+)\bar{\sigma}(\bar{t}_k^-)} \bar{\mathbf{x}}_{\bar{\sigma}(\bar{t}_k^-)}(\bar{t}_k^-) - b_{\bar{\sigma}(\bar{t}_k^+)\bar{\sigma}(\bar{t}_k^-)} = 0,$$
$$\mathbf{J}_{\bar{\sigma}(\bar{t}_k^+)\bar{\sigma}(\bar{t}_k^-)} \dot{\bar{\mathbf{x}}}_{\bar{\sigma}(\bar{t}_k^-)}(\bar{t}_k^-) > 0;$$

3) no multiple switching events at the same time: for all $k \in \mathcal{I}_N$ there does not exist a pair (i_1, i_2) with $i_1, i_2 \in \mathcal{P}$ and $i_1 \neq i_2$ such that
$$\mathbf{J}_{i_1\bar{\sigma}(\bar{t}_k^-)} \bar{\mathbf{x}}_{\bar{\sigma}(\bar{t}_k^-)}(\bar{t}_k^-) - b_{i_1\bar{\sigma}(\bar{t}_k^-)} = 0,$$
$$\mathbf{J}_{i_2\bar{\sigma}(\bar{t}_k^-)} \bar{\mathbf{x}}_{\bar{\sigma}(\bar{t}_k^-)}(\bar{t}_k^-) - b_{i_2\bar{\sigma}(\bar{t}_k^-)} = 0.$$

where in 1), 2) and 3), $\bar{\sigma}(t)$ and $\bar{\mathbf{x}}_{\bar{\sigma}(t)}(t)$ denote the nominal switching signal and the nominal value of the plant state vector, respectively, when $\mathbf{y}(t) = \bar{\mathbf{y}}(t)$. The existence and uniqueness of $\bar{\mathbf{x}}_{\bar{\sigma}(t)}(t)$ is guaranteed by Assumptions 2, 3, under the hypothesis that the subsystems P_i are square.

Throughout the paper, whenever the reference (or nominal, or desired) trajectory is considered, it is implicitly assumed to be admissible according to Definition 1.

III. ASYMPTOTIC TRACKING PROBLEM: DEFINITION AND SOLUTION

The goal of this section is the design of a control law such that the actual trajectory asymptotically tracks the desired one. More specifically the classical tracking problem (see, e.g., [23]) is considered and, in the following, it is properly amended in order to deal with the considered class of hybrid systems. The presence of discontinuities due to the switching events complicates the trajectory tracking problem as compared with the case of unconstrained systems. For a simple mechanical system subject to impacts, in [19] (see also [20], [21]) it is shown that the error on the velocity immediately after the impact times has in general absolute value greater than a given positive quantity. For this reason, the classical stability and attractivity properties are difficult (if not impossible) to be obtained and the tracking problem has been properly defined in order to neglect in the analysis the times belonging to infinitesimal intervals about the switching times (see Fig. 1), thus ensuring a sort of asymptotic stability for the error dynamics, similarly to what is proposed in [22] for impulsive differential systems.

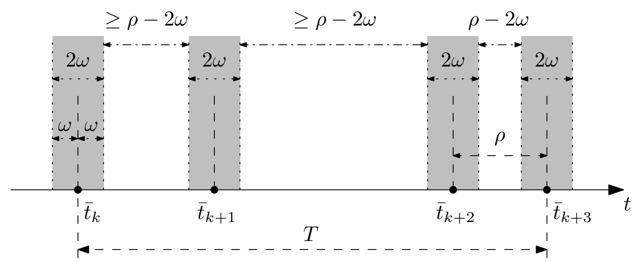


Fig. 1. Example of possible switching times for a trajectory with period T and $N = 3$ switching events per period. Time intervals identified by the grey blocks are neglected in the stability analysis.

By following the approach introduced in [20] and [21], a controller based on a discontinuous-version of the classical internal model principle (see, e.g., [23] and [24]) is considered. It is well known that in absence of discontinuities (i.e., during the free-motion phases) a continuous-time internal model of the desired trajectory is needed in the forward path of the feedback control system. The presence of such an internal model is guaranteed through a dynamic precompensator, whose state vector is subject to discontinuities at the desired switching times. By defining the error vectors relevant to the plant and the precompensator as follows $\tilde{\mathbf{x}}_{\sigma(t)}(t) := \mathbf{x}_{\sigma(t)}(t) - \bar{\mathbf{x}}_{\sigma(t)}(t)$, $\tilde{\mathbf{x}}_a(t) := \mathbf{x}_a(t) - \bar{\mathbf{x}}_a(t)$, the output error as $\mathbf{e}(t) = \mathbf{y}(t) - \bar{\mathbf{y}}(t)$, and the error at time $\bar{t}_k$ between the correct jump in the precompensator and the estimated one as $\tilde{\boldsymbol{\Lambda}}_{\bar{t}_k} = \boldsymbol{\Lambda}_{\bar{t}_k} - \bar{\boldsymbol{\Lambda}}_{\bar{t}_k}$, the control problem solved in this paper can be stated as follows.

Problem 1. *Find, if any, a piecewise continuous control law such that for all $\varepsilon > 0$, $t_0 \in \mathbb{R}$ and $\omega \in (0, \rho/2)$, there exists $\delta_{\varepsilon,\omega} > 0$ and a neighborhood $\boldsymbol{\Phi} \subseteq \boldsymbol{\Theta}$ of $\bar{\boldsymbol{\beta}}$ such that if $\|\tilde{\mathbf{x}}^e_{\sigma(t_0^+)}(t_0^+)\| < \delta_{\varepsilon,\omega}$, $\|\tilde{\mathbf{x}}_a(t_0^+)\| < \delta_{\varepsilon,\omega}$ and $\|\tilde{\boldsymbol{\Lambda}}_{\bar{t}_1}\| < \delta_{\varepsilon,\omega}, \cdots, \|\tilde{\boldsymbol{\Lambda}}_{\bar{t}_{N-1}}\| < \delta_{\varepsilon,\omega}$, then the following properties hold for the closed-loop system and for all $\boldsymbol{\beta} \in \boldsymbol{\Phi}$:*

$$\textbf{1)}\ \ \|\mathbf{e}(t)\| < \varepsilon, \quad \forall t \in \mathbb{R} \setminus \Omega,\ t > t_0,$$

where $\Omega := \bigcup_{k \in \mathbb{Z}} \Omega_k$ *and* $\Omega_k := \{\, t : \ |t - \bar{t}_k| \leq \omega \,\}$;

$$\textbf{2)}\ \ \lim_{k \to +\infty} \|\mathbf{e}((\bar{t}_k + \tau)^+)\| = 0, \quad \forall \tau \in (0, \rho),$$

where the limit is taken with k being integer.

The control scheme considered in this paper is depicted in Fig. 2, where the dashed arrows denote the switching times for the blocks they point to. In the following, all the steps for the design of such a control scheme are detailed.

Step 1: Precompensator design (IM)
The internal model $\bar{\phi}^{-1}(s)\mathbf{I}_q$ of the class of trajectories to be tracked, with $\bar{\phi}(s)$ defined in (7), can be realized as

$$\begin{cases} \dot{\mathbf{x}}_a(t) = \mathbf{A}_a\mathbf{x}_a(t) + \mathbf{B}_a\mathbf{e}(t) \\ \mathbf{y}_a(t) = \mathbf{x}_a(t) \end{cases}, \quad \forall t \in [\bar{t}_k, \bar{t}_{k+1}),$$

where $\mathbf{A}_a = \mathbf{blockdiag}\{\underbrace{\boldsymbol{\Upsilon}, \cdots, \boldsymbol{\Upsilon}}_{q-\text{tuple}}\} \in \mathbb{R}^{qm\times qm}$ and $\mathbf{B}_a = \mathbf{blockdiag}\{\underbrace{\boldsymbol{\upsilon}, \cdots, \boldsymbol{\upsilon}}_{q-\text{tuple}}\} \in \mathbb{R}^{qm\times q}$ with

$$\boldsymbol{\Upsilon} := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_m & -\alpha_{m-1} & -\alpha_{m-2} & \cdots & -\alpha_1 \end{bmatrix}, \ \boldsymbol{\upsilon} := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Step 2: Static gains design ($\mathbf{K}_i, \mathbf{K}_i^a$)
Consider the tandem connection of the plant followed by the precompensator. During the free-motion phases its composite dynamical equation is

$$\underbrace{\begin{bmatrix} \dot{\mathbf{x}}_i \\ \dot{\mathbf{x}}_a \end{bmatrix}}_{\dot{\mathbf{x}}_i^e} = \underbrace{\begin{bmatrix} \mathbf{A}_i(\boldsymbol{\beta}) & \mathbf{0} \\ \mathbf{B}_a\mathbf{C}_i(\boldsymbol{\beta}) & \mathbf{A}_a \end{bmatrix}}_{\mathbf{A}_i^e(\boldsymbol{\beta})} \underbrace{\begin{bmatrix} \mathbf{x}_i \\ \mathbf{x}_a \end{bmatrix}}_{\mathbf{x}_i^e} + \underbrace{\begin{bmatrix} \mathbf{B}_i(\boldsymbol{\beta}) \\ \mathbf{B}_a\mathbf{D}_i(\boldsymbol{\beta}) \end{bmatrix}}_{\mathbf{B}_i^e(\boldsymbol{\beta})} \mathbf{u} = $$
$$= \mathbf{A}_i^e(\boldsymbol{\beta})\mathbf{x}_i^e + \mathbf{B}_i^e(\boldsymbol{\beta})\mathbf{u}. \qquad (10)$$

This connection is controllable and observable (see, e.g., [24]) if and only if no root of the polynomial $\bar{\phi}(s)$ is a transmission zero of the plant, or, in other words, (10) is controllable if Assumption 3 is satisfied. In particular, if (8) is verified for every plant P_i, then the eigenvalues of the closed-loop system can be arbitrarily assigned by the state feedback

$$\mathbf{u}(t) = \underbrace{\begin{bmatrix} \mathbf{K}_i & \mathbf{K}_i^a \end{bmatrix}}_{\mathbf{K}_i^e} \begin{bmatrix} \mathbf{x}_i(t) \\ \mathbf{x}_a(t) \end{bmatrix}, \quad \forall t \in [t_k^M, t_{k+1}^m),$$

where $t_k^m := \min\{t_k, \bar{t}_k\}$, $t_k^M := \max\{t_k, \bar{t}_k\}$ and for all $i \in \mathcal{P}$ the gain $\mathbf{K}_i^e \in \mathbb{R}^{q\times(n_i+mq)}$ is chosen such that, for all $\boldsymbol{\beta} \in \boldsymbol{\Phi}_d$ with $\boldsymbol{\Phi}_d \subseteq \boldsymbol{\Theta}$ being a neighborhood of $\bar{\boldsymbol{\beta}}$, all eigenvalues of $(\mathbf{A}_i^e(\boldsymbol{\beta}) + \mathbf{B}_i^e(\boldsymbol{\beta})\mathbf{K}_i^e)$ have real part less than $-\eta$, with $\eta \in \mathbb{R}^+$, i.e.,

$$\mathrm{Re}(\lambda) < -\eta \text{ for all } \lambda \in \Lambda[\mathbf{A}_i^e(\boldsymbol{\beta}) + \mathbf{B}_i^e(\boldsymbol{\beta})\mathbf{K}_i^e], \qquad (11)$$

with $\mathrm{Re}(\lambda)$ and $\Lambda[\cdot]$ being the real part of λ and the set of the eigenvalues of the matrix at argument, respectively. Note that in view of Assumption 1, if (11) is satisfied in the nominal parameters, i.e., for $\beta = \bar{\beta}$, then it is automatically satisfied for all $\beta \in \boldsymbol{\Phi}$, with $\boldsymbol{\Phi} \subseteq (\boldsymbol{\Phi}_d \cap \boldsymbol{\Phi}_{abc}) \subseteq \boldsymbol{\Theta}$ and $\boldsymbol{\Phi}_{abc}$ defined in Remark 4.

Step 3: Precompensator reset values ($\boldsymbol{\Lambda}_{\bar{t}_k}$)
At the desired switching time $\bar{t}_k$ the state vector $\mathbf{x}_a$ is reset to $\mathbf{x}_a(\bar{t}_k^+) = \boldsymbol{\Lambda}_{\bar{t}_k}$, where $\boldsymbol{\Lambda}_{\bar{t}_k}$ denotes the reset value for the precompensator state vector at time $\bar{t}_k$.

Case 1 – without uncertainties: If there are no uncertainties on the plant modeling, i.e., $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$, then $\boldsymbol{\Lambda}_{\bar{t}_k} = \bar{\boldsymbol{\Lambda}}_{\bar{t}_k}$, with $\bar{\boldsymbol{\Lambda}}_{\bar{t}_k}$ being the nominal reset value for the precompensator state vector at time $\bar{t}_k$. In order to compute such a vector one can proceed as follows. By assuming perfect tracking, one has

$$\begin{cases} t_k = \bar{t}_k, & \forall k, \\ \mathbf{y}(t) = \bar{\mathbf{y}}(t) \Rightarrow \mathbf{e}(t) = \mathbf{0}, & \forall t \geq t_0, \end{cases}$$

so that, by considering the nominal state vector $\bar{\mathbf{x}}_i^e = [\bar{\mathbf{x}}_i^T, \bar{\mathbf{x}}_a^T]^T$, the nominal dynamics can be written as

$$\begin{cases} \dot{\bar{\mathbf{x}}}_i^e(t) = \bar{\mathbf{A}}_i^e(\boldsymbol{\beta})\bar{\mathbf{x}}_i^e(t) \\ \bar{\mathbf{y}}_i(t) = \bar{\mathbf{C}}_i^e(\boldsymbol{\beta})\bar{\mathbf{x}}_i^e(t) \end{cases}, \quad \forall t \in [\bar{t}_k, \bar{t}_{k+1}), \qquad (12)$$

where $\bar{\mathbf{A}}_i^e(\boldsymbol{\beta}) = \begin{bmatrix} \mathbf{A}_i(\boldsymbol{\beta}) + \mathbf{B}_i(\boldsymbol{\beta})\mathbf{K}_i & \mathbf{B}_i(\boldsymbol{\beta})\mathbf{K}_i^a \\ \mathbf{0} & \mathbf{A}_a \end{bmatrix}$ and $\bar{\mathbf{C}}_i^e(\boldsymbol{\beta}) = \begin{bmatrix} \mathbf{C}_i(\boldsymbol{\beta}) + \mathbf{D}_i(\boldsymbol{\beta})\mathbf{K}_i & \mathbf{D}_i(\boldsymbol{\beta})\mathbf{K}_i^a \end{bmatrix}$, whereas at the switching time $\bar{t}_k$ the *jump* in the augmented state vector is

$$\bar{\mathbf{x}}_i^e(\bar{t}_k^+) = \underbrace{\begin{bmatrix} \boldsymbol{\Gamma}_{ij} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\bar{\boldsymbol{\Gamma}}_{ij}^e} \bar{\mathbf{x}}_j^e(\bar{t}_k^-) + \underbrace{\begin{bmatrix} \mathbf{0} \\ \bar{\boldsymbol{\Lambda}}_{\bar{t}_k} \end{bmatrix}}_{\bar{\boldsymbol{\Lambda}}_{\bar{t}_k}^e} =$$
$$= \bar{\boldsymbol{\Gamma}}_{ij}^e \bar{\mathbf{x}}_j^e(\bar{t}_k^-) + \bar{\boldsymbol{\Lambda}}_{\bar{t}_k}^e,$$

where $i = \sigma(\bar{t}_k^+) = \sigma(\bar{t}_k) \in \mathcal{P}$ and $j = \sigma(\bar{t}_k^-) \in \mathcal{P}$.

The problem to compute $\bar{\boldsymbol{\Lambda}}_{\bar{t}_k}$ can be turned into an observability problem. In fact, since it is possible to find stabilizing gains $\mathbf{K}_i^e$ for the closed loop system and all the eigenvalues of IM are in the closed right half plane, then it is clear that the state vector $\bar{\mathbf{x}}_a$ has to be observable from the output. Since observability is in general not preserved under constant state feedback, if necessary, one has to carry out a canonical decomposition in order to separate the observable modes from the others (see, e.g., [24]). More precisely, $\bar{\boldsymbol{\Lambda}}_{\bar{t}_k}$ can be obtained as follows:

1) Define $\bar{\mathbf{A}}_i^e(\boldsymbol{\beta})$ and $\bar{\mathbf{C}}_i^e(\boldsymbol{\beta})$ as in (12) where $i = \sigma(\bar{t}_k^+)$;
2) Let $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$ (for the sake of readability in the following steps dependence on $\boldsymbol{\beta}$ is omitted);
3) Define[1]
$$\mathbf{Y}(\bar{t}_k^+) = \begin{bmatrix} \bar{\mathbf{y}}^T(\bar{t}_k^+) & \dot{\bar{\mathbf{y}}}^T(\bar{t}_k^+) & \cdots & \bar{\mathbf{y}}^{(n_o-1)T}(\bar{t}_k^+) \end{bmatrix}^T$$
and
$$\mathbf{O}_i = \begin{bmatrix} (\bar{\mathbf{C}}_i^e)^T & (\bar{\mathbf{C}}_i^e\bar{\mathbf{A}}_i^e)^T & \cdots & (\bar{\mathbf{C}}_i^e(\bar{\mathbf{A}}_i^e)^{n_o-1})^T \end{bmatrix}^T,$$
where $n_o = n_i + mq$. It is easy to see that $\mathrm{rank}(\mathbf{O}_i) = n_o$;

[1] In the following, $\bar{\mathbf{y}}^{(l)}(\bar{t}_k^+) := \left(\frac{d^l}{dt^l}(\bar{\mathbf{y}}_j(t))\right)\Big|_{t=\bar{t}_k^+}$, where $\bar{\mathbf{y}}(t) = \bar{\mathbf{y}}_j(t)$ for all $t \in [\bar{t}_k, \bar{t}_{k+1})$ (see (6)).

4) Finally, since $\mathbf{Y}(\bar{t}_k^+) = \mathbf{O}_i \bar{\mathbf{z}}_i^e(\bar{t}_k^+)$, it follows that

$$\bar{\mathbf{z}}_i^e(\bar{t}_k^+) = (\mathbf{O}_i^T \mathbf{O}_i)^{-1} \mathbf{O}_i^T \mathbf{Y}(\bar{t}_k^+),$$

which yields

$$\bar{\mathbf{x}}_a(\bar{t}_k^+) = \begin{bmatrix} \mathbf{0}_{qm \times n_i} & \mathbf{I}_{qm} \end{bmatrix} \bar{\mathbf{z}}_i^e(\bar{t}_k^+) = \bar{\mathbf{\Lambda}}_{\bar{t}_k}. \qquad (13)$$

Case 2 – with uncertainties: By (13), it is clear that $\bar{\mathbf{\Lambda}}_{\bar{t}_k}$ depends on the plant matrices, so that it can be computed only when such matrices are exactly known. In order to deal with possible uncertainties on the plant description, i.e., $\boldsymbol{\beta} \neq \bar{\boldsymbol{\beta}}$, the following rule is used

$$\mathbf{x}_a(\bar{t}_k^+) = e^{-\mathbf{A}_a(\bar{t}_{k-N+1} - \bar{t}_{k-N})} \mathbf{x}_a(\bar{t}_{k-N+1}^-) = \mathbf{\Lambda}_{\bar{t}_k}.$$

Main result

Under the assumption of sufficiently small initial errors, for all $k \in \mathbb{Z}^+$, the switching time t_k of the actual trajectory can be forced to be close to the switching time $\bar{t}_k$ of the desired trajectory so that, by following the guidelines of the control strategy proposed in [20] and [21], the precompensator input $\mathbf{u}_a$ and the control input $\mathbf{u}$ can be expressed by the *switching* laws reported below (see Fig. 2)

$$\mathbf{u}_a(t) = \begin{cases} \mathbf{e}(t), & \forall t \in (t_k^M, t_{k+1}^m),\ k \in \mathbb{Z}^+ \\ \mathbf{0}, & \text{otherwise} \end{cases}, \qquad (14)$$

$$\mathbf{u}(t) = \begin{cases} \mathbf{K}_i^e \mathbf{x}_i^e(t), & \forall t \in (t_k^M, t_{k+1}^m),\ k \in \mathbb{Z}^+ \\ \mathbf{0}, & \text{otherwise} \end{cases}, \qquad (15)$$

where $t_0^M := t_0$, $i = \sigma(t_k^M)$ and, for all $k \in \mathbb{Z}^+$, one defines $t_k^m := \min\{t_k, \bar{t}_k\}$ and $t_k^M := \max\{t_k, \bar{t}_k\}$.

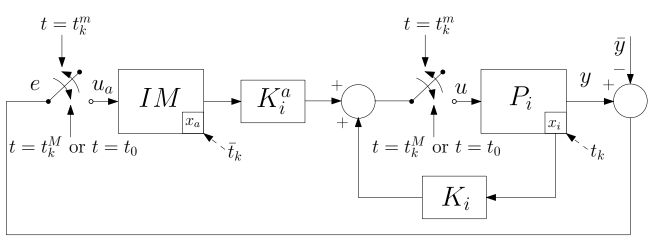


Fig. 2. Structure of the control scheme based on the internal model principle with switching control laws (14) and (15).

Remark 5. Although the proof of our main result goes through even if the control is never switched off, it has been observed through many simulations that the switching control laws given by (14) and (15) improve the behavior of the controlled system as compared with the case in which the control is never switched off, especially when the initial conditions are not particularly close to the desired ones.

At this point, the following result, whose proof has been omitted for lack of space, can be stated.

Theorem 1. *Under assumptions (1)–(3), for all $i \in \mathcal{P}$ there exist $\mathbf{K}_i$ and $\mathbf{K}_i^a$ such that the control law depicted in Fig. 2 is a solution of Problem 1.*

IV. A NUMERICAL EXAMPLE

In this section, a numerical example is given in order to illustrate the effectiveness of the proposed method; for lake of space only the case with uncertainties and estimate of the precompensator jumps is taken into account.

By using the notation introduced in Section II and by defining the number of modes $M = 2$, the switching surfaces parameters

$$\begin{array}{lll} 1 \to 2: & \mathbf{J}_{21} = \begin{bmatrix} -1 & 0 \end{bmatrix}, & b_{21} = 2, \\ 2 \to 1: & \mathbf{J}_{12} = -1, & b_{12} = 1, \\ 1 \to 1: & \mathbf{J}_{11} = \begin{bmatrix} 1 & 0 \end{bmatrix}, & b_{11} = 4, \end{array}$$

and the reset maps

$$\mathbf{\Gamma}_{21} = \begin{bmatrix} 1/2 & 0 \end{bmatrix}, \ \mathbf{\Gamma}_{12} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \ \mathbf{\Gamma}_{11} = \begin{bmatrix} 0 & 0 \\ 0 & -1/2 \end{bmatrix},$$

the hybrid system considered in this example can be represented as shown in Fig. 3.

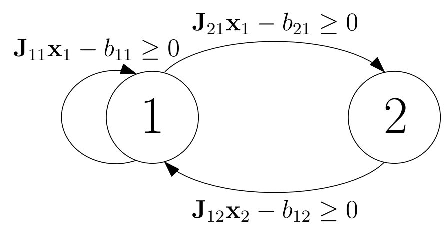


Fig. 3. Linear hybrid system with 2 modes considered for the example.

On the other hand, the continuous dynamics during the free-motion phases are characterized by

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 \\ -1 + \epsilon_1 & -2 + \epsilon_2 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 \\ 1 + \epsilon_3 \end{bmatrix},$$
$$\mathbf{C}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$\mathbf{A}_2 = 3 + \epsilon_4, \quad \mathbf{B}_2 = 2 + \epsilon_5, \quad \mathbf{C}_2 = 1, \quad \mathbf{D}_2 = 0,$$

where $\epsilon_i \in \mathbb{R}$, $i \in \{1, 2, 3, 4, 5\}$ denote possible uncertainties on the parameters of the plant. In the present example, $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = 0$ in the nominal parameters, whereas $\epsilon_1 = -0.5$, $\epsilon_2 = 1$, $\epsilon_3 = 0.7$, $\epsilon_4 = -0.8$ and $\epsilon_5 = 0.5$ in the actual parameters.

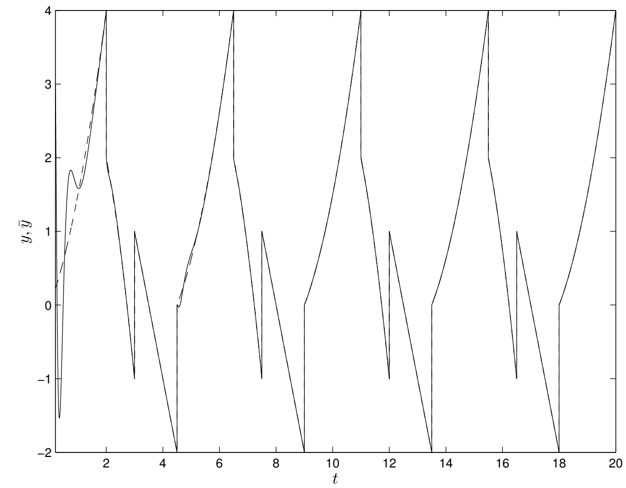
As for the reference trajectory, it can be given in the form of (6) as follows

$$\bar{\mathbf{y}}(t) = \begin{cases} 0.5t^2 + t, & t \in [0, 2), \\ -t^2 + 2t + 2, & t \in [2, 3), \\ -2t + 7, & t \in [3, 4.5), \end{cases}$$

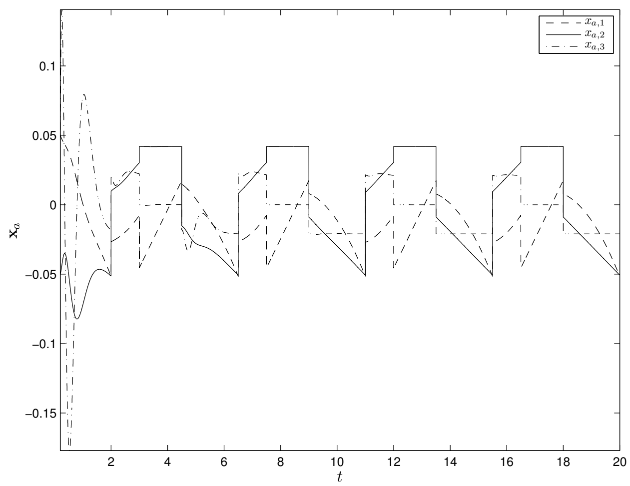
with period $T = 4.5$ and $N = 3$ switching events per period.

At this point, by using the control strategy described in Section III (it is easy to see that all the required assumptions are satisfiable) the static gains $\mathbf{K}_1$, $\mathbf{K}_1^a$ and $\mathbf{K}_2$, $\mathbf{K}_2^a$ are chosen such that η in (11) is at least equal to 3. Starting from the initial time $t_0 = 0.2$ with initial conditions $\mathbf{x}_1(t_0) =$

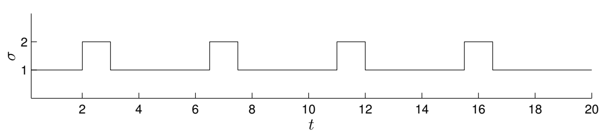
$\begin{bmatrix} 3 & -2 \end{bmatrix}^T$, $\mathbf{x}_2(t_0) = 0$ and $\mathbf{x}_a(t_0) = \begin{bmatrix} 0.05 & -0.05 & 0.05 \end{bmatrix}^T$, the behavior of the controlled trajectory during the first 20 seconds of motion can be observed in Fig. 4.



(a) $\mathbf{y}(t)$ and $\bar{\mathbf{y}}(t)$



(b) Precompensator state vector: $\mathbf{x}_a(t)$



(c) Switching signal: $\sigma(t)$

Fig. 4. The desired (dashed) trajectory and the actual (solid) one (a); time behavior of the precompensator state variables $\mathbf{x}_a(t) = [x_{a,1}\ x_{a,2}\ x_{a,3}]^T$ (b); switching signal relevant to the controlled trajectory (c).

V. CONCLUSIONS

In this paper, a robust trajectory tracking problem of periodic trajectories in hybrid systems having linear dynamics in each operating mode with isolated discrete switching events and linear reset maps is stated and solved. A controller whose state is subject to discontinuities and whose structure is based on the internal model principle is used. Due to the possible presence of uncertainties on the system description, an algorithm to estimate the correct jumps for such a controller is implemented.

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