2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Global Stabilization at Arbitrary Eigenstates of N-dimensional Quantum Spin Systems via Continuous Feedback*

Koji Tsumura

Abstract— **In this paper, we deal with a stabilization problem on quantum spin systems in general dimensions. The spin systems are supposed under continuous measurement by mutual interference with laser beams and a magnetic field is applied for control. The quantum states of spin systems can be estimated by quantum filtering and the intension of the magnetic field is controlled according to the estimation. We show that the global stabilization at arbitrary eigenstates of the spin systems is possible by continuous control inputs. Our proposing control input is the sum of two terms: a term which attracts the quantum states to an objective eigenstate and the other term which draws apart from the other equilibrium points. We also demonstrate the results by a numerical simulation.**

I. Introduction

Quantum information technologies have been actively investigated in the broad fields of physics or information theory [12]. On of problems for their realization is the generation or preservation of quantum bit (q-bit) under noisy environment. Quantum feedback control is indispensable for those purposes and the theory for it has been constructed. Belavkin [2] and others [22] showed that the time evolution of estimated quantum states under continuous measurement can be described by a classical stochastic differential equation in the early 1990's. This is called "quantum filtering." After that, research on feedback control by using estimated quantum states has been actively investigated [21], [4], [23] and its effectiveness has been also demonstrated by actual experiments [6].

Quantum spin system is one of possible realizations as the quantum bits and a recent notable result [19] is on feedback control of single spin 1/2 systems by using a continuous control input. This result is important for a possibility or a theoretical guarantee of feedback control, however its application is limited by the dimension of the systems. The generalization of the dimension was attained by Mirrahimi & van Handel [11]. They proposed a switching control for a group of atoms to globally stabilize the angular moments at arbitrary eigenstates. The proof is done by the strict analysis on the sample paths of the quantum state. This is the first result to show the global stability for quantum spin systems in general dimensions.

With those results, our interest naturally moves to a question on the global stabilizability of the quantum spin systems by continuous feedback. From an engineering sense, this problem is important for the realizability of apparatus because precise switching is actually difficult and also switching usually causes undesirable noise. Another, however more crucial, motivation is that this problem is also significant from a sense of pure physics and mathematics, and it is the main subject of this paper.

Recently, Tsumura [16], [17] proved that global stabilizability is possible by a continuous control input. This control scheme was firstly proposed in [9] and its effectiveness was demonstrated by numerical examples. However, the target state is limited to the maximum energy eigenstates. This paper solves this limitation by introducing another new control input and proves that the global stability at arbitrary eigenstates of N-dimensional quantum spin systems is possible by continuous control. The proposing continuous control signal is a sum of two terms: a control signal which attracts the quantum states to a target eigenstate and another signal which draws away from the other equilibrium points. The related work on an almost global stability was reported in [1] and global stability except for some special points was shown to be possible.

This paper is organized as follows. In Section II, we introduce the problem setting and some preliminaries. In Section III, we give the main result of this paper and its proof. The procedure of the proof is similar to that of Mirrahimi & van Handel [11] and Tsumura [16], [17]. In Section IV, we show a numerical example in order to demonstrate the efficiency of our proposing control scheme and we conclude this paper in Section V.

II. Formulation

In this paper, we deal with the system in Fig. 1 [19], [20], [11] with *continuous measurement*. A group of atoms is held in a cavity and the purpose is to control their spin moment. A laser beam is applied to the atoms to cause mutual interaction between them and the interacted laser is observed by a photo detector. The intensity of the interacted laser brings the information on the angular moment of the atoms, however the observation of this indirect information causes an inevitable back action on the quantum state of the atoms. A magnetic field is also applied to the group of the atoms and its intension is controlled. By using the history of the indirectly observed information, the conditional expectation of the observable can be calculated [2]. This is called *quantum filtering* and the time evolution of the

$*$ This paper is a modified version of [18].

The author is with Department of Information Physics and Computing, The University of Tokyo, Hongo 7–3–1, Bunkyo-ku, Tokyo 113–8656, Japan, tel: +81–3–5841–6891, fax: +81–3–5841–6886, e-mail: tsumura@i.u-tokyo.ac.jp

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

estimated quantum state; density matrix, becomes a quantum version of a classical Kushner-Stratonovich equation [2], [3], [19].

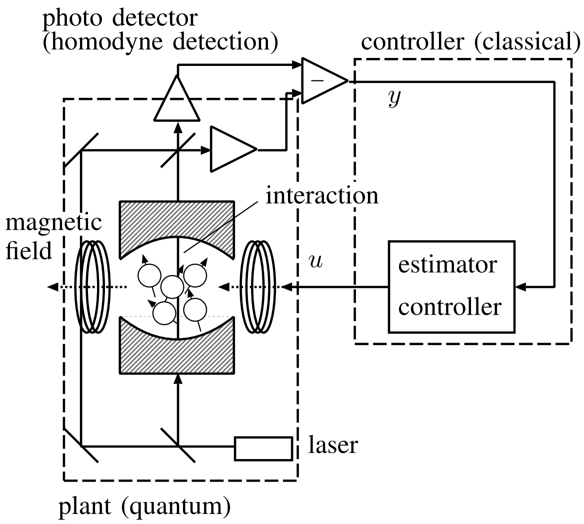

Fig.1: Quantum spin system under continuous measurement

A density matrix ρ_t represents the quantum state on the spin. When the number of atoms is n, the dimension of the quantum state on the angular moment is $N = 2J+1$ where $J = \frac{1}{2}n$ is the absolute value of the moment. When we observe the angular moment on z-axis and apply the magnetic field along y-axis, the corresponding quantum filter becomes a nonlinear Itô stochastic differential equation of ρ_t:

$$\begin{aligned} d\rho_t =& -iu_t[F_y, \rho_t]dt - \frac{1}{2}[F_z, [F_z, \rho_t]]dt \\ &+ \sqrt{\eta}(F_z\rho_t + \rho_t F_z - 2\mathrm{tr}\,(F_z\rho_t)\rho_t)dW_t, \quad (1) \\ dy =& 2\sqrt{\eta}\mathrm{tr}\,(F_z\rho)dt + dW_t \quad (2) \end{aligned}$$

where

$$\begin{aligned} \mathcal{S} &: \{\rho \in \mathcal{C}^{N\times N} : \rho = \rho^*,\ \mathrm{tr}\,(\rho) = 1,\ \rho \geq 0\}, \\ \rho_t &: \rho_t \in \mathcal{S}, \text{a quantum state at time } t, \\ dW_t &: \text{an infinitesimal Wiener increment satisfying} \\ &\quad \mathrm{E}[(dW_t)^2] = dt,\ \mathrm{E}[dW_t] = 0, \\ u_t &: \text{control input } (u_t \in \mathcal{R}), \\ y_t &: \text{output } (y_t \in \mathcal{R}), \\ \eta &: \text{the detector efficiency } (0 < \eta \leq 1), \end{aligned}$$

F_y: the angular momentum along the axis y of the form [10]:

$$F_y = \frac{1}{2i}\begin{bmatrix} 0 & -c_1 & & & \\ c_1 & 0 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{2J-1} & 0 & -c_{2J} \\ & & & c_{2J} & 0 \end{bmatrix},$$

$$c_m = \sqrt{(2J+1-m)m}, \quad (3)$$

F_z: the angular momentum along the axis z of the form [10]:

$$F_z = \begin{bmatrix} J & & & & \\ & J-1 & & & \\ & & \ddots & & \\ & & & -J+1 & \\ & & & & -J \end{bmatrix}. \quad (4)$$

This is called SME (*stochastic master equation*) and it has been mainly investigated in the research field of quantum control. It should be noted that the solution of (1) is continuous in time [13] if u_t is continuous. We also define some notations:

$$\psi_i := [0\cdots0\underbrace{1}_{i-\text{th}}0\cdots0]^*, \quad (5)$$

$$\rho_{\psi_i} := \psi_i\psi_i^*, \quad (6)$$

$$V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho) := 1 - \mathrm{tr}\,(\rho\rho_{\mathrm{f}}), \quad (7)$$

$$V^{\mathrm{II}}_{\rho_{\mathrm{f}}}(\rho) := 1 - (\mathrm{tr}\,(\rho\rho_{\mathrm{f}}))^2, \quad (8)$$

$$V^{\mathrm{III}}_{\rho_{\mathrm{f}}}(\rho) := \lambda_i - \mathrm{tr}\,(F_z\rho), \quad (9)$$

$$\lambda_i := J - (i-1), \quad (10)$$

where $\rho_{\mathrm{f}} \in \mathcal{S}$ is one of eigenstates ρ_{ψ_i}, $i = 1, 2, \ldots, N$. Note that $0 \leq V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho) \leq 1$ ($0 \leq V^{\mathrm{II}}_{\rho_{\mathrm{f}}}(\rho) \leq 1$), and $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho) = 0$ ($V^{\mathrm{II}}_{\rho_{\mathrm{f}}}(\rho) = 0$) iff $\rho = \rho_{\mathrm{f}}$. Moreover, for $\epsilon > 0$, define

$$\mathcal{S}^{<\epsilon}_{\rho_{\mathrm{f}}} := \left\{\rho \,|\, 0 \leq V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho) < \epsilon\right\}, \quad (11)$$

$$\mathcal{S}^{\epsilon}_{\rho_{\mathrm{f}}} := \left\{\rho \,|\, V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho) = \epsilon\right\}, \quad (12)$$

$$\mathcal{S}^{\epsilon<}_{\rho_{\mathrm{f}}} := \left\{\rho \,|\, \epsilon < V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho)\right\}. \quad (13)$$

The control objective is to globally stabilize the quantum state ρ_t on an eigenstate $\rho_{\mathrm{f}} = \rho_{\psi_i}$ by controlling the intensity u_t of the magnetic field, which is a function of ρ_t or its record.

We define the stochastic stability of (1) as follows.

Definition 2.1: [8] Let ρ_{e} be an equilibrium point of (1), i.e. $d\rho_t|_{\rho_t=\rho_{\mathrm{e}}} = 0$. Then

1. the equilibrium ρ_{e} is said to be *stable in probability* if
$$\lim_{\rho_0\to\rho_{\mathrm{e}}} \mathrm{Pr}\left(\sup_{0<t<\infty} \|\rho_t - \rho_{\mathrm{e}}\| \geq \epsilon\right) = 0,\ \forall\epsilon > 0, \quad (14)$$
where $\|\cdot\|$ is an arbitrary norm of a matrix in $\mathcal{C}^{N\times N}$.
2. The equilibrium ρ_{e} is *globally stable* if it is stable in probability and additionally
$$\mathrm{Pr}\left(\lim_{t\to\infty} \rho_t = \rho_{\mathrm{e}}\right) = 1,\ \forall\rho_0 \in \mathcal{S}. \quad (15)$$

For showing the stochastic stabilities of (1), a stochastic version of the Lyapunov theorem is available. At first define a nonnegative real-valued continuous function $V(\cdot)$ on $\mathcal{S}$. Also define $\rho_t^{\rho_\iota} := \rho_t$ for $\rho_0 = \rho_\iota$, a level set $\mathcal{Q}_\epsilon := \{\rho \in \mathcal{S} : V(\rho) < \epsilon\}$, $\tau_\epsilon := \inf\{t : \rho_t^{\rho_\iota} \notin \mathcal{Q}_\epsilon\}$, $\tilde{\rho}_t^{\rho_\iota} := \rho_{t\wedge\tau_\epsilon}^{\rho_\iota}$ where $t \wedge \tau_\epsilon := \min(t, \tau_\epsilon)$, $\mathcal{L}$: infinitesimal operator, and $\mathcal{L}_\epsilon$: restriction of $\mathcal{L}$ on $\tilde{\rho}_t^{\rho_\iota}$, respectively. Then, we get the following propositions.

Proposition 2.1: [8] Suppose $\mathcal{L}_\epsilon V \leq 0$ in $\mathcal{Q}_\epsilon$. Then, the following hold:

1. $\lim_{t\to\infty} V(\tilde{\rho}_t^{\rho_\iota})$ exists a.s., so $V(\rho_t^{\rho_\iota})$ converges for a.e. path remaining in $\mathcal{Q}_\epsilon$.
2. Pr–$\lim_{t\to\infty} \mathcal{L}_\epsilon V(\tilde{\rho}_t^{\rho_\iota}) = 0$, so $\mathcal{L}_\epsilon V(\rho_t^{\rho_\iota}) \to 0$ in probability as $t \to \infty$ for almost all paths which never leave $\mathcal{Q}_\epsilon$.
3. For $\rho_\iota \in \mathcal{Q}_\epsilon$ and $\alpha \leq \epsilon$, we have the uniform estimate

$$\begin{aligned} &\Pr\left(\sup_{0\leq t<\infty} V(\rho_t^{\rho_\iota}) \geq \alpha\right) \\ &= \Pr\left(\sup_{0\leq t<\infty} V(\tilde{\rho}_t^{\rho_\iota}) \geq \alpha\right) \leq \frac{V(\rho_\iota)}{\alpha}. \end{aligned} \tag{16}$$

4. If $V(\rho_\mathrm{s}) = 0$ and $V(\rho) \neq 0$ for $\rho \neq \rho_\mathrm{s}$, where $\rho_\mathrm{s} \in \mathcal{Q}_\epsilon$, then ρ_s is stable in probability.

Definition 2.2: An *invariant set* $\mathcal{I} \subseteq \mathcal{S}$ is defined such as $\rho_t^{\rho_\iota} \in \mathcal{I}$, $\forall t \geq 0$ whenever $\rho_\iota \in \mathcal{I}$.

Proposition 2.2: [11] Assume the following:

1. $\mathcal{Q}_\epsilon$ is bounded and $\mathcal{L}_\epsilon V(\rho) \leq 0$, $\forall \rho \in \mathcal{Q}_\epsilon$.
2. For any bounded scalar continuous function $f(\rho)$ and a fixed t, $\mathrm{E}[f(\rho_t^{\rho_\iota})]$ is continuous on $\rho_\iota = \rho_0$.
3. For any positive real number κ and $\rho_\iota \in \mathcal{Q}_\epsilon$, $\Pr(\|\rho_t^{\rho_\iota} - \rho_\iota\| > \kappa) \to 0$, $t \to 0$.

Let $\mathcal{Q}_o$ be the set of all points within $\mathcal{Q}_\epsilon$ where $\mathcal{L}_\epsilon V(\rho) = 0$, and let $\mathcal{I}$ be the largest invariant set in $\mathcal{Q}_o$. Then, every solution ρ_t in $\mathcal{Q}_\epsilon$ tends to $\mathcal{I}$ as $t \to \infty$.

Here we consider the control problem:

Problem 2.1: For the controlled spin system (1) and (2), find a globally stabilizing controller u_t on an eigenstate $\rho_\mathrm{f} = \rho_{\psi_i}$.

This is not a trivial problem from the following viewpoints: (i) (1) is a nonlinear stochastic system, (ii) there exist plural locally stable equilibrium points when $u = 0$ because of the nonlinearity, (iii) because of a kind of symmetry of the dynamics, many of locally stabilizing control scheme on one of the equilibrium points also preserve the other equilibrium points.

On this stabilization problem, van Handel *et al.* [19] firstly introduced a globally stabilizing feedback controller for a special case of single spin 1/2 systems by using a continuous control rule. The limitation on the dimension of the systems and the difficulties mentioned before were solved by Mirrahimi & van Handel [11]. They rigorously proved that there exists a globally stabilizing control scheme for N-dimensional quantum spin systems on arbitrary target eigenstates by introducing a switching rule:

Proposition 2.3: [11] Consider the system (1) evolving in the set $\mathcal{S}$ and let $\gamma > 0$, $\rho_\mathrm{f} = \rho_{\psi_i}$ and

$$u_1(\rho_t) := -\mathrm{tr}\,(i[F_y, \rho_t]\rho_\mathrm{f}). \tag{17}$$

Moreover, consider the following control scheme:

1. $u_t = u_1(\rho_t)$ if $V_{\rho_\mathrm{f}}^\mathrm{I}(\rho_t) \leq 1 - \gamma$;
2. $u_t = 1$ if $V_{\rho_\mathrm{f}}^\mathrm{I}(\rho_t) \geq 1 - \gamma/2$;
3. If $\rho_t \in \mathcal{B} = \{\rho : 1 - \gamma < V_{\rho_\mathrm{f}}^\mathrm{I}(\rho_t) < 1 - \gamma/2\}$, then $u_t = u_1(\rho_t)$ if ρ_t last entered $\mathcal{B}$ through the boundary $V_{\rho_\mathrm{f}}^\mathrm{I}(\rho) = 1 - \gamma$, and $u_t = 1$ otherwise.

Then $\exists \gamma > 0$ s.t. u_t globally stabilizes (1) around ρ_f and $\mathrm{E}[\rho_t] \to \rho_\mathrm{f}$ as $t \to \infty$.

This is the first and important result to show the global stability for quantum spin systems in general dimensions.

The scheme in Proposition 2.3 is a complex switching control and it should be avoided from the view point of practical use. Moreover, the essential question whether the quantum spin systems can be globally stabilized by continuous feedback is interesting itself in a sense of pure physics or mathematics and it is one of main research subjects in this field. With this motivation, Tsumura showed global stabilizability by using a continuous control signal [16], [17], however, the target state is limited to the maximum energy eigenstate:

Proposition 2.4: [16], [17] Consider the system (1) evolving in the set $\mathcal{S}$. Let $\rho_\mathrm{f} = \rho_{\psi_1}$ and $\eta > 0$. Then,

$$\begin{aligned} u_t &= \alpha u_1(\rho_t) + \beta V_{\rho_\mathrm{f}}^\mathrm{I}(\rho_t) \\ &\alpha,\ \beta > 0 \end{aligned} \tag{18}$$

globally stabilizes (1) around ρ_f and $\mathrm{E}[\rho_t] \to \rho_\mathrm{f}$ as $t \to \infty$ when

$$\frac{\beta^2}{8\alpha\eta} < 1. \tag{19}$$

The main purpose of this paper is to remove the limitation of the target state.

III. Main Result

In this section, we show the stabilizability of (1) with a continuous feedback by modifying (18) to a new one and provide the strict proof for the global stabilizability at arbitrary eigenstates $\rho_\mathrm{f} = \rho_{\psi_i}$, $i = 1, 2, \ldots, N$. We get the following theorem:

Theorem 3.1: Consider the system (1) evolving in the set $\mathcal{S}$. Let $\rho_\mathrm{f} = \rho_{\psi_i}$, $i = 1, 2, \ldots, N$ and $\eta > 0$. Then,

$$\begin{aligned} u_t &= \alpha u_1(\rho_t) + \beta V_{\rho_\mathrm{f}}^\mathrm{III}(\rho_t) \\ &\alpha,\ \beta > 0 \end{aligned} \tag{20}$$

globally stabilizes (1) around ρ_f and $\mathrm{E}[\rho_t] \to \rho_\mathrm{f}$ as $t \to \infty$ when

$$\frac{\beta^2}{8\alpha\eta} < 1. \tag{21}$$

Remark 3.1: This is the first result to show the global stabilizability of general finite dimensional quantum systems at arbitrary eigenstates by continuous feedback for the type of the master equation (1). Note that α and β are design parameters and we can always find them satisfying the condition (21) if $\eta > 0$.

We prove Theorem 3.1 in the followings. The procedure of the proof is similar to that in [11], [16], [17] and it is composed of showing the following three statements:

Step 1: $\rho_{\mathrm{f}} = \rho_{\psi_i}$ is stable in probability.

Step 2: there exists $0 < \gamma < 1$ and almost all sample paths which never leave the domain $\mathcal{S}_{\rho_{\mathrm{f}}}^{<1-\gamma}$ converge to ρ_{f}.

Step 3: for almost all sample paths there exists a finite time T and after it, they never leave $\mathcal{S}_{\rho_{\mathrm{f}}}^{<1-\gamma}$.

We prove each statement in the following.

Step 1

In order to show the statement of Step 1, we find a Lyapunov function which satisfies the conditions of Proposition 2.1 around ρ_{f}. We show a key lemma for it.

Lemma 3.1: With the control input (20),

$$\mathcal{L}_{\epsilon} V_{\rho_{\mathrm{f}}}^{\mathrm{II}} \le 0 \tag{22}$$

is satisfied in the subsets $\mathcal{S}_{\rho_{\mathrm{f}}}^{<1-\gamma_o}$, where

$$\gamma_o = \frac{\beta^2}{8\alpha\eta} < 1. \tag{23}$$

Proof: By the direct calculation of $\mathcal{L}V_{\rho_{\mathrm{f}}}^{\mathrm{II}}$, we get the following:

$$\begin{aligned} \mathcal{L}V_{\rho_{\mathrm{f}}}^{\mathrm{II}} =& -2\mathrm{tr}\,(\rho_t\rho_{\mathrm{f}})\,u_t\,\mathrm{tr}\,(-i[F_y,\rho_t]\rho_{\mathrm{f}}) \\ & -4\eta(\lambda_i - \mathrm{tr}\,(F_z\rho_t))^2(\mathrm{tr}\,(\rho_t\rho_{\mathrm{f}}))^2 \\ =& -2\mathrm{tr}\,(\rho_t\rho_{\mathrm{f}})(\alpha u_1 + \beta V_{\rho_{\mathrm{f}}}^{\mathrm{III}})u_1 \\ & -4\eta(\lambda_i - \mathrm{tr}\,(F_z\rho_t))^2(\mathrm{tr}\,(\rho_t\rho_{\mathrm{f}}))^2 \\ =& -2\mathrm{tr}\,(\rho_t\rho_{\mathrm{f}})\left\{(\alpha u_1 + \beta V_{\rho_{\mathrm{f}}}^{\mathrm{III}})u_1 \right. \\ & \left. + 2\eta(\lambda_i - \mathrm{tr}\,(F_z\rho_t))^2\mathrm{tr}\,(\rho_t\rho_{\mathrm{f}})\right\}. \end{aligned} \tag{24}$$

The factor $\mathrm{tr}\,(\rho_t\rho_{\mathrm{f}})$ is always nonnegative, therefore, the factor:

$$(\alpha u_1 + \beta V_{\rho_{\mathrm{f}}}^{\mathrm{III}})u_1 + 2\eta(\lambda_i - \mathrm{tr}\,(F_z\rho_t))^2\mathrm{tr}\,(\rho_t\rho_{\mathrm{f}}) \tag{25}$$

should be nonnegative for $\mathcal{L}V_{\rho_{\mathrm{f}}}^{\mathrm{II}}$ to be nonpositive. It can be reduced as

$$\begin{aligned} &\left(\alpha u_1 + \beta V_{\rho_{\mathrm{f}}}^{\mathrm{III}}\right)u_1 + 2\eta(\lambda_i - \mathrm{tr}\,(F_z\rho_t))^2\mathrm{tr}\,(\rho\rho_{\mathrm{f}}) \\ &= \alpha\left(u_1 + \frac{\beta}{\alpha}\frac{V_{\rho_{\mathrm{f}}}^{\mathrm{III}}}{2}\right)^2 - \frac{\beta^2}{\alpha}\frac{(V_{\rho_{\mathrm{f}}}^{\mathrm{III}})^2}{4} + 2\eta(V_{\rho_{\mathrm{f}}}^{\mathrm{III}})^2\mathrm{tr}\,(\rho\rho_{\mathrm{f}}) \\ &= \alpha\left(u_1 + \frac{\beta}{\alpha}\frac{V_{\rho_{\mathrm{f}}}^{\mathrm{III}}}{2}\right)^2 + 2\eta(V_{\rho_{\mathrm{f}}}^{\mathrm{III}})^2\left(\mathrm{tr}\,(\rho\rho_{\mathrm{f}}) - \frac{\beta^2}{8\alpha\eta}\right). \end{aligned} \tag{26}$$

Therefore, when $\frac{\beta^2}{8\alpha\eta} < 1$ is satisfied, we can set

$$\gamma_o := \frac{\beta^2}{8\alpha\eta} \tag{27}$$

and for the case: $\gamma_o < \rho_{ii} \le 1$, we conclude $\mathcal{L}V_{\rho_{\mathrm{f}}}^{\mathrm{II}} \le 0$. This implies

$$\mathcal{L}_{1-\gamma_o} V_{\rho_{\mathrm{f}}}^{\mathrm{II}} \le 0,\ \forall \rho \in \mathcal{S}_{\rho_{\mathrm{f}}}^{<1-\gamma_o}. \tag{28}$$

■

From Lemma 3.1, the statements in Proposition 2.1 are concluded in the subset $\mathcal{S}_{\rho_{\mathrm{f}}}^{<1-\gamma_o}$. In particular, $\rho = \rho_{\mathrm{f}}$ is stable in probability.

Step 2

At first, we show the following lemma:

Lemma 3.2: The largest invariant set in $\{\rho \,|\, \mathcal{L}_{<1-\gamma_o} V_{\rho_{\mathrm{f}}}^{\mathrm{II}} = 0\}$ in the subset $\mathcal{S}_{\rho_{\mathrm{f}}}^{<1-\gamma_o}$ is $\{\rho_{\mathrm{f}}\}$.

Proof: In the proof of Lemma 3.1, the case $\mathcal{L}_{<1-\gamma_o} V_{\rho_{\mathrm{f}}}^{\mathrm{II}} = 0$ in the subset $\mathcal{S}_{\rho_{\mathrm{f}}}^{<1-\gamma_o}$ is only when $\mathrm{tr}\,(\rho\rho_{\mathrm{f}}) = 0$ or $V_{\rho_{\mathrm{f}}}^{\mathrm{III}} = 0$. However, $\{\rho \,|\, \mathrm{tr}\,(\rho\rho_{\mathrm{f}}) = 0, \rho_{ii} \neq 1\}$ is not an invariant set of (1) from Lemma A.2. Similarly, $\{\rho \,|\, V_{\rho_{\mathrm{f}}}^{\mathrm{III}}(\rho) = 0, \rho_{ii} \neq 1\}$ is not also an invariant set of (1) from Lemma A.1. On the contrary, we can check $V_{\rho_{\mathrm{f}}}^{\mathrm{II}} = 0$, $\mathcal{L}_{<1-\gamma_o} V_{\rho_{\mathrm{f}}}^{\mathrm{II}} = 0$ at $\rho = \rho_{\mathrm{f}}$ and it is an invariant set of (1). ■

Then, the key lemma in this step is given as follows.

Lemma 3.3: The solution ρ_t of (1) converges to ρ_{f} as $t \to \infty$ for almost all paths that never exit the set $\mathcal{S}_{\rho_{\mathrm{f}}}^{<1-\gamma_o}$.

Proof: From Lemma 3.1, the master equation (1) with the control input (20) satisfies the conditions in Proposition 2.2, therefore, with Lemma 3.2, the sample paths which never leave the subset $\mathcal{S}_{\rho_{\mathrm{f}}}^{<1-\gamma_o}$ converge to ρ_{f} in probability. Moreover, $V_{\rho_{\mathrm{f}}}^{\mathrm{II}}$ converges almost surely from Proposition 2.1. With this, the boundedness of $V_{\rho_{\mathrm{f}}}^{\mathrm{II}}$ and Lebesgue's dominated convergence, we can show that almost all paths converge to ρ_{f} by employing the similar discussion in the proof to Lemma 4.9 in [11]. ■

Step 3

We finally examine the behavior of the paths when they leave $\mathcal{S}_{\rho_{\mathrm{f}}}^{<1-\gamma_o}$ or the initial state is outside it. We get the following lemma:

Lemma 3.4: The solution $\rho_t^{\rho_\iota}$ of (1) where $\rho_0 = \rho_\iota \in \mathcal{S}_{\rho_{\mathrm{f}}}^{>1-\gamma_o}$ satisfies

$$\sup_{\rho_\iota \in \mathcal{S}_{\rho_{\mathrm{f}}}^{>1-\gamma_o}} \mathrm{E}[\min\ t : \rho_t^{\rho_\iota} \notin \mathcal{S}_{\rho_{\mathrm{f}}}^{>1-\gamma_o}] < \infty. \tag{29}$$

For the proof of this lemma, we introduce the following two propositions:

Proposition 3.1: [15], [7] Consider a Stratonovich stochastic differential equation:

$$d\varphi_t = f_0(\varphi_t, t)dt + \sum_{l=1}^{n} f_l(\varphi_t, t) \circ dW^l(t). \tag{30}$$

Assume that the coefficients $f_l(x,t)$, $l = 0, 1, 2, \ldots, n$ are of the class $C_b^{k+1,\delta}$ for some $k \ge 2$ and $\delta > 0$ (see Appendix for the definition of $C_b^{k+1,\delta}$). Let φ_t be the Brownian flow determined by (30). Then the support of $\varphi(t) = \varphi_t$ as the C^{k-1}-flow is equal to the closure $\{\varphi_t : \xi \in \Xi\}$ of

$$\frac{d\varphi_t}{dt} = f_0(\varphi_t, t) + \sum_{l=1}^{n} f_l(\varphi_t, t)\xi^l(t) \tag{31}$$

in the space W_{k-1}, where Ξ is the set of all deterministic piecewise smooth function and $W_k = C([0,T] : C^k)$.

Proposition 3.2: [5] Consider diffusion process $x_t \in E$ starting from x where E is the domain of x_t. Let Γ be a subset of E and $\tau_x(\Gamma)$ be the first exit time of x_t from Γ. Then for all $T \geq 0, x \in E$,

$$\mathrm{E}[\tau_x(\Gamma)] \leq \frac{T}{1 - \sup_{x\in E} \Pr\{\tau_x(\Gamma) > T\}}. \tag{32}$$

Proof of Lemma 3.4 At first, we claim that the support of $V^{\mathrm{I}}_{\rho_\mathrm{f}}(\rho_t)$ contains $[0,\gamma]$ when $V^{\mathrm{I}}_{\rho_\mathrm{f}}(\rho_0) = \gamma$ by using Proposition 3.1.

By employing Proposition A.1 on the Stratonovich form of (1), the corresponding deterministic differential equation of ρ_t is

$$\begin{aligned}\frac{d}{dt}\rho_t = {}& \mathcal{D}_{F_z}(\rho_t) - \frac{1}{2}\eta\left(-2\mathcal{E}_{F_z}(\rho_t)\mathcal{H}_{F_z}(\rho_t) + \mathcal{K}_{F_z}(\rho_t)\right) \\ & + u\mathcal{G}_{F_y}(\rho_t) + \sqrt{\eta}\mathcal{H}_{F_z}(\rho_t)\xi \end{aligned} \tag{33}$$

where ξ is an associated input. With this (33), we get

$$\begin{aligned}\frac{d}{dt}V^{\mathrm{I}}_{\rho_\mathrm{f}}(\rho) &= -\operatorname{tr}\left(\frac{d\rho}{dt}\rho_\mathrm{f}\right) \\ &= -\operatorname{tr}\left(\left\{-\frac{1}{2}\eta(-2\mathcal{E}_{F_z}(\rho)\mathcal{H}_{F_z}(\rho) + \mathcal{K}_{F_z}(\rho))\right.\right. \\ &\qquad \left.\left. + u\mathcal{G}_{F_y}(\rho) + \sqrt{\eta}\mathcal{H}_{F_z}(\rho)\xi\right\}\rho_\mathrm{f}\right). \end{aligned} \tag{34}$$

The term which includes ξ in (34) is

$$\begin{aligned}\operatorname{tr}\left(\mathcal{H}_{F_z}(\rho)\xi\rho_\mathrm{f}\right) &= \operatorname{tr}\left((F_z\rho + \rho F_z - 2\operatorname{tr}(F_z\rho)\rho)\rho_\mathrm{f}\right)\xi \\ &= 2\left(\lambda_i - \operatorname{tr}(F_z\rho)\right)\operatorname{tr}(\rho\rho_\mathrm{f})\xi \\ &= 2V^{\mathrm{III}}_{\rho_\mathrm{f}}(\rho)\rho_{ii}\xi. \end{aligned} \tag{35}$$

The case (35) = 0 for $\xi \neq 0$ is when $\rho_{ii} = 0$ or $V^{\mathrm{III}}_{\rho_\mathrm{f}}(\rho) = 0$. When $\rho_{ii} = 0$ and $V^{\mathrm{III}}_{\rho_\mathrm{f}}(\rho) \neq 0$, $V^{\mathrm{I}}_{\rho_\mathrm{f}}(\rho) = 1$ and $u = V^{\mathrm{III}}_{\rho_\mathrm{f}}(\rho) \neq 0$, however, it is known that $\{\rho \,|\, V^{\mathrm{I}}_{\rho_\mathrm{f}}(\rho) = 1\}$ is not an invariant set of (1) when $u_t \neq 0$ [11]. Next, $\{\rho \,|\, \rho_{ii} \neq 1, V^{\mathrm{III}}_{\rho_\mathrm{f}}(\rho) = 0\}$ is not an invariant of (1) from Lemma A.1. Finally, when $\rho_{ii} = 1$ ($V^{\mathrm{II}}_{\rho_\mathrm{f}}(\rho) = 0$), $\rho = \rho_\mathrm{f}$ and it is the target point. In the other cases, (35) except for ξ is nonzero.

From above and Proposition 3.1, the assertion that the support of $V^{\mathrm{I}}_{\rho_\mathrm{f}}(\rho_t)$ contains $[0,\gamma]$ when $V^{\mathrm{I}}_{\rho_\mathrm{f}}(\rho_0) = \gamma$. Therefore, for any finite time T, there exists a measurable set of sample paths ρ_t, which leave $\mathcal{S}^{>1-\gamma_0}_{\rho_\mathrm{f}}$ in $[0,T]$. Finally with Proposition 3.2, we can conclude the statement [11]. ■

By using Lemma 3.4 and employing the similar discussion of [11], we can derive the following lemma.

Lemma 3.5: For almost every sample path of ρ_t there exists a time $T < \infty$ after which the path never exits the set $\mathcal{S}^{<1-\gamma_o}_{\rho_\mathrm{f}}$.

We omit the proof.

Proof of Theorem 3.1 By unifying the results of Step 1–3, we can conclude the convergence of the solution to the target point. The convergence of the expectation can be also derived by dominated convergence. ■

IV. Numerical Example

We demonstrate the efficiency of the proposing continuous feedback by using a numerical simulation. Here we consider a spin system where $N = 4$. The initial and the target states are

$$\rho_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \rho_\mathrm{f} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

respectively. We simulate the solution ρ_t with $\eta = 0.8$, $\alpha = 1$, and $\beta = 1$, 10 times. This case satisfies the condition (21) and the global stability is guaranteed. Fig. 2 shows the average of 10 transitions of $V^{\mathrm{I}}_{\rho_\mathrm{f}}$, which indicates the gap between the target ρ_f and ρ_t ($V^{\mathrm{I}}_{\rho_\mathrm{f}}(\rho) = 0$ means $\rho = \rho_\mathrm{f}$), with the above case.

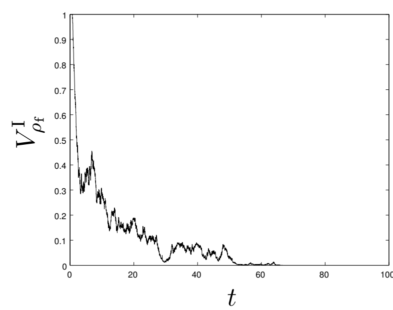

Fig. 2: Average of 10 transitions of $V^{\mathrm{I}}_{\rho_\mathrm{f}}$ with $\eta = 0.8$, $\alpha = 1$, $\beta = 1$

From the simulation, we can confirm the efficiency of our proposing continuous feedback. Note that (21) is a sufficient condition for the global stability, therefore, even if it is not satisfied, the system may be stable. However, we recognized the significance of the condition (21) with respect to the convergence rate of ρ_t to the target points by several simulations.

V. Conclusion

In this paper, we considered control problem of N-dimensional quantum spin systems and showed that continuous feedback is possible to stochastically globally stabilize the systems on arbitrary eigenstates. The control scheme is composed of two distinctive terms and the stability is proved by following the sample paths of the stochastic master equation strictly.

Acknowledgement: This work was supported in part by Grant-in-Aid for Scientific Research (C) (19560436), Japan Society for the Promotion of Science.

APPENDIX

Definition A.1: The notation $C_b^{m,\delta}$ is the set $\{f \in C^{k+1}$, $D^\alpha f$ $(|\alpha| = m)$: δ-Hölder continuous, $\|f\|_{m+\delta} < \infty$ $\}$ and

$$\|f\|_{m+\delta} := \|f\|_m + \sum_{|\alpha|=m} \sup \frac{D^\alpha f(x) - D^\alpha f(y)}{|x-y|^\delta}. \quad (36)$$

Proposition A.1: [14] The Stratonovich form of (1) is given by

$$\begin{aligned} d\rho_t = \mathcal{D}_{F_z}(\rho_t)dt - \frac{1}{2}\eta\left(-2\mathcal{E}_{F_z}(\rho_t)\mathcal{H}_{F_z}(\rho_t) + \mathcal{K}_{F_z}(\rho_t)\right)dt \\ + u_t\mathcal{G}_{F_y}(\rho_t)dt + \sqrt{\eta}\mathcal{H}_{F_z}(\rho_t) \circ dW, \end{aligned} \quad (37)$$

where

$$\begin{aligned} \mathcal{D}_{F_z}(\rho) &:= -\frac{1}{2}[F_z, [F_z, \rho]] \\ \mathcal{E}_{F_z}(\rho) &:= 2\mathrm{tr}\,(F_z\rho) \\ \mathcal{H}_{F_z}(\rho) &:= F_z\rho + \rho F_z - 2\mathrm{tr}\,(F_z\rho)\rho \\ \mathcal{K}_{F_z}(\rho) &:= F_z^2\rho + 2F_z\rho F_z^* + \rho(F_z^*)^2 \\ &\quad - \mathrm{tr}\,(F_z^2\rho + 2F_z\rho F_z^* + \rho(F_z^*)^2)\rho \\ \mathcal{G}_{F_y}(\rho) &:= -i[F_y, \rho]. \end{aligned} \quad (38)$$

Lemma A.1: $\{\rho \,|\, V_{\rho_\mathrm{f}}^{\mathrm{III}}(\rho) = 0, \rho_{ii} \neq 1\}$ is not an invariant set of (1).

Proof: With (33), we differentiate $V_{\rho_\mathrm{f}}^{\mathrm{III}}(\rho)$ as

$$\begin{aligned} \frac{d}{dt}V_{\rho_\mathrm{f}}^{\mathrm{III}}(\rho) &= -\mathrm{tr}\left(\frac{d\rho}{dt}F_z\right) \\ &= -\mathrm{tr}\left(\left\{-\frac{1}{2}\eta(-2\mathcal{E}_{F_z}(\rho)\mathcal{H}_{F_z}(\rho) + \mathcal{K}_{F_z}(\rho))\right.\right. \\ &\quad \left.\left. + u\mathcal{G}_{F_y}(\rho) + \sqrt{\eta}\mathcal{H}_{F_z}(\rho)\xi\right\} F_z\right). \end{aligned} \quad (39)$$

The term which includes ξ in (39) is

$$\begin{aligned} \mathrm{tr}\,(\mathcal{H}_{F_z}(\rho)\xi F_z) &= \mathrm{tr}\,((F_z\rho + \rho F_z - 2\mathrm{tr}\,(F_z\rho)\rho)F_z)\xi \\ &= 2\left(\mathrm{tr}\,F_z^2\rho - (\mathrm{tr}\,F_z\rho)^2\right)\xi. \end{aligned} \quad (40)$$

The case eq. (40) = 0 for $\xi \neq 0$ is only at $\rho = \rho_{\psi_j}$, however, when $\rho = \rho_{\psi_j} \neq \rho_\mathrm{f}$, $u = \beta V_{\rho_\mathrm{f}}^{\mathrm{III}}(\rho) = \beta(\lambda_\mathrm{f} - (J-(j-1))) \neq 0$. Therefore, $\rho = \rho_{\psi_j} \neq \rho_\mathrm{f}$ is not an invariant set of (1) [11]. With this and from Proposition 3.1, we can conclude the statement of this lemma. ■

Lemma A.2: $\{\rho \,|\, \mathrm{tr}\,(\rho\rho_\mathrm{f}) = 0, \rho_{ii} \neq 1\}$ is not an invariant set of (1).

Proof: With (33), we get

$$\begin{aligned} \frac{d}{dt}\mathrm{tr}\,(\rho_t\rho_\mathrm{f}) &= \mathrm{tr}\left(\left\{-\frac{1}{2}\eta(-2\mathcal{E}_{F_z}(\rho)\mathcal{H}_{F_z}(\rho) + \mathcal{K}_{F_z}(\rho))\right.\right. \\ &\quad \left.\left. + u\mathcal{G}_{F_y}(\rho) + \sqrt{\eta}\mathcal{H}_{F_z}(\rho)\xi\right\} \rho_\mathrm{f}\right). \end{aligned} \quad (41)$$

The term which includes ξ in (41) is

$$\begin{aligned} \mathrm{tr}\,(\mathcal{H}_{F_z}(\rho)\xi\rho_\mathrm{f}) &= \mathrm{tr}\,((F_z\rho + \rho F_z - 2\mathrm{tr}\,(F_z\rho)\rho)\rho_\mathrm{f})\xi \\ &= 2\,(\lambda_i - \mathrm{tr}\,(F_z\rho))\,\mathrm{tr}\,(\rho\rho_\mathrm{f})\xi \\ &= 2V_{\rho_\mathrm{f}}^{\mathrm{III}}(\rho)\rho_{ii}\xi. \end{aligned} \quad (42)$$

The case (42) = 0 for $\xi \neq 0$ is when $\rho_{ii} = 0$ or $V_{\rho_\mathrm{f}}^{\mathrm{III}}(\rho) = 0$. When $\rho_{ii} = 0$ and $V_{\rho_\mathrm{f}}^{\mathrm{III}}(\rho) \neq 0$, $V_{\rho_\mathrm{f}}^{\mathrm{I}}(\rho) = 1$ and $u = V_{\rho_\mathrm{f}}^{\mathrm{III}}(\rho) \neq 0$, however, it is known that $\{\rho \,|\, V_{\rho_\mathrm{f}}^{\mathrm{I}}(\rho) = 1\}$ is not an invariant set of (1) when $u_t \neq 0$ [11]. Next, $\{\rho \,|\, \rho_{ii} \neq 1, V_{\rho_\mathrm{f}}^{\mathrm{III}}(\rho) = 0\}$ is not an invariant of (1) from Lemma A.1. With this and from Proposition 3.1, we can conclude the statement of this lemma. ■

REFERENCES

[1] C. Altafini and F. Ticozzi, Almost global stochastic feedback stabilization of conditional quantum dynamics, *arXiv:quant-ph/0510222 v1*, 28 Oct., 2005.

[2] V. P. Belavkin, Quantum stochastic calculus and quantum nonlinear filtering, *J. Multivariate Anal.*, 42, 171–202, 1992.

[3] L. M. Bouten, M. I. Guţă, and H. Massen, Stochastic Schrödinger equations, *J.Phys. A: Math. Gen.*, 37, 3189–3209, 2004.

[4] L. M. Bouten, S. C. Edwards, and V. P. Belavkin, Bellman equations for optimal feedback control of qubit states, *J. Phys. B, at. Mol. Opt. Phys.,*, 38, 151–160, 2005.

[5] E. B. Dynkin, *Markov Process*, volume I, Springer-Verlag, 1965.

[6] J.M. Geremia, J.K. Stockton, and H. Mabuchi, Real-time quantum feedback control of atomic spin-squeezing, *SCIENCE*, 304, 270–273, 2004.

[7] H. Kunita, *Stochastic flows and stochastic differential equations*, Cambridge University Press, 1990.

[8] H. J. Kushner, *Stochastic Stability and Control*, Academic Press, 1967.

[9] K. Matsumoto, *On Stabilization of Quantum Spin Systems via Non-continuous/continuous Feedback*, Bachelor thesis of University of Tokyo, 2006.

[10] E. Merzbachr, *Quantum mechanics*, Wiley, third edition, 1998.

[11] M. Mirrahimi and R. van Handel, Stabilizing feedback controls for quantum systems, *SIAM J. Control Optimization*, 46(2), 445–467, 2007 (also in *arXiv:math-ph/0510066 v2*, 2005)

[12] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000

[13] B. Øksendal, *Stochastic Differential Equations*, Springer, fifth edition, 1998.

[14] T. Sasaki, S. Hara and K. Tsumura, Analysis of controlled quantum dynamics by a complex matrix representation, *Proc. 34th SICE Symposium on Control Theory*, 159–164, October 31–November 2, Osaka, 2005.

[15] D. W. Stroock and S. R. S. Varadhan, On the support of diffusion processes with applications to the strong maximum principle, *Proc. 6th Berkeley Symp. Math. Statist. Probab. III*, 333–359, Univ. California Press, Berkeley, 1972.

[16] K. Tsumura, Stabilization of Quantum Spin Systems via Continuous Feedback, *Proc. 35th SICE Symposium on Control Theory*, 105–110, September 19–21, Osaka, 2006.

[17] K. Tsumura, Global stabilization of N-dimensional quantum spin systems via continuous feedback, *Proc. the 2007 American Control Conference*, 2129–2134, 2007.

[18] K. Tsumura, Global stabilization at arbitrary eigenstates of N-dimensional quantum spin systems via continuous feedback, *Mathematical Engineering Technical Reports*, METR 2007-48, The University of Tokyo, August, 2007 (also in *Proc. 36th SICE Symposium on Control Theory*, 311–316, September 5–7, Sapporo, 2007).

[19] R. van Handel, J. K. Stockton, and H. Mabuchi, Feedback control of quantum state reduction, *IEEE Transactions on Automatic Control*, 50(6), Jun 2005.

[20] R. van Handel, J. K. Stockton, and H. Mabuchi, Modeling and feedback control design for quantum state preparation, *J. Opt. B: Quantum Semiclass. Opt. 7*, S179–S197, 2005.

[21] J. Wang and H. M. Wiseman, Feedback-stabilization of an arbitrary pure state of a two-level atom, *Phys. Rev. A*, 64, 063810, 2001.

[22] H. M. Wiseman, Quantum theory of continuous feedback, *Phys. Rev. A*, 49(3), 2133–2150, 1994.

[23] N. Yamamoto, K.Tsumura, and S. Hara, Feedback control of quantum entanglement in a two-spin system, *Automatica*, 43, 981–992, 2007.