

\mathcal{H}_∞ Gain Scheduling for Discrete-Time Systems with Control Delays and Time-Varying Parameters: a BMI Approach

Renato A. Borges, Ricardo C. L. F. Oliveira, Chaouki T. Abdallah and Pedro L. D. Peres

Abstract—In this paper, the problem of gain scheduling for time-varying systems with time delays is investigated. By using a memory at the feedback loop, a discrete gain scheduled controller which minimizes an upper bound to the \mathcal{H}_∞ performance of the closed loop system is determined. The design conditions, expressed in terms of bilinear matrix inequalities, are obtained from the Finsler's Lemma combined with the Lyapunov theory. The extra variables introduced by the Finsler's Lemma represent an alternative way in the search of better system behavior. The time-varying uncertainties are modeled using polytopic domains. The controller is obtained by the solution of an optimization problem formulated only in terms of the vertices of the polytope. No grids in the parametric space are used. Numerical examples illustrate the efficiency of the proposed approach.

I. INTRODUCTION

When it comes to designing controllers for time-varying systems, there is no denying the fact that gain scheduling has represented an important issue within control system theory [1, 2]. As shown in [3], this technique can extend the validity of the linearization approach of non-linear systems to a range of operating points. The main idea is to model the system in such a way that these different operating points are parametrized by one or more variables, commonly called scheduling variables [3]. The stability is then guaranteed by a family of linear controllers, whose parameters are changed in accordance with the scheduling rules.

Recently, linear parameter dependent (LPV) systems have been brought into focus due to, primarily, the fact that they are good not only to represent certain classes of nonlinear systems but also to provide an interesting framework for gain scheduling control by means of convex optimization [2, 4–6]. Although there are other articles first addressing the topic of gain scheduling, [7–9] are considered pioneering works. It is worth to stress that for this class of linear systems the dynamic matrices depend on time-varying parameters that are measured online. The parameters, restricted to vary in pre-specified sets, bring extra information during the synthesis step, what may lead to less conservative results when contrasted to robust control structures, for example.

This work is partially supported by the Brazilian agencies CAPES, FAPESP and CNPq.

R. A. Borges (working at the Electrical and Computer Engineering Department, University of New Mexico), R. C. L. F. Oliveira and P. L. D. Peres are with the School of Electrical and Computer Engineering, University of Campinas, CP 6101, 13081-970, Campinas, SP, Brazil {renato, ricfow, peres}@dt.fee.unicamp.br.

C. T. Abdallah is with the Electrical and Computer Engineering Department, MSC01 1100 1 University of New Mexico, Albuquerque, NM, 87131-0001 USA chaouki@ece.unm.edu.

Lately, the Lyapunov theory has been used as a main tool to deal with synthesis of gain scheduled controllers. In many cases, it might be possible to express the design conditions as an optimization problem in terms of linear matrix inequalities (LMIs), which can be numerically handled by powerful softwares [10, 11]. As a way to guarantee robustness against practical disturbances, the \mathcal{H}_2 and \mathcal{H}_∞ norms have been frequently applied as indexes of performance. Recent works include [12] where the problem of stabilizability and \mathcal{H}_∞ control of discrete-time LPV systems is investigated by means of gain scheduled state feedback, [13] in which gain scheduling for linear fractional transformation (LFT) systems is designed by using parameter dependent Lyapunov functions, [14] where gain scheduled \mathcal{H}_2 controllers for affine LPV systems are proposed, [15] in which robust and gain scheduled controllers for LFT parameter-dependent systems are designed by using duality theory, [16] where switching \mathcal{H}_∞ controllers for a class of LPV systems scheduled along a measurable parameter trajectory are addressed, among others.

Bilinear matrix inequalities (BMIs) have also been applied in the study of control of LPV systems. It is well-known that optimization problems expressed in terms of BMIs are non-convex. Nevertheless, the use of BMIs may represent a good strategy to face problems with either no solution or only sufficient conditions available in the literature. See, for instance, [17–20] and references therein.

Another important aspect observed in a large number of dynamic models, including LPV plants, is the presence of time delays. A good characterization of time delays is always required since they may represent a source of instability to the system trajectories. Considering this framework, the Lyapunov-Krasovskii functional has been widely used to cope with system delays. However, in general, this strategy requires a bigger computational burden due to its more complex structure. An alternative way to surpass this difficulty is the use of memory controllers in the feedback loop.

The aim of this paper is to provide gain scheduled memory controllers to stabilize LPV systems subject to time delays. The Lyapunov theory is applied in order to obtain the stability conditions of the closed-loop system. An \mathcal{H}_∞ guaranteed cost, which reflects the worst-case energy gain of the system, provides robustness with respect to unmodeled uncertainties. A parameter dependent Lyapunov function is used to reduce the conservatism of the proposed method, resulting in a more general approach when compared to methods based on quadratic stability. Extra variables introduced by the Finsler's Lemma, that may be freely explored in the search

for better performance of the LPV system, lead to design conditions expressed in terms of BMIs. By incorporating the LMIs related to the bounded real lemma to the BMIs conditions provided, \mathcal{H}_∞ robust memory controllers can be obtained. Some results from the literature concerned with stability without time delays can be obtained as a particular case of the proposed method. The use of a memory in the feedback loop allows one to cope with time delays, even when they are time-varying, without making use of more complex Lyapunov functionals. All the system matrices are assumed to be affected by the time-varying parameters, which are supposed to lie inside polytopic domains. The gain scheduled memory controller is then obtained by the solution of an optimization problem that minimizes an upper bound to the \mathcal{H}_∞ index of performance subject to a finite number of BMI constraints formulated only in terms of the vertices of a polytope. No grids in the parametric space are used. Numerical examples illustrate the efficiency of the proposed results.

II. PRELIMINARIES AND PROBLEM STATEMENT

Consider the time-varying discrete-time system, $x(0) = 0$ and $u(\theta) = 0$ for $\theta \in \{-\tau, 0\}$

$$\begin{aligned} x(k+1) &= A(\alpha(k))x(k) + B_{du}(\alpha(k))u(k-\tau) \\ &\quad + B_u(\alpha(k))u(k) + B_w(\alpha(k))w(k) \\ y(k) &= C(\alpha(k))x(k) + D_{du}(\alpha(k))u(k-\tau) \\ &\quad + D_u(\alpha(k))u(k) + D_w(\alpha(k))w(k) \end{aligned} \quad (1)$$

where τ represents the discrete-time delay, $x(k) \in \mathbb{R}^n$ is the state space vector, $u(k) \in \mathbb{R}^m$ is the control signal, $w(k) \in \mathbb{R}^r$ is the $l_2[0, \infty)$ noise and $y(k) \in \mathbb{R}^q$ is the output. The time-varying vector of parameters $\alpha(k)$ belongs to the unit simplex for all $k \geq 0$

$$\mathcal{U} = \left\{ \alpha \in \mathbb{R}^N : \sum_{i=1}^N \alpha_i = 1, \alpha_i \geq 0, i = 1, \dots, N, \right\}$$

All matrices are real, with appropriate dimensions, belonging to the polytope¹

$$\mathcal{P} \triangleq \left\{ \left[\begin{array}{c|c} A(\alpha) & B_u(\alpha) \\ \hline B_{du}(\alpha) & B_w(\alpha) \\ \hline C(\alpha) & D_u(\alpha) \\ \hline D_{du}(\alpha) & D_w(\alpha) \end{array} \right] = \sum_{i=1}^N \alpha_i \left[\begin{array}{c|c} A_i & B_{ui} \\ \hline B_{dui} & B_{wi} \\ \hline C_i & D_{ui} \\ \hline D_{dui} & D_{wi} \end{array} \right] \right\} \quad (2)$$

More specifically, the system matrices are given, for any time $k \geq 0$, by the convex combination of the well-defined vertices of the polytope \mathcal{P} . It is also assumed that the parameters $\alpha(k)$ are measured online, and their variation rates, $\Delta\alpha$, are unknown.

In order to guarantee the stability of system (1), a memory state feedback controller with a parameter-dependent gain is designed. Using an extra state variable $z(k)$ to store the delayed value of the control signal, $u(k-\tau)$, system (1) can

be rewritten as follows [21]

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}(\alpha)\tilde{x}(k) + \tilde{B}_u(\alpha)u(k) + \tilde{B}_w(\alpha)w(k) \\ y(k) &= \tilde{C}(\alpha)\tilde{x}(k) + \tilde{D}_u(\alpha)u(k) + \tilde{D}_w(\alpha)w(k) \end{aligned} \quad (3)$$

where $\tilde{x}(k) = [x(k)' \ z(k)']'$ and

$$\begin{aligned} \tilde{A}(\alpha) &= \begin{bmatrix} A(\alpha) & B_{du}(\alpha) \\ 0 & 0 \end{bmatrix}, \tilde{B}_u(\alpha) = \begin{bmatrix} B_u(\alpha) \\ I \end{bmatrix}, \\ \tilde{B}_w(\alpha) &= [B_w(\alpha)' \ 0]', \tilde{C}(\alpha) = [C(\alpha) \ D_{du}(\alpha)], \\ \tilde{D}_u(\alpha) &= D_u(\alpha), \tilde{D}_w(\alpha) = D_w(\alpha) \end{aligned} \quad (4)$$

The memory control law is given by

$$\begin{aligned} u(k) &= K_x(\alpha)x(k) + K_d(\alpha)u(k-\tau) \\ &= [K_x(\alpha) \ K_d(\alpha)] \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} \end{aligned} \quad (5)$$

and the closed-loop system by

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}_{cl}(\alpha)\tilde{x}(k) + \tilde{B}_{wcl}(\alpha)w(k) \\ y(k) &= \tilde{C}_{cl}(\alpha)\tilde{x}(k) + \tilde{D}_{wcl}(\alpha)w(k) \end{aligned} \quad (6)$$

with

$$\begin{aligned} \tilde{A}_{cl}(\alpha) &= \tilde{A}(\alpha) + \tilde{B}_u(\alpha)K(\alpha), \tilde{B}_{wcl}(\alpha) = \tilde{B}_w(\alpha), \\ \tilde{C}_{cl}(\alpha) &= \tilde{C}(\alpha) + \tilde{D}_u(\alpha)K(\alpha), \tilde{D}_{wcl}(\alpha) = \tilde{D}_w(\alpha) \end{aligned} \quad (7)$$

where $K(\alpha) = [K_x(\alpha) \ K_d(\alpha)]$. The whole of possible outcomes for the parameter-dependent gain (5) belongs to the polytope

$$\tilde{\mathcal{P}} \triangleq \left\{ K(\alpha) = \sum_{i=1}^N \alpha_i K_i = \sum_{i=1}^N \alpha_i [K_{xi} \ K_{di}] \right\}, \alpha \in \mathcal{U} \quad (8)$$

The control problem to be dealt with can be stated as follows.

Problem 1: Find matrices $K_{xi} \in \mathbb{R}^{m \times n}$ and $K_{di} \in \mathbb{R}^{m \times n}$ of the control law (5), such that the closed-loop system (6) is asymptotically stable, and an upper bound $\gamma > 0$ to the \mathcal{H}_∞ performance is minimized, that is, for all $k \in \mathbb{N}$

$$\sup_{w(k) \neq 0} \frac{\|y(k)\|_2^2}{\|w(k)\|_2^2} < \gamma^2 \quad (9)$$

with $w(k) \in l_2[0, \infty)$.

Before proceeding to the solution of Problem 1, a previous result is needed.

Lemma 1: (Finsler) Let $\xi \in \mathbb{R}^a$, $\mathcal{Q} = \mathcal{Q}' \in \mathbb{R}^{a \times a}$, $\mathcal{B} \in \mathbb{R}^{b \times a}$ with $\text{rank}(\mathcal{B}) < a$, and \mathcal{B}^\perp a basis for the null-space of \mathcal{B} (i.e. $\mathcal{B}\mathcal{B}^\perp = 0$). The following statements are equivalent.

- i) $\xi' \mathcal{Q} \xi < 0, \forall \mathcal{B} \xi = 0, \xi \neq 0$;
- ii) $\mathcal{B}^{\perp'} \mathcal{Q} \mathcal{B}^\perp < 0$;
- iii) $\exists \mu \in \mathbb{R} : \mathcal{Q} - \mu \mathcal{B}' \mathcal{B} < 0$;
- iv) $\exists \mathcal{X} \in \mathbb{R}^{a \times b} : \mathcal{Q} + \mathcal{X} \mathcal{B} + \mathcal{B}' \mathcal{X}' < 0$.

Proof: See [22]. ■

By applying the Bounded Real Lemma [10], combined with the Finsler's Lemma (1), condition (9) can be guaranteed as follows.

¹The time dependence of $\alpha(k)$ will be omitted to lighten the notation.

Lemma 2: For a given $\gamma > 0$, if there exists a parameter-dependent matrix $P(\alpha)' = P(\alpha) > 0$ such that the statements of Lemma 1 are satisfied for

$$\mathcal{Q} = \begin{bmatrix} P(\alpha_+) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -P(\alpha) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma^{-1} \tilde{B}_{wcl}(\alpha) \tilde{B}_{wcl}(\alpha)' & \gamma^{-1} \tilde{B}_{wcl}(\alpha) \tilde{D}_{wcl}(\alpha)' \\ \mathbf{0} & \gamma^{-1} \tilde{D}_{wcl}(\alpha) \tilde{B}_{wcl}(\alpha)' & \gamma^{-1} \tilde{D}_{wcl}(\alpha) \tilde{D}_{wcl}(\alpha)' - \gamma \mathbf{I} \end{bmatrix}$$

$$\mathcal{B} = [-\mathbf{I} \quad \tilde{A}_{cl}(\alpha)' \quad \tilde{C}_{cl}(\alpha)'],$$

$$\mathcal{B}^\perp = \begin{bmatrix} \tilde{A}_{cl}(\alpha)' & \tilde{C}_{cl}(\alpha)' \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \xi = [\tilde{x}(k+1)' \quad \tilde{x}(k)' \quad w(k)']'$$

where $\alpha_+ = \alpha(k+1)$, then the closed-loop system (6) is asymptotically stable with an upper bound $\gamma > 0$ to the \mathcal{H}_∞ performance.

Proof: Let $v(k) = \tilde{x}(k)' P(\alpha) \tilde{x}(k)$ be a parameter-dependent Lyapunov function. Considering the dual system (i.e. $\tilde{A}_{cl} = \tilde{A}'_{cl}$, $\tilde{B}_{wcl} = \tilde{C}'_{cl}$, $\tilde{C}_{cl} = \tilde{B}'_{wcl}$ and $\tilde{D}_{wcl} = \tilde{D}'_{wcl}$), it is straightforward from statement *i*) of Lemma 1 that Lemma 2 ensures $v(k) > 0$ and

$$\Delta v(k) < -\gamma^{-1} y(k)' y(k) + \gamma w(k)' w(k)$$

with the choice $\xi = [\tilde{x}(k+1)' \quad \tilde{x}(k)' \quad w(k)']'$. The last inequality comes from $\Delta v(k) < 0$ and

$$y(k)' y(k) - \gamma^2 w(k)' w(k) < 0$$

by applying the Bounded Real Lemma. Therefore, system (6) has an upper bound γ to the \mathcal{H}_∞ performance and, from the Lyapunov theory [23], is asymptotically stable. ■

The conditions of Lemma 2 exhibit nonlinearities and must be tested at all points of the simplex \mathcal{U} , i.e., at an infinite number of points. Hence, the main goal hereafter is to obtain finite-dimensional BMI conditions in terms of the vertices of the polytope \mathcal{P} to solve Problem 1. Using Schur complement, change of variables and exploring the extra variables provided by Lemma 1, parameter-dependent BMIs assuring the existence of such controllers are given in the next section.

III. MAIN RESULTS

Theorem 1: (\mathcal{H}_∞ MEMORY GAIN SCHEDULING) Given the augmented discrete-time system (3), if there exist matrices $L_i \in \mathbb{R}^{m \times (n+m)}$, $H_i \in \mathbb{R}^{q \times (n+m)}$, F_i , G_i , $P_i = P'_i > 0 \in \mathbb{R}^{(n+m) \times (n+m)}$, $i = 1, \dots, N$ and a scalar $\gamma > 0$ such that²

$$\Xi_{ij} \triangleq \begin{bmatrix} P_j - F_i - F'_i & \mathcal{F}_{12} & F_i \tilde{C}'_i + L'_i \tilde{D}'_{ui} - F'_i H'_j & \mathbf{0} \\ (\star) & \mathcal{F}_{22} & \mathcal{F}_{23} & \tilde{B}_{wi} \\ (\star) & (\star) & \mathcal{F}_{33} & \tilde{D}_{wi} \\ (\star) & (\star) & (\star) & -\gamma \mathbf{I} \end{bmatrix} < 0 \quad (10)$$

²The term (\star) indicates symmetric blocks in the LMIs.

$$i = 1, \dots, N, \quad j = 1, \dots, N$$

$$\mathcal{F}_{12} = F_i \tilde{A}'_i + L'_i \tilde{B}'_{ui} - F'_i G'_j,$$

$$\mathcal{F}_{22} = G_j F_i \tilde{A}'_i + \tilde{A}_i F'_i G'_j + G_j L'_i \tilde{B}'_{ui} + \tilde{B}_{ui} L_i G'_j - P_i,$$

$$\mathcal{F}_{23} = G_j F_i \tilde{C}'_i + G_j L'_i \tilde{D}'_{ui} + \tilde{A}_i F'_i H'_j + \tilde{B}_{ui} L_i H'_j,$$

$$\mathcal{F}_{33} = H_j F_i \tilde{C}'_i + \tilde{C}_i F'_i H'_j + H_j L'_i \tilde{D}'_{ui} + \tilde{D}_{ui} L_i H'_j - \gamma \mathbf{I}$$

$$\Xi_{ikj} \triangleq \begin{bmatrix} \hat{\mathcal{F}}_{11} & \hat{\mathcal{F}}_{12} & \hat{\mathcal{F}}_{13} & \mathbf{0} \\ (\star) & \hat{\mathcal{F}}_{22} & \hat{\mathcal{F}}_{23} & \tilde{B}_{wi} + \tilde{B}_{wk} \\ (\star) & (\star) & \hat{\mathcal{F}}_{33} & \tilde{D}_{wi} + \tilde{D}_{wk} \\ (\star) & (\star) & (\star) & -2\gamma \mathbf{I} \end{bmatrix} < 0 \quad (11)$$

$$i = 1, \dots, N-1, \quad k = i+1, \dots, N, \quad j = 1, \dots, N,$$

$$\hat{\mathcal{F}}_{11} = 2P_j - F_i - F'_i - F_k - F'_k$$

$$\hat{\mathcal{F}}_{12} = F_i \tilde{A}'_k + F_k \tilde{A}'_i + L'_i \tilde{B}'_{uk} + L'_k \tilde{B}'_{ui} - F'_i G'_j - F'_k G'_j,$$

$$\hat{\mathcal{F}}_{13} = F_i \tilde{C}'_k + F_k \tilde{C}'_i + L'_i \tilde{D}'_{uk} + L'_k \tilde{D}'_{ui} - F'_i H'_j - F'_k H'_j$$

$$\hat{\mathcal{F}}_{22} = G_j (F_i \tilde{A}'_k + F_k \tilde{A}'_i) + (\tilde{A}_i F'_k + \tilde{A}_k F'_i) G'_j$$

$$+ G_j (L'_i \tilde{B}'_{uk} + L'_k \tilde{B}'_{ui}) + (\tilde{B}_{ui} L_k + \tilde{B}_{uk} L_i) G'_j - P_i - P_k,$$

$$\hat{\mathcal{F}}_{23} = G_j (F_i \tilde{C}'_k + F_k \tilde{C}'_i) + G_j (L'_i \tilde{D}'_{uk} + L'_k \tilde{D}'_{ui})$$

$$+ (\tilde{A}_i F'_k + \tilde{A}_k F'_i) H'_j + (\tilde{B}_{ui} L_k + \tilde{B}_{uk} L_i) H'_j,$$

$$\hat{\mathcal{F}}_{33} = H_j (F_i \tilde{C}'_k + F_k \tilde{C}'_i) + (\tilde{C}_i F'_k + \tilde{C}_k F'_i) H'_j$$

$$+ H_j (L'_i \tilde{D}'_{uk} + L'_k \tilde{D}'_{ui}) + (\tilde{D}_{ui} L_k + \tilde{D}_{uk} L_i) H'_j - 2\gamma \mathbf{I}$$

then there exists a memory control law (5), ensuring the asymptotic stability of the closed-loop system (6) and an \mathcal{H}_∞ guaranteed cost γ , with matrices K_i (8) given by

$$K_i = L_i (F'_i)^{-1}, \quad i = 1, \dots, N. \quad (12)$$

Proof: Firstly, applying the following operation [24]

$$\Xi(\alpha) = \sum_{j=1}^N \alpha_j \left\{ \sum_{i=1}^N \alpha_i^2 \Xi_{ij} + \sum_{i=1}^{N-1} \sum_{k=i+1}^N \alpha_i \alpha_k \Xi_{ikj} \right\} \quad (13)$$

to the BMIs (10) and (11) with the change of variables $L(\alpha) = K(\alpha) F(\alpha)'$ it follows that

$$\Xi(\alpha) = \begin{bmatrix} \hat{\mathcal{F}}_{11} & \hat{\mathcal{F}}_{12} & \hat{\mathcal{F}}_{13} & \mathbf{0} \\ (\star) & \hat{\mathcal{F}}_{22} & \hat{\mathcal{F}}_{23} & \tilde{B}_{wcl}(\alpha) \\ (\star) & (\star) & \hat{\mathcal{F}}_{33} & \tilde{D}_{wcl}(\alpha) \\ (\star) & (\star) & (\star) & -\gamma \mathbf{I} \end{bmatrix} < 0 \quad (14)$$

$$\hat{\mathcal{F}}_{11} = P(\alpha_+) - F(\alpha) - F(\alpha)',$$

$$\hat{\mathcal{F}}_{12} = F(\alpha) \tilde{A}_{cl}(\alpha)' - F(\alpha)' G(\alpha_+)$$

$$\hat{\mathcal{F}}_{13} = F(\alpha) \tilde{C}_{cl}(\alpha)' - F(\alpha)' H(\alpha_+)$$

$$\hat{\mathcal{F}}_{22} = G(\alpha_+) F(\alpha) \tilde{A}_{cl}(\alpha)' + \tilde{A}_{cl}(\alpha) F(\alpha)' G(\alpha_+) - P(\alpha)$$

$$\hat{\mathcal{F}}_{23} = G(\alpha_+) F(\alpha) \tilde{C}_{cl}(\alpha)' + \tilde{A}_{cl}(\alpha) F(\alpha)' H(\alpha_+)$$

$$\hat{\mathcal{F}}_{33} = H(\alpha_+) F(\alpha) \tilde{C}_{cl}(\alpha)' + \tilde{C}_{cl}(\alpha) F(\alpha)' H(\alpha_+) - \gamma \mathbf{I}$$

Using Schur complement, inequality (14) can be rewritten as follows

$$\begin{bmatrix} \hat{\mathcal{F}}_{11} & \hat{\mathcal{F}}_{12} & F \tilde{C}_{cl}(\alpha)' - F' H(\alpha_+)' \\ (\star) & \hat{\mathcal{F}}_{22} & \hat{\mathcal{F}}_{23} \\ (\star) & (\star) & \hat{\mathcal{F}}_{33} \end{bmatrix} + \gamma^{-1} \hat{\mathcal{F}}_4(\alpha) \hat{\mathcal{F}}_4(\alpha)' < 0 \quad (15)$$

where

$$\hat{\mathcal{F}}_4(\alpha) = [\mathbf{0} \quad \tilde{B}_{wcl}(\alpha)' \quad \tilde{D}_{wcl}(\alpha)']'$$

By defining $\mathcal{X} = [F(\alpha)' \quad F(\alpha)' G(\alpha_+)' \quad F(\alpha)' H(\alpha_+)]'$ inequality (15) yields statement *iv*) of Lemma 1 with \mathcal{L}

\mathcal{B} and ξ as in Lemma 2. Lastly, the parameter-dependent gain $K(\alpha)$ is obtained by the change of variables $L(\alpha) = K(\alpha)F(\alpha)'$, what concludes the proof. ■

Corollary 1: The minimum γ attainable by the conditions of Theorem 1 is given by the optimization problem

$$\min \gamma \text{ s.t. (10), (11)} \quad (16)$$

From this point, some remarks are in order.

A. Remarks

The use of memory controller brings some advantages when dealing with discrete time-delay systems. Using a new variable to store the past values of the control signal, it was possible to cope with Problem 1 without applying more complex Lyapunov functions, (for instance, the Lyapunov-Krasovskii functional). Sophisticated Lyapunov functionals may lead to conditions that requires a bigger computational effort to be solved.

By setting $\tilde{A}(\alpha) = A(\alpha)$, $\tilde{B}_u(\alpha) = B_u(\alpha)$, $\tilde{B}_w(\alpha) = B_w(\alpha)$, $\tilde{C}(\alpha) = C(\alpha)$ and $K(\alpha) = K_x(\alpha)$ the conditions of Theorem 1 can be directly applied when no time-delays are considered. In this context, gain scheduled control of discrete-time systems with time-varying parameters was also addressed by means of affinely parameter-dependent Lyapunov functions in [25, 26] and improved in [12] to cope with systems in which all state space matrices are supposed to be affected by time-varying parameters. In the above works, the design conditions are given in terms of LMIs. In this paper, however, statement *iv*) in Lemma 1 is applied to reach more general BMI conditions with multipliers defined as in Lemma 2 and $\mathcal{X} = [F(\alpha)' \ F(\alpha)'G(\alpha_+) \ F(\alpha)'H(\alpha_+)']'$. The advantages of this approach are due to the extra variables that can be used in the search for better performance of the closed-loop system. For example, a lower \mathcal{H}_∞ guaranteed cost may be obtained exploring the new variables $G(\alpha_+)$ and $H(\alpha_+)$. In this sense, Lemma 2 encompasses the conditions in [25]. Further, by choosing $G(\alpha_+) = \mathbf{0}$ and $H(\alpha_+) = \mathbf{0}$ the conditions of Theorem 1 reduce to the ones proposed in [12].

Although other methods could be applied to solve problem (16), the following algorithm is proposed. Fix the variables H_i and G_i , minimize w.r.t. γ , F_i , L_i and P_i , get the new values of F_i , L_i and P_i . Then, fix the variables F_i , L_i and P_i , minimize w.r.t. γ , H_i and G_i , get the new values of H_i and G_i . Repeat this procedure until no significant changes in the value of γ occur. This approach is sometimes called an Alternating Semi-Definite Programming (or Gauss-Seidel) method [17]. At each step a convex optimization problem in terms of LMI conditions is solved. It is worth stressing here that Theorem 1 is not concerned with new strategies to solve BMIs. Whenever feasible, other methods from the literature can be applied to solve Corollary 1, as the ones appeared in [17–20].

By setting the variables $G(\alpha_+)$ and $H(\alpha_+)$ at time $k+1$ ($\alpha_+ = \alpha(k+1)$) all products involving three parameter-dependent matrices appeared at the BMIs (10) and (11) occur with one matrix at a different instant of time. As a consequence, the number of BMIs and the computational

time required to solve the optimization problem (16) are reduced. If Theorem 1 was written with $G(\cdot)$ and $H(\cdot)$ at time k , a more sophisticated procedure, as the one proposed in [27], should be applied in order to get the BMI conditions expressed only in terms of the vertices of the polytope, resulting in a larger number of BMIs.

The conditions of Theorem 1 are directly applicable to discrete-time systems whose matrices depend affinely on the vector of time-varying parameters, since this class of systems has a polytopic representation whenever the parameters are bounded [28].

B. Robust Control Design

Finally, by fixing the variable matrices $F_i = F$ and $L_i = L$ (not depending on α), \mathcal{H}_∞ robust memory controllers can be obtained using the conditions of Theorem 1, as stated in the next corollary.

Corollary 2: (\mathcal{H}_∞ ROBUST MEMORY CONTROLLER) Given the augmented discrete-time system (3), if BMI (10) of Theorem 1 is feasible with fixed variable matrices $L \in \mathbb{R}^{m \times (n+m)}$, and $F \in \mathbb{R}^{(n+m) \times (n+m)}$ then the closed-loop system (6) is asymptotically stable with a robust memory controller $K = L(F')^{-1}$ and an \mathcal{H}_∞ guaranteed cost γ .

Note that BMI (11) is not necessary anymore, since in this case there is no product involving three parameter-dependent variables. All products appeared in the conditions of Corollary 2 occur at different instants of time.

IV. NUMERICAL EXPERIMENTS

All the experiments have been performed in a Pentium IV 2.6 GHz, 512 MB RAM, using the LMI Control Toolbox [11].

Example 1

Consider the discrete-time system (3) with vertices (borrowed from [12, Example 2]) given by

$$\begin{aligned} \tilde{A}_1 &= \begin{bmatrix} 0.28 & -0.315 \\ 0.63 & -0.84 \end{bmatrix}, \tilde{A}_2 = \begin{bmatrix} 0.52 & 0.77 \\ -0.7 & -0.07 \end{bmatrix}, \\ \tilde{B}_{w1} = \tilde{B}_{w2} &= [1 \ 0]', \tilde{B}_{u1} = [1 \ 0]', \tilde{B}_{u2} = [0 \ 1]', \\ \tilde{C}_1 = \tilde{C}_2 &= [1 \ 0], \tilde{D}_{w1} = \tilde{D}_{w2} = \tilde{D}_{u1} = \tilde{D}_{u2} = [0]' \end{aligned}$$

This system was also studied in [26], but in a simpler case where matrix \tilde{B}_u was fixed and time-invariant (i.e. $\tilde{B}_{u1} = \tilde{B}_{u2}$). The aim here is to compare the gain-scheduling design conditions from [12] with the BMI approach proposed in Theorem 1 by solving an example from the literature. Table I gives the details concerning the improvements of the BMI approach over [12] as the number of iterations evolve. The computational times are given in seconds and only the time required to solve the LMIs is considered. The time necessary to build the set of LMIs is not considered since it highly depends on the LMI parser interface.

As it can be seen in Table I, the \mathcal{H}_∞ upper bound γ was reduced in approximately 24.78% with 5 iterations, providing better rejection of disturbances.

TABLE I

RESULTS AND NUMERICAL COMPLEXITY ASSOCIATED TO THE METHODS [12] AND THE CONDITIONS OF THEOREM 1 IN THE GAIN-SCHEDULING CONTROL DESIGN GIVEN IN EXAMPLE I. THE COMPUTATIONAL TIME (IN SECONDS) RESULTING FROM THEOREM 1 IS THE ACCUMULATED TIME AS THE NUMBER OF ITERATIONS EVOLVE.

Method	γ	Improvement	Time
[12]	3.8754	–	0.07
$T1_{it=1}$	3.5400	8.65 %	0.06
$T1_{it=2}$	2.8756	25.79 %	0.14
$T1_{it=3}$	2.8698	25.94 %	0.21
$T1_{it=4}$	2.8562	26.29 %	0.31
$T1_{it=5}$	2.8508	26.43 %	0.39
$T1_{it=6}$	2.8434	26.62 %	0.46

A. Example II

Consider a discrete-time system in the form (3), with vertex matrices given by

$$\tilde{A}_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \tilde{A}_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, [\tilde{B}_{u1} \quad \tilde{B}_{u2}] = \begin{bmatrix} -4 & 1 \\ 2 & 3 \end{bmatrix},$$

$$\tilde{B}_{wi} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tilde{C}_i = [3 \quad 1], \tilde{D}_{wi} = 0, \tilde{D}_{ui} = -0.1, i = 1, 2$$

The conditions proposed in Theorem 1 and the conditions proposed in [12, Theorem 2] are compared again. Although there is no difference in terms of system characteristics with respect to Example I, this case emphasis the improvement provided by the BMI approach, where the \mathcal{H}_∞ upper bound was reduced from 27.3961 to 7.0508. The results are shown in Table II.

TABLE II

RESULTS AND NUMERICAL COMPLEXITY ASSOCIATED TO THE METHODS [12] AND TO THE CONDITIONS OF THEOREM 1 IN THE GAIN-SCHEDULING CONTROL DESIGN GIVEN IN EXAMPLE II. THE COMPUTATIONAL TIME (IN SECONDS) RESULTING FROM THEOREM 1 IS THE ACCUMULATED TIME AS THE NUMBER OF BMI ITERATIONS EVOLVES.

Method	γ	Improvement	Time
[12]	27.3961	–	0.17
$T1_{it=1}$	24.6767	9.92 %	0.06
$T1_{it=2}$	12.8722	53.01 %	0.14
$T1_{it=3}$	11.6024	57.64 %	0.21
$T1_{it=4}$	10.7193	60.87 %	0.29
$T1_{it=5}$	9.7705	64.33 %	0.37
$T1_{it=6}$	8.9520	67.32 %	0.45
$T1_{it=7}$	8.2562	69.86 %	0.53
$T1_{it=8}$	7.7623	71.66 %	0.60
$T1_{it=9}$	7.3568	73.14 %	0.68
$T1_{it=10}$	7.0508	74.26 %	0.76

Again, it is clear that the BMI approach of Theorem 1 can significantly improve the results when compared to the method from [12]. The price to be paid is the increase in the computational burden accordingly to the number of iterations.

V. CONCLUSION

The \mathcal{H}_∞ gain scheduled memory controller for LPV systems with time delays belonging to a polytope has been

addressed in this paper. A sufficient condition has been proposed in terms of BMIs described only at the vertices of the polytope. Extra variables provided by the Finsler's Lemma were used to derive the BMI conditions. The controller design is accomplished by means of an optimization problem where all system matrices are considered to be affected by time-varying parameters. Efficient numerical algorithms can be used in the solution of the proposed method. The memory of the controller, used to store the previous values of the control signal, was modeled as a new state-space variable leading to an augmented system representation. The proposed approach also provides some improvements when compared with other methods from the literature in the context of discrete-time systems without delays. An extension to deal with the design of \mathcal{H}_∞ robust memory controllers has also been given.

REFERENCES

- [1] D. J. Leith and W. E. Leithead, "Survey of gain-scheduling analysis and design," *Int. J. Contr.*, vol. 73, no. 11, pp. 1001–1025, July 2000.
- [2] W. J. Rugh and J. S. Shamma, "Research on gain scheduling," *Automatica*, vol. 36, no. 10, pp. 1401–1425, October 2000.
- [3] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice Hall, 1996.
- [4] J. S. Shamma and M. Athans, "Analysis of gain scheduled control for nonlinear plants," *IEEE Trans. Automat. Contr.*, vol. 35, no. 8, pp. 898–907, August 1990.
- [5] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, UK: Cambridge University Press, 2004.
- [6] R. T. Rockafellar, *Convex Analysis*. Princeton University Press, 1970.
- [7] J. S. Shamma and M. Athans, "Guaranteed properties of gain scheduled control for linear parameter-varying plants," *Automatica*, vol. 27, no. 3, pp. 559–564, 1991.
- [8] P. Apkarian and P. Gahinet, "A convex characterization of gain-scheduled \mathcal{H}_∞ controllers," *IEEE Trans. Automat. Contr.*, vol. 40, no. 5, pp. 853–864, May 1995.
- [9] P. Apkarian, P. Gahinet, and G. Becker, "Self-scheduled \mathcal{H}_∞ control of linear parameter-varying systems: a design example," *Automatica*, vol. 31, no. 9, pp. 1251–1261, 1995.
- [10] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM Studies in Applied Mathematics, 1994.
- [11] P. Gahinet, A. Nemirovskii, A. J. Laub, and M. Chilali, *LMI Control Toolbox User's Guide*. The Math Works Inc., Natick, MA, 1995.
- [12] V. F. Montagner, R. C. L. F. Oliveira, V. J. S. Leite, and P. L. D. Peres, "Gain scheduled state feedback control of discrete systems with time-varying uncertainties: an LMI approach," in *Proc. 44th IEEE Conf. Decision Contr. — Eur. Control Conf. ECC 2005*, Seville, Spain, December 2005, pp. 4305–4310.
- [13] F. Wu and K. Dong, "Gain-scheduled control of LFT systems using parameter-dependent Lyapunov functions," *Automatica*, vol. 42, no. 1, pp. 39–50, January 2006.
- [14] C. E. de Souza and A. Trofino, "Gain-scheduled \mathcal{H}_2 controller synthesis for linear parameter varying systems via parameter-dependent Lyapunov functions," *Int. J. Robust Nonlinear Contr.*, vol. 16, no. 5, pp. 243–257, March 2006.
- [15] K. Dong and F. Wu, "Robust and gain-scheduling control of LFT systems through duality and conjugate Lyapunov functions," *Int. J. Contr.*, vol. 80, no. 4, pp. 555–568, 2007.
- [16] P. Yan and H. Ozbay, "On switching \mathcal{H}_∞ controllers for a class of linear parameter varying systems," *Syst. Contr. Lett.*, vol. 56, no. 7–8, pp. 504–511, 2007.
- [17] M. Fukuda and M. Kojima, "Branch-and-cut algorithms for the bilinear matrix inequality eigenvalue problem," *Computat. Optim. Appl.*, vol. 19, no. 1, pp. 79–105, 2001.
- [18] F. Zheng, Q. G. Wang, and T. H. Lee, "A heuristic approach to solving a class of bilinear matrix inequality problems," *Syst. Contr. Lett.*, vol. 47, no. 2, pp. 111–119, October 2002.

- [19] H. D. Tuan and P. Apkarian, "Low nonconvexity-rank bilinear matrix inequalities: algorithms and applications in robust controller and structure designs," *IEEE Trans. Automat. Contr.*, vol. 45, no. 11, pp. 2111–2117, November 2002.
- [20] S. Kanev, C. Scherer, M. Verhaegen, and B. D. Schutter, "Robust output-feedback controller design via local BMI optimization," *Automatica*, vol. 40, no. 7, pp. 1115–1127, July 2004.
- [21] K. J. Åström and B. Wittenmark, *Computer Controlled Systems: Theory and Design*. Englewood Cliffs, NJ: Prentice Hall Inc., 1984.
- [22] M. C. de Oliveira and R. E. Skelton, "Stability tests for constrained linear systems," in *Perspectives in Robust Control*, ser. Lecture Notes in Control and Information Science, S. O. Reza Moheimani, Ed. New York: Springer-Verlag, 2001, vol. 268, pp. 241–257.
- [23] M. Vidyasagar, *Nonlinear Systems Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1993.
- [24] D. C. W. Ramos and P. L. D. Peres, "An LMI condition for the robust stability of uncertain continuous-time linear systems," *IEEE Trans. Automat. Contr.*, vol. 47, no. 4, pp. 675–678, April 2002.
- [25] J. Daafouz and J. Bernussou, "Parameter dependent Lyapunov functions for discrete time systems with time varying parameter uncertainties," *Syst. Contr. Lett.*, vol. 43, no. 5, pp. 355–359, August 2001.
- [26] —, "Poly-quadratic stability and \mathcal{H}_∞ performance for discrete systems with time varying uncertainties," in *Proc. 40th IEEE Conf. Decision Contr.*, vol. 1, 2001, pp. 267–272.
- [27] D. C. W. Ramos and P. L. D. Peres, "A less conservative LMI condition for the robust stability of discrete-time uncertain systems," *Syst. Contr. Lett.*, vol. 43, no. 5, pp. 371–378, August 2001.
- [28] W. Xie, " \mathcal{H}_2 gain scheduled state feedback for LPV system with new LMI formulation," *IEE Proc. — Contr. Theory and Appl.*, vol. 152, no. 6, pp. 693–697, November 2005.